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ON AN EXTENSION OF FINITE FUNCTIONALS BY THE TRANSFINITE INDUCTION

MICHAL ŠABO

The very well known theorem concerning the extension of a σ-finite measure defined on a Boolean ring of subsets of an abstract set may be proved by means of the transfinite induction [1]. The main subject of this paper is to show how the transfinite induction can be used to extend a finite functional defined on a certain lattice. The main result can be used for an extension both of a measure and an integral. A similar result was obtained in [3] but with the help of another construction.

1

Let S be a system in which a limit is defined for some sequences $\{x_n\}$, $x_n \in S$. Let $A \subset S$ and $A^* = \{x \in S$, for which there is $\{a_n\}$, $a_n \in A$ and $x = \lim a_n\}$. Define a transfinite sequence: $A = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_\omega \subset \ldots$ as follows $A_\alpha = \bigcup_{\beta<\alpha} A^*_\beta$. The following statement holds: There exists the smallest ordinal σ such that $A^*_\sigma = A_\sigma$ i. e. A_σ is closed with regard to limits. The statement is an immediate consequence of the fact that for the ordinal Ω we have $A_\Omega = A^*_\Omega$ [2].

2

Let S be a lattice. Denote by $x \cup y$ $(x \cap y)$ the supremum (infimum) of $x, y \in S$. Let $\{x_n\}$ be a sequence of elements of S. Denote by $\bigcup x_n (\bigcap x_n)$ the supremum (infimum) of the sequence $\{x_n\}$ if it exists. Denote $\lim x_n = x$ or $x_n \to x (x_n, x \in S)$ if $\bigcap_{n\geq 1} \bigcup_{i\geq n} x_i$, $\bigcup_{n\geq 1} \bigcap_{i\geq n} x_i$ exist and $x = \bigcap_{n\geq 1} \bigcup_{i\geq n} x_i = \bigcup_{n\geq 1} \bigcap_{i\geq n} x_i$. If $\{x_n\}$ is an increasing (decreasing) sequence; i. e. $x_n \leq x_{n+1} (x_n \geq x_{n+1})$ and $x = \bigcup x_n$ $(x = \bigcap x_n)$, we shall write $x_n \nearrow x$ $(x_n \searrow x)$.

Assume, that S fulfills:

I.1. Let $\{x_n\}$, $x_n \in S$ be an increasing (decreasing) sequence bounded above (below). Then there exists $x \in S$ such that $x = \lim x_n$.

2. There are two binary operations $+$, $-$ on S satisfying the following conditions:

(a) $a, b, c \in S$, $a \leqslant b \Rightarrow b - a \geqslant c - c$.
(b) $a, b, c \in S$, $a \leqslant b \Rightarrow a + c \leqslant b + c$; $a - c \leqslant b - c$; $c - a \geqslant c - b$.
(c) $x_n, y_n, x, y \in S$, $x_n \to x$, $y_n \to y \Rightarrow x_n \cup y_n \to x \cup y$

$$x_n \cap y_n \to x \cap y$$
$$x_n - y_n \to x - y$$
$$x_n + y_n \to x + y.$$

Let A be a sublattice of S such that:

II.1. A is closed under $+$, $-$.

2. If $x \in S$, then there exist $a, b \in A$ such that $a \leqslant x \leqslant b$.

Let a real-valued finite functional J be defined on A such that:

III. 1. $a, b \in A$, $a \leqslant b \Rightarrow J(a) \leqslant J(b)$.

2. $a, b \in A$, $a \leqslant b \Rightarrow J(a) + J(b-a) = J(b)$.

3. $a, b \in A \Rightarrow J(a) - J(b) \leqslant J(a-b)$.

4. $a, b \in A \Rightarrow J(a \cap b) + J(a \cup b) = J(a) + J(b)$.

5. $a_n \in A$, $a_n \searrow o \Rightarrow \lim J(a_n) = 0$.

o is such element of S that $x - x = o$ for any $x \in S$. (I.2.a) implies that such element is uniquely defined. Evidently (by II.1.), $o \in A$.

We want to prove the existence of sublattice A_σ with $A \subset A_\sigma \subset S$ and of a finite real-valued functional $\bar{J}$ defined on A_σ, which is the unique extension of J such that for A_σ, $\bar{J}$ satisfies the conditions II, III and the property:

$$x_n \to x;\ x_n \in A_\sigma \Rightarrow x \in A_\sigma \text{ and } \lim \bar{J}(x_n) = \bar{J}(x). \tag{P}$$

3

The main purpose of this section is to prove that for any convergent sequence in A the following statement holds: $J(\lim a_n) = \lim J(a_n)$.

Lemma 1. $J(o) = 0$. It follows from (III.2.).

Lemma 2. *If $x_n \to x$; $x, x_n \in S$, then there exist $a, b \in A$ such that $a \leqslant x_n \leqslant b$ $n = 1, 2, 3, \ldots$*

Proof. $x = \bigcap_{n \geqslant 1} \bigcup_{i \geqslant n} x_i = \bigcup_{n \geqslant 1} \bigcap_{i \geqslant n} x_i$. (II.2.) implies the existence of $a, b \in A$ such that

$x_n \leqslant \bigcup_{i > 1} x_i \leqslant b$, $x_n \geqslant \bigcap_{i \geqslant 1} x_i \geqslant a$ $n = 1, 2, 3, \ldots$

Lemma 3. *If $a_n \nearrow o$, $a_n \in A$ then* $\lim J(a_n) = 0$.

Proof. If $o - a_n \searrow o$, then $J(o) - J(a_n) \leqslant J(o - a_n) \to 0$. Hence $\lim J(a_n) \geqslant J(o) = 0$.

Theorem 1. *Let $a_n \nearrow a$ ($a_n \searrow a$); $a_n, a \in A$. Then* $\lim J(a_n) = J(a)$.

Proof. $a-a_n\searrow o(a_n-a\searrow o)$ implies: $\lim J(a-a_n)=0$ ($\lim J(a_n-a)-0$). Hence $J(a)-J(a_n)\leqslant J(a-a_n)\to 0(J(a_n)-J(a)\leqslant J(a_n-a)\to 0)$; $\lim J(a_n)\geqslant J(a)$ ($\lim J(a_n)\leqslant J(a)$) .

Theorem 2. *Let* $a_n\to o$; $a_n\in A$. *Then* $\lim J(a_n)=0$.

Proof. If $a_n\to o$, $a_n\in A$, then $a_n\cup o\to o$. It implies: $a_n\cup o-a_n\to o$ and $a_n\cup o-a_n\geqslant o$. We have (see III.2.) $J(a_n)=J(a_n\cup o)-J((a_n\cup o)-a_n)$. Thus, we can assume that $a_n\geqslant o$. It is sufficient to show that $\limsup J(a_n)\leqslant 0$, because then $0\leqslant\liminf J(a_n)\leqslant\limsup J(a_n)\leqslant 0$; i. e. $\lim J(a_n)=0$.

From the definition of the limit $o=\bigcap_{n\geqslant 1}\bigcup_{i\geqslant n}a_i$. Denote $s_n=\bigcup_{i\geqslant n}a_i$. T ie sequer ce { } is decreasing and $s_n\searrow o$. We shall find a decreasing sequence $\{t_n\}$, t $o\leqslant t_n\leqslant s_n$ $n=1, 2, 3, \dots$ with the property:

$$\varepsilon\geqslant 0,\ x\in A,\ x\leqslant s_n\Rightarrow J(x)-J(t_n)\leqslant\varepsilon\sum_{i=1}^{n}\frac{1}{2^i}\,. \qquad \text{(A}$$

Assuming this, $\lim J(t_n)=0$ and if we put $x=a_n$, then $\limsup J(a_n)\leqslant\varepsilon$.

There is $c\in A$ (see Lemma 2.) such that $\bigcup_{i=n}^{m}a_i\leqslant\bigcup_{i\geqslant n}a_i\leqslant c$. Therefore the sequence $\left\{J\left(\bigcup_{i=n}^{m}a_i\right)\right\}_m$ is bounded by $J(c)$. For any n there exists n', $n'\geqslant n$ such that for any $m\geqslant n'$ $J\left(\bigcup_{i=n}^{m}a_i\right)-J\left(\bigcup_{i=n}^{n'}a_i\right)\leqslant\frac{\varepsilon}{2^n}$. Put $k_n=\bigcup_{i=n}^{n'}a_i$ and $t_n=t_{n-1}\cap k_n$ $n=1, 2, 3, \dots$; $t_0=k_1$.

(A) is satisfied for $n=1$. In fact, if $s_1\geqslant x\in A$, then $x\cap\bigcup_{i=1}^{m}a_i\nearrow x$ Thus $J(x)-J(t_1)=\lim J(x\cap\bigcup_{i=1}^{m}a_i)-J(k_1)\leqslant\lim J\left(\bigcup_{i=1}^{m}a_i\right)-J(k_1)\leqslant\frac{\varepsilon}{2}$.

Suppose (A) is satisfied for $n-1$. Let $x\leqslant s_n$, $x\in A$. Then $J(x)-J(k_n)=\lim J(x\cap\bigcup_{i=n}^{m}a_i)-J(k_n)\leqslant\lim J\left(\bigcup_{i=n}^{m}a_i\right)-J(k_n)\leqslant\frac{\varepsilon}{2^n}$. As (III.4.) holds and $t_{n-1}\cup k_n\leqslant s_{n-1}$, we have: $J(x)-J(t_n)=J(x)-J(t_{n-1}\cap k_n)=J(x)-J(k_n)+$ $+J(t_{n-1}\cup k_n)-J(t_{n-1})\leqslant\sum_{i=1}^{n}\frac{\varepsilon}{2^i}$.

Theorem 3. Let a_n, $a\in A$; $a_n\to a$. *Then* $\lim J(a_n)=J(a)$.

Proof. $a_n\to a\Rightarrow a_n-a\to o$ and $a-a_n\to o$. Hence $\lim J(a_n-a)-\lim J(a-a_n)=0$. It implies $\limsup J(a_n)\leqslant J(a)$ and $\liminf J(a_n)\geqslant J(a)$.

4

In this section we are going to extend the functional J onto J^*, defined on a system A^*.

Lemma 4. *Let* $x_n \to x$; $x, x_n \in S$; *let* $\{x_{k_n}\}: k_1 < k_2 < k_3 < \ldots$ *be a subsequence of* $\{x_n\}$. *Then* $x_{k_n} \to x$.

Proof. $\bigcup_{i \geq n} x_{k_i} \leq \bigcup_{i \geq n} x_i \Rightarrow \limsup x_{k_n} \leq \limsup x_n$, $\bigcap_{i \geq n} x_{k_i} \geq \bigcap_{i > n} x_i \Rightarrow \liminf x_{k_n} \geq \liminf x_n$.

Definition 1. $A^* = \{x \in S$, *for which there is* $\{a_n\}$, $a_n \in A$ *and* $= \lim a_n\}$.

Definition 2. *Let* $x \in A^*$ *i. e.* $x = \lim a_n$; $a_n \in A$, *then* $J^*(x) = \lim J(a_n)$.

It is necessary to prove that the limit in Definition 2 exists and depends only on x.

Let $a_n \to x$; $a_n \in A$; let $\{a_{k_n}\}_{n=1}^{\infty}$ be a subsequence of $\{a_n\}$ then (Lemma 4) $a_{k_n} \to x$. Hence $a_{k_n} - a_n \to o$, $a_n - a_{k_n} \to o$. Thus $\lim J(a_n - a_{k_n}) = \lim J(a_{k_n} - a_n) = 0$. Hence $0 \leq |J(a_n) - J(a_{k_n})| = \max\{J(a_n) - J(a_{k_n}), J(a_{k_n}) - J(a_n)\} \leq \max\{J(a_n - a_{k_n}), J(a_{k_n} - a_n)\}$. For $\varepsilon > 0$ there are n_1, n_2: $n > n_1 \Rightarrow J(a_n - a_{k_n}) < \varepsilon$, $n > n_2 \Rightarrow J(a_{k_n} - a_n) < \varepsilon$. Denote $n_0 = \max\{n_1, n_2\}$. For $n > n_0$ we have: $|J(a_n) - J(a_{k_n})| < \varepsilon$. Hence $\{J(a_n)\}$ is a Cauchy sequence.

Let $a_n \to x$, $b_n \to x$; $a_n, b_n \in A$. Then $a_n - b_n \to o$ and $b_n - a_n \to o$. Hence $J(a_n) - J(b_n) \leq J(a_n - b_n) \to 0$, $J(b_n) - J(a_n) \leq J(b_n - a_n) \to 0$. Thus, $\lim J(a_n) = \lim J(b_n)$.

Denote $L = \{x: x \in A^*, x = \bigcap_{n \geq 1} a_n, a_n \in A\}$. Evidently, any element of L can be expressed as a limit of a decreasing sequence of elements of A and $A \subset L \subset A^*$. If $l_n \searrow l$, $l_n \in L$, then $l \in L$.

Lemma 5. *For any* $x \in A^*$ *and* $\varepsilon > 0$ *there exists* $l \in L$, $l \leq x$ *such that* $|J^*(x) - J^*(l)| < \varepsilon$.

Proof. Let $x \in A^*$, i. e. $x = \lim a_n$, $a_n \in A$. From the definition of the limit $x = \bigcup_{n \geq 1} \bigcap_{i \geq n} a_i$, $a_i \in A \Rightarrow \bigcap_{i \geq n} a_i = l_n \in L$. As $l_n = \bigcap_{i \geq n} a_i = \bigcap_{i \geq n} a_i'$, where $a_i' = a_n \cap a_{n+1} \cap a_{n+2} \cap \ldots \cap a_i$, $i = n, n+1, n+2, \ldots$, $a_i' \in A$ we have $\lim J(a_i') = J^*(l_n)$. Hence, for any n and $\varepsilon > 0$ there exists an element a_n' such that $|J(a_n') - J^*(l_n)| < \frac{\varepsilon}{2}$, $l_n \leq a_n' \leq a_n$. Thus, there exists a sequence $\{a_n'\}$, $a_n' \in A$ such that $a_n' \to x$. We have: for any $\varepsilon > 0$ there exists n_0 such that $|J^*(x) - J(a_n')| < \frac{\varepsilon}{2}$ for any $n > n_0$. But this implies: for $n > n_0$ $|J^*(x) - J^*(l_n)| = |J^*(x) - J(a_n') + J(a_n') - J^*(l_n)| \leq |J^*(x) - J(a_n')| + |J(a_n') - J^*(l_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Remark. If $x \geq o$, one can choose $l \geq o$.

Lemma 6. Let $l_n \searrow l$; $l, l_n \in L$. *Then* $\lim J^*(l_n) = J^*(l)$.

Proof. For any n there exists a sequence $\{a_m^n\}$, $a_m^n \searrow l_n$. Denote $c_n = \bigcap_{m=1}^{n} a_n^m$, i. e. $c_1 = a_1^1$, $c_2 = a_2^2 \cap a_1^2$ etc. Evidently $c_n \in A$, $c_n \geqslant c_{n+1}$, $c_n \geqslant l_n$.

For any a_i^k there exists an element $c_{ik} \in \{c_n\}$ such that $a_i^k \geqslant c_{ik}$ ($i < k \Rightarrow a_i^k \geqslant c_k$; $a_i \geqslant c_i$; $i > k \Rightarrow a_i^k \geqslant c_i$. Thus $l = \bigcap_n \bigcap_m a_m^n \geqslant \bigcap_n c_n \geqslant \bigcap_n l_n = l$. It implies $c_n \searrow l$ and therefore $\lim J(c_n) = J^*(l)$. Since $c_n \geqslant l_n \geqslant l$, $J^*(l) = \lim J(c_n) \geqslant \lim J^*(l_n) \geqslant J^*(l)$.

Theorem 4. *Let A^*, J^* have the same meaning as before. Then A^* is a sublattice of S; J^* is a finite functional defined on A^*; A^*, J^* satisfy the following conditions:*

II.1. *A^* is closed with respect to $+$, $-$.*

2. *$x \in S$ implies the existence of a, $b \in A^*$ such that $a \leqslant x \leqslant b$.*

III.1. a, $b \in A^*$, $a \leqslant b \Rightarrow J^*(a) \leqslant J^*(b)$.

2. a, $b \in A^*$, $a \leqslant b \Rightarrow J^*(b) = J^*(a) + J^*(b - a)$.

3. a, $b \in A^* \Rightarrow J^*(a) - J^*(b) \leqslant J^*(a - b)$.

4. a, $b \in A^* \Rightarrow J^*(a) + J^*(b) = J^*(a \cap b) + J^*(a \cup b)$.

5. $a_n \searrow o$, $a_n \in A^* \Rightarrow \lim J^*(a_n) = 0$.

Proof. It is sufficient to show (III.5.). Let $a_n \searrow o$, $a_n \in A^*$. Then $a_n \geqslant o$. For any n there exists an element $l_n \in L$ such that $o \leqslant l_n \leqslant a_n$ and $J^*(a_n) - J^*(l_n) \leqslant \frac{\varepsilon}{2^n}$ (*Lemma* 5). *We introduce* $l_n' = l_1 \cap l_2 \cap l_3 \cap \ldots \cap l_n$. Then $l_n' \geqslant o$ and $\lim l_n' = \cap l_n' = \cap l_n$. Since $o \leqslant l_n' \leqslant l_n \leqslant a_n$, we have $\lim l_n' = o$. It implies $\lim J^*(l_n') = 0$. To complete the proof we shall show that $\lim J^*(a_n) - \lim J^*(l_n') \leqslant \varepsilon$, for any $\varepsilon > 0$. We prove that

$$J^*(a_n) - J^*(l_n') \leqslant \varepsilon \sum_{i=1}^{n} \frac{1}{2^i} \qquad \text{(M)}$$

using induction. Evidently, (M) is satisfied for $n = 1$. Applying (III.4.), $J^*(l_n') = J^*(l_n \cap l_{n-1}') = J^*(l_n) + J^*(l_{n-1}') - J^*(l_n \cup l_{n-1}')$ and since $l_n \cup l_{n-1}' \leqslant a_{n-1}$, we have: $J^*(a_n) - J^*(l_n) = J^*(a_n) - J^*(l_n) + J^*(l_n \cup l_{n-1}) - J^*(l_{n-1}) \leqslant J^*(a_n) - J^*(l_n) + J^*(a_{n-1}) - J^*(l_{n-1}') \leqslant \varepsilon \sum_{i=1}^{n} \frac{1}{2^i}$.

Corollary. If $a_n \to a$, $a_n \in A^*$, then $\lim J^*(a_n) = J^*(a)$. The proof is analogous to the proof of Theorem 3.

5

In this section we shall obtain the main result of the paper: There exists a lattice A_σ, $A \subset A_\sigma \subset S$ which is closed with respect to limit, and a finite functional $\bar{J}$ defined on A_σ, which is an extension of the functional J, where the property (P) is satisfied.

Introduce this transfinite sequence:

$$A = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_\omega \subset \ldots,$$

where $A_\alpha = A^*_{\alpha-1}$ if α is an ordinal of the first kind, and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ if α is an ordinal of the second kind.

Theorem 5. *For any α it is true:*

(i) The functional J defined on A extends uniquely to a functional J_α on A_α.

(ii) If $x \in A_\alpha$, $x \geq o$, $\varepsilon > 0$ then there exists $l \in L$ such that $o \leq l \leq x$ and $J_\alpha(x) - J_\alpha(l) < \varepsilon$.

(iii) A_α is the sublattice of S, J_α is a finite functional defined on A_α and for J_α, A_α the conditions II *and* III *hold.*

Proof. We are using the transfinite induction. Obviously, the theorem is true for $\alpha = 1$ (Theorem 4).

Let α be of the first kind. We define $J_\alpha(x) = \lim J_{\alpha-1}(a_n)$, $x \in A_\alpha$, $a_n \in A_{\alpha-1}$. As $A_\alpha = A^*_{\alpha-1}$, the first statement is obvious. There is a set $L_{\alpha-1}$ related to $A_{\alpha-1}$ as L to A, i. e. $A_{\alpha-1} \subset L_{\alpha-1} \subset A_\alpha$. Let $x \in A_\alpha$, $x \geq o$, then there exist $k \in L_{\alpha-1}$ such that $o \leq k \leq x$ and $J_\alpha(x) - J_\alpha(k) < \frac{\varepsilon}{2}$. The definition of $L_{\alpha-1}$ implies the existence of a sequence $\{a_n\}$, $a_n \in A_{\alpha-1}$, $a_n \geq o$, $a_n \searrow k$. Since for $\alpha - 1$ the statement is true, we find for any n an element $l_n \in L$ such that $o \leq l_n \leq a_n$ and $J_{\alpha-1}(a_n) - J_{\alpha-1}(l_n) < \frac{\varepsilon}{2^{n+1}}$.

Denote $l'_n = l_1 \cap l_2 \cap l_3 \cap \ldots \cap l_n$. Then $o \leq l'_n \leq a_n$, $l'_n \searrow l$, $l \in L$ and $o \leq l \leq k$. It can be shown by induction:

$$J_{\alpha-1}(a_n) - J_{\alpha-1}(l'_n) < \frac{\varepsilon}{2} \sum_{i=1}^{n} \frac{1}{2^i}.$$

In fact, $J_{\alpha-1}(a_n) - J_{\alpha-1}(l'_n) = J_{\alpha-1}(a_n) - J_{\alpha-1}(l_n) + J_{\alpha-1}(l_n \cup l'_{n-1}) - J_{\alpha-1}(l'_{n-1}) < \frac{\varepsilon}{2} \sum_{i=1}^{n-1} \frac{1}{2^i} + \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} \sum_{i=1}^{n} \frac{1}{2^i}$. It implies $J_\alpha(x) - J_\alpha(l) = J_\alpha(x) - J_\alpha(k) + J_\alpha(k) - J_\alpha(l) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. The proof of statement *(iii)* is analogous to the proof of Theorem 4.

Let now α be of the second kind. For any $x \in A_\alpha$ we define $J_\alpha(x) = J_\beta(x)$, where $\beta < \alpha$. It is necessary to prove that $J_\alpha(x)$ depends only on x. Contrary, suppose that there exist ordinals β_1, β_2: $\beta_1 < \beta_2 < \alpha$ such that $x \in A_{\beta_1} \subset A_{\beta_2}$ and $J_{\beta_1}(x) \neq J_{\beta_2}(x)$. This statement contradicts the induction hypothesis.

The implication $x \in A_\alpha \Rightarrow x \in A_\beta$, $\beta < \alpha$ implies the validity of the statement *(ii)*. For the proof of the statement *(iii)* it is sufficient to show (III.5.) only. This proof is analogous to the proof used in Theorem 4 using the statement *(ii)*.

Corollary. *For any α it is true: $a_n \to a$; a_n, $a \in A_\alpha \Rightarrow \lim J_\alpha(a_n) = J_\alpha(a)$* (Theorem 3).

If we put $J_\sigma = \bar{J}$, we have a sublattice A_σ of S and a finite functional $\bar{J}$ defined on A_σ that is a unique extension of J. For A_σ and $\bar{J}$ the conditions II, III and the property (P) hold, i. e. $x_n \in A_\sigma$, $x_n \to x \Rightarrow x \in A_\sigma$ and $\lim \bar{J}(x_n) = \bar{J}(x)$.

6

Let $\mathcal{S}$ be the system of all subsets of a set X. Define $A + B = A \cup B = \{x : X \in A$ or $x \in B\}$; $A - B = \{x : x \in A$ and $x \bar{\in} B\}$; $A \cap B = \{X : X \in A$ and $x \in B\}$; A, $B \in \mathcal{S}$.

Let $\mathcal{A}$ be an algebra of subsets of X and J be a finite measure defined on $\mathcal{A}$. Then the system $\mathcal{A}_\sigma$ constructed in the same way as in section 5 is a σ-algebra generated by $\mathcal{A}$ (see [4]. p. 26). By means of the results of previous sections we get the theorem on an extension of a finite measure:

Let $\mathcal{A}$ be an algebra of subsets of X, J a finite measure defined on $\mathcal{A}$. Then the measure J can be uniquely extended on the σ-algebra generated by $\mathcal{A}$.

Let now $\mathcal{S}$ be a system of all finite real functions defined on a set X. Let μ be a finite measure defined on a σ-algebra $\mathcal{M}$ of subsets of X. The limit is defined as follows: $f_n \to f$; $f, f_n \in \mathcal{S} \Leftrightarrow f_n(x) \to f(x)$, for all $x \in X$. The relations $+$, $-$ and $\leq$ have the usual meaning. Let $\mathcal{B}$ be a system of simple integrable functions, i. e. $f \in \mathcal{B} \Leftrightarrow f = \sum_{i=1}^{n} c_i \chi_{E_i}$, $E_i \in \mathcal{M}$, $E_i \cap E_j = \emptyset$, $i \neq j$. Denote $J_0(f) = \sum_{i=1}^{n} c_i \mu(E_i)$, for $f \in \mathcal{B}$. We get the following theorem on an extension of a finite integral: There exist a system $\mathcal{N}$: $\mathcal{B} \subset \mathcal{N} \subset \mathcal{S}$ and a finite integral J defined on $\mathcal{N}$, which is a unique extension of J_0 such that: $f_n \to f$; $f_n \in \mathcal{N}$, $f \in \mathcal{S} \Rightarrow f \in \mathcal{N}$ and $J(f) = \lim J(f_n)$.

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О ПРОДОЛЖЕНИИ КОНЕЧНЫХ ФУНКЦИОНАЛОВ МЕТОДОМ ТРАНСФИНИТНОЙ ИНДУКЦИИ

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Резюме

Теорема о продолжении меры была доказана также методом трансфинитнои индукции [1]. В статье этим методом доказана теорема об единственном продолжении непрерывного монотонного функционала J, определенного на некоторой подструктуре A условно σ-монотонной структуры S, замкнутой относительно операций $+$, $-$. Первый индукционный шаг излагаетвя в части 4. В части 5 мы докажем существование непрерывного монотонного функционала $\bar{J}$, определенного на подструктуре A_σ: $A \subset A_\sigma \subset S$. $\bar{J}$ единственное продолжение J, $\bar{J}$ и A_σ выполняют свойство (P), т. е. $x_n \in A_\sigma$, $\lim x_n = x \Rightarrow x \in A_\sigma$, $\bar{J}(x) = \lim \bar{J}(x_n)$. Часть 6 посвящена применению главной теоремы для продолжений меры и интеграла.