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ON THE STRUCTURE OF THE SET OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR ODEs ON UNBOUNDED INTERVALS

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ABSTRACT. For boundary value problem $\dot{x}(t) - A(t)x(t) = f\big(t, x(t)\big)$, $Tx = r$, defined on unbounded intervals we have established sufficient conditions that the set of solutions be an R_δ-set.

Introduction

The aim of this paper is to investigate the set of the solutions for the boundary value problem

$$\begin{aligned} \dot{x}(t) - A(t)x(t) &= f\big(t, x(t)\big) \\ Tx &= r\,, \qquad r \in \mathbb{R}^\nu \ \ (\nu \le n)\,, \end{aligned} \tag{BVP}$$

with linear boundary conditions on a non-compact interval $[a, \infty)$. The existence of a solution (generally unbounded) for (BVP) has been studied in [9], [10]. To prove that the set of all solutions of (BVP) is an R_δ-set we use a theorem of Z. Kubáček [13], [16], which is a generalization of a theorem proved by Vidossich [18]. Some applications of this theory to initial value problems on unbounded intervals are presented in [14], [15]. Another approach to investigate the structure of the set of all solutions for a certain integral equation in an unbounded domain is used in [6]. In [1]–[4], [9], [10] results about the topological structure of the set of solutions to multi-valued asymptotic problems can be found. One of the methods used there consists in studying the topological structure of fixed point sets of limit maps induced by maps of inverse systems. The obtained results were applied to differential inclusions on noncompact intervals.

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First, we show that the system (BVP) is equivalent to the equation

$$Lx = Nx\,, \tag{OE}$$

where L is a linear operator, which need not be Fredholm, N is, generally, nonlinear. Using the theorems of M. Cecchi, M. Marini and P. L. Zezza [5] about equivalence between the set of solutions for (OE) and the set of fixed points of operator $M = P + K_P N$ we may reduce our investigation to the set of fixed points of the operator M. The main tool of the proof that the set of fixed points of M is an R_δ-set is a theorem of Z. Kubáček.

1. Preliminaries

Let a be a real number and let $C = C([a,\infty),\mathbb{R}^n)$ be the vector space of continuous functions from $[a,\infty)$ into $\mathbb{R}^n$, the topology of which is given by the system of seminorms

$$p_m(x) = \sup_{t\in[a,a+m]} \|x(t)\| \qquad \text{for each} \quad x \in C\,,$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^n$. The space C is a Fréchet space and the metric in C can be given by

$$d(x,y) = \sum_{m=1}^{\infty} 2^{-m} \frac{p_m(x-y)}{1+p_m(x-y)}\,, \qquad x, y \in C\,.$$

At first, we present some theorems which will be used later. We recall that a non-empty subset F of a metric space X is said to be an *R_δ-set in the space X* if it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. By [7; p. 92], a metric space Z is called an *absolute retract* when each continuous map $f\colon W \to Z$ has a continuous extension $g\colon Y \to Z$ for each metric space Y and each closed $W \subset Y$.

PROPOSITION 1.1. ([13; p. 350]) *Let the Fréchet space C have the same meaning as above and let $\varphi, \varphi_k, \varphi_{*k} \in C([a,\infty),(0,\infty))$, $k \in \mathbb{N}$, satisfy the conditions*

(i) *for each $t \in [a,\infty)$ the sequence $\{\varphi_k(t)\}_{k=1}^{\infty}$ is non-increasing and $\lim\limits_{k\to\infty} \varphi_k(t) = 0$.*

(ii) *Let $\Omega = \{x \in C : \ \|x(t)\| \le \varphi(t)\,, \ t \ge a\}$. Suppose that $Q\colon \Omega \to C$ is a compact map and there exists a sequence $\{Q_k\}_{k=1}^{\infty}$ of compact maps $Q_k\colon \Omega \to C$ such that*

$$\|Q_k x(t) - Qx(t)\| \le \varphi_k(t)\,, \qquad x \in \Omega\,, \ \ t \ge a\,;$$

(iii) *for each* $k \in \mathbb{N}$ *there exists a function* $\varphi_{*k} \in C\big([a,\infty),[0,\infty)\big)$ *such that*

$$\varphi_{*k}(t)+\varphi_k(t) \le \varphi(t)\,, \quad t \ge a \qquad \text{and} \qquad \|Q_k x(t)\| \le \varphi_{*k}(t)\,, \quad x \in \Omega\,, \ t \ge a\,;$$

(iv) *the map* $S_k = I - Q_k$ *is injective on* Ω.

Then the set F *of all fixed points of the map* Q *is an* R_δ*-set.*

PROPOSITION 1.2. ([8; p. 168]) *Suppose that* E *is a (partially) ordered set. We assume that any majorized, increasing sequence* $\{x_n\}_{n=1}^{\infty}$ *of elements of* E *has a supremum in* E, *and that any minorized, decreasing sequence* $\{y_n\}_{n=1}^{\infty}$ *of elements of* E *has an infimum in* E. *If, under these conditions,* x *(resp.* y*) is the supremum (resp. infimum) of* $\{x_n\}_{n=1}^{\infty}$ *(resp.* $\{y_n\}_{n=1}^{\infty}$*), we shall write* $x_n \uparrow x$ *(resp.* $y_n \downarrow y$*). We consider a self-map* $u\colon E \to E$ *is increasing (i.e.* $u(x) \le u(y)$ *whenever* $x \le y$*), and that is such that* $u(x_n) \uparrow u(x)$ *and* $u(y_n) \downarrow u(y)$ *whenever* $x_n \uparrow x$ *and* $y_n \downarrow y$. *Suppose further that there exist* x_0 *and* y_0 *of* E *such that*

$$x_0 \le y_0\,, \qquad x_0 \le u(x_0)\,, \qquad u(y_0) \le y_0\,.$$

Define $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *by*

$$x_{n+1} = u(x_n)\,, \qquad y_{n+1} = u(y_n)\,.$$

Then there exist elements x *and* y *of* E *such that* $x_n \uparrow x$, $y_n \downarrow y$, $x \le y$, *and both* x *and* y *are fixed points of* u. *Moreover, if* x^* *(resp.* y^**) is a fixed point of* u *satisfying* $x^* \ge x_0$ *(resp.* $y^* \le y_0$*), then* $x^* \ge x$ *(resp.* $y^* \le y$*).*

We consider the system of differential equations

$$\dot{x}(t) - A(t)x(t) = f\big(t, x(t)\big)\,, \tag{1.1}$$

and the boundary conditions:

$$Tx = r\,, \qquad r \in \mathbb{R}^\nu\,, \tag{1.2}$$

where $1 \le \nu \le n$, $A(t)$ is an $n \times n$ matrix function which is continuous in $[a,\infty)$, $f\colon [a,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $T\colon \operatorname{dom} T \subset C \to \mathbb{R}^\nu$ is a linear continuous operator; it means that there exist $\gamma > 0$, $m_0 \in \mathbb{N}$ such that:

$$\big\|Tx\big\|_1 \le \gamma \cdot p_{m_0}(x) \qquad \text{for each} \quad x \in \operatorname{dom} T\,, \tag{1.3}$$

where $\|\cdot\|_1$ is a norm in $\mathbb{R}^\nu$.

Remark 1.1. It is known that a linear operator from locally convex space into finite dimensional space is continuous if and only if (1.3) holds ([20]).

Let D be the space of all continuous solutions of the linear system

$$\dot{y}(t) - A(t)y(t) = 0\,, \tag{1.4}$$

and let us assume that T satisfies the condition

$$D \subset \operatorname{dom} T\,, \qquad T(D) = \mathbb{R}^\nu\,. \tag{1.5}$$

Let $L\colon \operatorname{dom} L \subset C \to C \times \mathbb{R}^\nu$ be the linear operator defined by the relation: $x(\cdot) \mapsto \big(\dot{x}(\cdot) - A(\cdot)x(\cdot); Tx\big)$, where $\operatorname{dom} L = C^1\big([a,\infty), \mathbb{R}^n\big) \cap \operatorname{dom} T$ and let $N\colon \operatorname{dom} N \subset C \to C \times \mathbb{R}^\nu$ be the operator given by: $x(\cdot) \mapsto \big(f(\cdot, x(\cdot)); r\big)$.

Then the system (1.1)–(1.2) is equivalent to

$$Lx = Nx\,. \tag{1.6}$$

Let $X(t) = \big(\omega_1(t); \dots; \omega_n(t)\big)$ be a fundamental matrix for the equation (1.4), where $\omega_1; \dots; \omega_l$ is a basis for $\operatorname{Ker} L$ ($l = \dim \operatorname{Ker} L$) and $\omega_1; \dots; \omega_l; \omega_{l+1}; \dots; \omega_n$ is a basis for D.

By the results of P. L. Zezza [21], and M. Cecchi, M. Marini, P. L. Zezza [5], the system (1.1)–(1.2) is also equivalent to the equation

$$x = Mx\,, \tag{1.7}$$

where

$$M = P + K_P N\,, \tag{1.8}$$

P is a continuous projection, $P\colon C \to \operatorname{Ker} L$, K_P is the inverse operator of $L\big|_{(\operatorname{dom} L \cap C_{I-P})}$, $C_{I-P} = R(I-P)$, $K_P\colon R(L) \to \operatorname{dom} L \cap C_{I-P}$ is defined by the relation

$$\begin{aligned} K_P\colon \big(b(t), r\big) \longmapsto X(t)JT_0^{-1}\left(r - T\left(\int_a^t X(t)X^{-1}(s)b(s)\,\mathrm{d}s\right)\right) \\ + \int_a^t X(t)X^{-1}(s)b(s)\,\mathrm{d}s\,, \end{aligned} \tag{1.9}$$

J is the immersion of $\mathbb{R}^\nu$ into $\mathbb{R}^n$

$$\begin{aligned} J(r) &= (0, \dots, 0, r_1, \dots, r_\nu)^\top\,, \qquad r = (r_1, \dots, r_\nu)^\top \in \mathbb{R}^\nu\,, \\ T_0 &= (T\omega_{l+1}, \dots, T\omega_n) \end{aligned}$$

and T_0^{-1} is the inverse of T_0.

Remark 1.2. ([5; p. 274]) Operator K_P defined in (1.9) depends on P, because the choice of P is related to the fundamental matrix $X(t)$. If $\nu = n$, then this construction can be simplified, matrix $TX(t)$ is invertible, and hence $J = I$, $JT_0^{-1} = \big(TX(t)\big)^{-1}$.

Remark 1.3. ([5; p. 275]) According to (1.9), M is defined on the set:

$$A = \left\{ g \in C : \int_a^t X(t)X^{-1}(s)f\big(s, g(s)\big)\,\mathrm{d}s \in \operatorname{dom} T \right\}.$$

2. Results

In this section we prove the theorem which guarantees that the set of all solutions of (1.1)–(1.2) belonging to C_{I-P} is an R_δ-set. Let

$$H_m = \sup_{t\in[a,a+m]} \|X(t)\| . \tag{2.1}$$

THEOREM 2.1. *Let the system* (1.1)–(1.2) *satisfy the following conditions:*
$A(t)$ is an $n\times n$ matrix defined and continuous on $[a,\infty)$, $X(t)$ is a fundamental matrix of (1.4) *with the following properties:*

(2.2) *For all $t\ge a$, $\|X(\tau)-X(t)\|$ is a non-increasing function of $\tau\in\langle a,t\rangle$,*

(2.3) *$f\colon [a,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function and it satisfies:*

$$\|X^{-1}(t)f(t,u)\| \le p(t)\|u\| + q(t) \qquad \textit{for each} \quad u\in\mathbb{R}^n\,,\quad t\ge a\,,$$

where p, q are locally integrable functions in $[a,\infty)$,

$$\int_a^{a+m} p(s)\,\mathrm{d}s = \Gamma_m < +\infty\,, \qquad \int_a^{a+m} q(s)\,\mathrm{d}s = \Lambda_m < +\infty\,,$$

(2.4) $\big(\forall t\in[a,a+m_0]\big)\big(\forall u,v\in\mathbb{R}^n\big)\Big(\big\|X^{-1}(t)\big[f(t,u)-f(t,v)\big]\big\| \le p(t)\|u-v\|\Big)$,
T is a linear continuous operator from $\operatorname{dom}T = C$ *onto $\mathbb{R}^\nu$; it means that there exist $\gamma>0$, $m_0\in\mathbb{N}$ such that:*

$$\|Tx\|_1 \le \gamma\cdot p_{m_0}(x) \qquad \textit{for each} \quad x\in\operatorname{dom}T\,,$$

and the rank of the matrix $TX(t)$ is ν,

(2.5) $\zeta = \|JT_0^{-1}\|\gamma H_{m_0}\Gamma_{m_0}\beta < 1$,
where $\beta = 3H_{m_0}\big(1+H_{m_0}\Gamma_{m_0}\exp(H_{m_0}\Gamma_{m_0})\big)$.

Then the set of all solutions for (1.1)–(1.2) *belonging to C_{I-P} is an R_δ-set.*

Remark 2.1. The constant m_0 in the (2.4) is given from the continuity of T, which is defined on Fréchet space and takes values in $\mathbb{R}^\nu$. Theorem 2.1 includes also the non-expansive case.

The proof of this theorem consists of several steps. At the first step we present the following lemma which assures that the operator

$$Q =: K_P N \tag{2.6}$$

is completely continuous.

LEMMA 2.1. ([11; pp. 49–53]) *Under the above hypotheses, if* $\operatorname{dom} T = C$, *then the operator* Q *is defined on* C *and it is completely continuous.*

Further, we construct Tonelli's sequence of operators Q_k which converges to the operator Q and the sequences $\{\varphi_k\}_{k=1}^{\infty}$ and $\{\varphi_{*k}\}_{k=1}^{\infty}$ with the following properties:

$$\begin{aligned} \|Q_k x(t) - Qx(t)\| &\le \varphi_k(t)\,, \qquad & x \in \Omega\,,\ t \ge a\,; \\ \|Q_k x(t)\| &\le \varphi_{*k}(t)\,, \qquad & x \in \Omega\,,\ t \ge a\,; \end{aligned}$$

where $\Omega = \big\{x \in C:\ \|x(t)\| \le \varphi(t)\,,\ t \ge a\big\}$, $\varphi(t)$ will be specified at the second step.

LEMMA 2.2. *Let* $\{Q_k\}_{k=1}^{\infty}$ *be the sequence of operators* Q_k *for all* $k \in \mathbb{N}$, $Q_k\colon \operatorname{dom}(Q_k) \subset C \to C$ *defined by the relation*

$$Q_k x(t) = \begin{cases} X(a)c(x)\,, & a \le t \le a + \frac{1}{k}\,, \\[2ex] X\big(t - \frac{1}{k}\big)\cdot c(x) \\ \quad + \displaystyle\int_a^{t-\frac{1}{k}} X(t)X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s\,, & a + \frac{1}{k} \le t < \infty \end{cases} \tag{2.7}$$

where $c(x) = JT_0^{-1}\Big(r - T\Big(\displaystyle\int_a^t X(t)X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s\Big)\Big)$.

If $\operatorname{dom} T = C$, *then* $(\forall k \in \mathbb{N})(\operatorname{dom} Q_k = C)$ *and* Q_k *is completely continuous.*

Proof of Lemma 2.2 can be done by a slight modification of the proof of Lemma 2.1.

LEMMA 2.3. *Let* $\{\varphi_k\}_{k=1}^{\infty}$ *be the sequence of non-negative functions defined by*

$$\varphi_k(t) = \begin{cases} \|X(a) - X(t)\|\, C(\varphi) \\ \quad + \|X(t)\| \displaystyle\int_a^t \big(p(s)\varphi(s) + q(s)\big)\,\mathrm{d}s\,, & a \le t \le a + \frac{1}{k}\,, \\[2ex] \big\|X\big(t - \frac{1}{k}\big) - X(t)\big\|\, C(\varphi) \\ \quad + \|X(t)\| \displaystyle\int_{t-\frac{1}{k}}^t \big(p(s)\varphi(s) + q(s)\big)\,\mathrm{d}s\,, & a + \frac{1}{k} \le t < \infty\,, \end{cases} \tag{2.8}$$

where

$$C(\varphi) = \|JT_0^{-1}\| \cdot \Big[\|r\|_1 + \gamma H_{m_0}\Big(\Gamma_{m_0} \sup_{t \in [a, a+m_0]} \varphi(t) + \Lambda_{m_0}\Big)\Big] \tag{2.9}$$

and $\varphi \in C([a,\infty),[0,\infty))$. *Then for each* $k \in \mathbb{N}$ *the function* $\varphi_k\colon [a,\infty) \to [0,\infty)$ *is continuous and the sequence* $\{\varphi_k\}_{k=1}^{\infty}$ *of continuous non-negative functions is non-increasing and converges to* 0 *uniformly on each compact subinterval of* $[a,\infty)$.

Proof. The continuity of φ_k is obvious. First, let us prove that $\{\varphi_k\}_{k=1}^{\infty}$ is non-increasing sequence. Let k be an arbitrary but fixed natural number, then

$$\varphi_k(t) - \varphi_{k+1}(t) =$$

$$= \begin{cases} 0\,, & a \le t \le a + \frac{1}{k+1}\,, \\ \Big(\|X(a) - X(t)\| - \big\|X\big(t - \frac{1}{k+1}\big) - X(t)\big\|\Big)C(\varphi) & \\ \qquad + \|X(t)\| \int\limits_a^{t-\frac{1}{k+1}} \big(p(s)\varphi(s) + q(s)\big)\,\mathrm{d}s\,, & a + \frac{1}{k+1} \le t \le a + \frac{1}{k}\,, \\ \Big(\big\|X\big(t - \frac{1}{k}\big) - X(t)\big\| - \big\|X\big(t - \frac{1}{k+1}\big) - X(t)\big\|\Big)C(\varphi) & \\ \qquad + \|X(t)\| \int\limits_{t-\frac{1}{k}}^{t-\frac{1}{k+1}} \big(p(s)\varphi(s) + q(s)\big)\,\mathrm{d}s\,, & a + \frac{1}{k} \le t < \infty\,. \end{cases}$$

From the hypothesis (2.2) there follows:

$$\|X(a) - X(t)\| \ge \big\|X\big(t - \tfrac{1}{k+1}\big) - X(t)\big\|\,,$$
$$\big\|X\big(t - \tfrac{1}{k}\big) - X(t)\big\| \ge \big\|X\big(t - \tfrac{1}{k+1}\big) - X(t)\big\|\,,$$

and so $\varphi_k(t) - \varphi_{k+1}(t) \ge 0$ for each $t \in [a,\infty)$. Then the sequence $\{\varphi_k\}_{k=1}^{\infty}$ is nonincreasing. □

Now we prove that $\{\varphi_k\}_{k=1}^{\infty}$ converges uniformly to 0 on each compact subinterval of $[a,\infty)$. $X(t) = (\omega_1;\dots;\omega_n)$ is a fundamental matrix of solutions of the linear system (1.4), so $\omega_i \in C^1([a,\infty);\mathbb{R}^n)$, $i = 1,\dots,n$, and for each $m \in \mathbb{N}$ there exists $L_m > 0$ such that:

$$\|X(t) - X(s)\| \le L_m \|s - t\| \qquad \text{for each} \quad s,t \in [a, a+m]\,.$$

Then there holds: for $a \le t \le a + \frac{1}{k}$

$$\begin{aligned} \varphi_k(t) &\le L_1|a - t|C(\varphi) + \|X(t)\| \int\limits_a^t \big(p(s)\varphi(s) + q(s)\big)\,\mathrm{d}s \\ &\le \frac{1}{k}\Big[L_1 C(\varphi) + H_1 \cdot \sup_{t\in[a,a+1]} \big(p(s)\varphi(s) + q(s)\big)\Big] \end{aligned}$$

and for $a + \frac{1}{k} \le t \le a + m$

$$\varphi_k(t) \le \frac{1}{k}\Big[L_m C(\varphi) + H_m \cdot \sup_{t\in[a,a+m]} \big(p(s)\varphi(s) + q(s)\big)\Big]$$

and hence $\varphi_k \rightrightarrows 0$ for $k \to \infty$ on each subinterval $[a, a+m] \subset [a,\infty)$.

In the next two lemmas, $C(\varphi)$ is given by (2.9).

Lemma 2.4. *Let $\varphi \in C\big([a,\infty),[0,\infty)\big)$ and let $\Omega = \big\{x \in C: \ \|x(t)\| \le \varphi(t)\,,\ t \ge a\big\}$. Further let $\{\varphi_{*k}\}_{k=1}^{\infty}$ be the following sequence of the functions:*

$$\varphi_{*k}(t) = \begin{cases} \|X(a)\|\, C(\varphi)\,, & a \le t \le a + \frac{1}{k}\,, \\ \big\|X\big(t-\frac{1}{k}\big)\big\|\, C(\varphi) & \\ \quad + \|X(t)\| \int\limits_a^{t-\frac{1}{k}} \big(p(s)\varphi(s) + q(s)\big)\, \mathrm{d}s\,, & a + \frac{1}{k} \le t < \infty\,. \end{cases} \tag{2.10}$$

Then the function $\varphi_{*k}\colon [a,\infty) \to [0,\infty)$ is continuous for an arbitrary $k \in \mathbb{N}$ and $\|Q_k x(t)\| \le \varphi_{*k}(t)$ for each $t \in [a,\infty)$, $x \in \Omega$.

Proof. It is easy to verify that for each $k \in \mathbb{N}$ the function $\varphi_{*k}(t)$ is continuous on $[a,\infty)$. The inequality $\|Q_k x(t)\| \le \varphi_{*k}(t)$ for all $t \in [a,\infty)$ and $x \in \Omega$ follows from the assumptions (2.1), (2.3). □

At the second step, we determine a function φ satisfying the following condition:

$$\varphi_{*k}(t) + \varphi_k(t) \le \varphi(t) \qquad \text{for each} \quad k \in \mathbb{N}, \quad t \in [a,\infty)\,.$$

We have:

$$\varphi_{*k}(t) + \varphi_k(t) =$$

$$= \begin{cases} \big(\|X(a)\| + \|X(a) - X(t)\|\big)\, C(\varphi) & \\ \quad + \|X(t)\| \int\limits_a^t \big(p(s)\varphi(s) + q(s)\big)\, \mathrm{d}s\,, & a \le t \le a + \frac{1}{k}\,, \\ \Big(\big\|X\big(t-\frac{1}{k}\big)\big\| + \big\|X\big(t-\frac{1}{k}\big) - X(t)\big\|\Big)\, C(\varphi) & \\ \qquad + \|X(t)\| \int\limits_a^t \big(p(s)\varphi(s) + q(s)\big)\, \mathrm{d}s\,, & a + \frac{1}{k} \le t < \infty\,. \end{cases} \tag{2.11}$$

Lemma 2.5. ([17; p. 85]) *Let $\varphi \in C\big([a,\infty),[0,\infty)\big)$, let $\chi(t)$ be the solution of the equation:*

$$\chi(t) = \begin{cases} \big(H_1 + \|X(a) - X(t)\|\big)\, C(\varphi) & \\ \quad + \|X(t)\| \int\limits_a^t \big(p(s)\chi(s) + q(s)\big)\, \mathrm{d}s\,, & a \le t \le a+1\,, \\ \Big(\sup\limits_{s\in[a,t]} \|X(s)\| + \|X(t-1) - X(t)\|\Big)\, C(\varphi) & \\ \qquad + \|X(t)\| \int\limits_a^t \big(p(s)\chi(s) + q(s)\big)\, \mathrm{d}s\,, & a + 1 \le t < \infty\,. \end{cases} \tag{2.12}$$

Then

$$\chi(t) = a(t) + b(t) \int_a^t p(r) a(r) \exp\left(\int_r^t b(s) p(s) \, \mathrm{d}s \right) \mathrm{d}r \tag{2.13}$$

where

$$a(t) = \begin{cases} \big(H_1 + \|X(a) - X(t)\|\big)\, C(\varphi) \\ \qquad + \|X(t)\| \int_a^t q(s) \, \mathrm{d}s \,, & a \le t \le a+1 \,, \\ \Big(\sup\limits_{s \in [a,t]} \|X(s)\| + \|X(t-1) - X(t)\| \Big)\, C(\varphi) \\ \qquad + \|X(t)\| \int_a^t q(s) \, \mathrm{d}s \,, & a+1 \le t < \infty \,, \end{cases}$$

and $b(t) = \|X(t)\|$.

If we denote

$$\psi_1(t) = \|X(t)\| \cdot \int_a^t q(s) \, \mathrm{d}s + \|X(t)\| \int_a^t \|X(r)\| \cdot$$
$$\cdot \left(\int_a^r q(\tau) \, \mathrm{d}\tau \right) p(r) \exp\left(\int_r^t \|X(s)\| p(s) \, \mathrm{d}s \right) \mathrm{d}r \,, \qquad a \le t < \infty \,,$$

and

$$\psi_2(t) = \begin{cases} H_1 + \|X(a) - X(t)\| + \|X(t)\| \int_a^t \big(\|X(a)\| \\ \quad + \|X(a) - X(r)\| \big) p(r) \exp\Big(\int_r^t \|X(s)\| p(s) \, \mathrm{d}s \Big) \, \mathrm{d}r \,, & a \le t \le a+1 \,, \\ \sup\limits_{s \in [a,t]} \|X(s)\| + \|X(t-1) - X(t)\| \\ \quad + \|X(t)\| \int_a^t \Big(\sup\limits_{s \in [a,r]} \|X(s)\| + \|X(r-1) - X(r)\| \Big) \cdot \\ \qquad \cdot\, p(r) \exp\Big(\int_r^t \|X(s)\| p(s) \, \mathrm{d}s \Big) \, \mathrm{d}r \,, & a+1 \le t < \infty \,, \end{cases}$$

then we may write the solution χ of the equation (2.12) in the form

$$\chi(t) = \psi_1(t) + \psi_2(t) \cdot C(\varphi) \,.$$

Now we want to find $\varphi \in C\big([a,\infty),[0,\infty)\big)$ such that for $\varphi_{*k}(t) + \varphi_k(t)$ given by (2.11) the inequality

$$\varphi_{*k}(t) + \varphi_k(t) \le \varphi(t)$$

holds, that is, φ is a solution of

$$\varphi(t) = \psi_1(t) + \psi_2(t) \cdot C(\varphi)\,. \tag{2.14}$$

We are going to seek the solution of (2.14) in $L_1([a, a+m_0], \mathbb{R})$. We use the usual partial order in $L_1([a, a+m_0], \mathbb{R})$: $u \le v$ if and only if $u(t) \le v(t)$ for all $t \in [a, a+m_0]$. The following lemma holds.

LEMMA 2.6. *Let* $U\colon \operatorname{dom} U \subset L_1([a, a+m_0], \mathbb{R}) \to L_1([a, a+m_0], \mathbb{R})$ *be defined by*

$$(Uu)(t) = \psi_1(t) + \psi_2(t) \cdot C(u)\,,$$

where $\operatorname{dom} U = \{u \in L_1([a, a+m_0], \mathbb{R}) \;:\; (\forall t \in [a, a+m_0])(0 \le u(t) \le \xi)\}$, *with* ξ *sufficiently large positive real constant. Let the following hypotheses be satisfied:*

$$\zeta = \|JT_0^{-1}\| \gamma H_{m_0} \Gamma_{m_0} \beta < 1\,,$$

$$\textit{where} \qquad \beta = 3H_{m_0}\bigl(1 + H_{m_0}\Gamma_{m_0} \exp(H_{m_0}\Gamma_{m_0})\bigr)\,. \tag{2.5}$$

Then there exists a fixed point of the operator U.

P r o o f . We use Proposition 1.2 to prove this lemma. We must verify the following hypotheses:

(i) U is an increasing operator,
(ii) there exists x_0 such that: $x_0 \le U(x_0)$,
(iii) there exists ξ such that: $\xi \ge U(\xi)$.

(i) Let $v(t) \le w(t)$, $t \in [a, a+a_{m_0}]$. Then

$$\begin{aligned}
&(Uv)(t) = \\
&= \psi_1(t) + \psi_2(t) \cdot \Bigl\{\|JT_0^{-1}\| \cdot \Bigl[\|r\|_1 + \gamma H_{m_0}\Bigl(\mathit{\Gamma}_{m_0} \sup_{t\in[a,a+m_0]} v(t) + \Lambda_{m_0}\Bigr)\Bigr]\Bigr\} \\
&\le \psi_1(t) + \psi_2(t) \cdot \Bigl\{\|JT_0^{-1}\| \cdot \Bigl[\|r\|_1 + \gamma H_{m_0}\Bigl(\mathit{\Gamma}_{m_0} \sup_{t\in[a,a+m_0]} w(t) + \Lambda_{m_0}\Bigr)\Bigr]\Bigr\} \\
&= U(w)(t)\,.
\end{aligned}$$

(ii) Let $x_0 = 0$, then

$$0 \le U(0)(t) = \psi_1(t) + \psi_2(t) \cdot \bigl\{\|JT_0^{-1}\| \cdot \bigl[\|r\|_1 + \gamma H_{m_0}\Lambda_{m_0}\bigr]\bigr\}\,.$$

(iii) Let ξ be a positive real constant. Then

$$U(\xi)(t) \le \psi_1(t) + \psi_2(t) \cdot C(\xi)$$

and

$$\sup_{[a,a+m_0]} U(\xi) \le \sup_{[a,a+m_0]} \psi_1(t) + \sup_{[a,a+m_0]} \psi_2(t) \cdot C(\xi)$$

where

$$\sup_{[a,a+m_0]} \psi_1(t) \le H_{m_0}\Lambda_{m_0}\Big(1 + H_{m_0}\Gamma_{m_0}\exp\big(H_{m_0}\Gamma_{m_0}\big)\Big) = \alpha$$

$$\begin{aligned}\sup_{[a,a+m_0]} \psi_2(t) \le \max\Big\{ &\Big(H_1 + \sup_{[a,a+1]} |X(a) - X(t)\|\Big)\Big(1 + \Gamma_1 H_1 \exp(H_1\Gamma_1)\Big)\,,\\ &\Big(H_{m_0} + \sup_{[a+1,a+m_0]} \|X(t-1) - X(t)\|\Big)\cdot\\ &\cdot\Big(1 + \sup_{[a+1,a+m_0]} \|X(t)\|\,\Gamma_{m_0}\exp\big(H_{m_0}\Gamma_{m_0}\big)\Big)\Big\} \le \beta\,.\end{aligned}$$

We are looking for ξ with the property $0 \le u \le \xi \implies U(u) \le \xi$. Put $\theta = \alpha + \big[\|JT_0^{-1}\|\big(\|r\|_1 + \gamma H_{m_0}\Lambda_{m_0}\big)\big]\cdot\beta$, then it suffices to choose any ξ satisfying the inequality $\theta + \zeta\xi \le \xi$. By Proposition 1.2 the operator U has a fixed point $u \in L_1\big([a, a+m_0], [0,\infty)\big)$, $0 \le u(t) \le \xi$, $t \in [a, a+a_{m_0}]$.

Since the functions ψ_1, ψ_2 are non-negative continuous functions, the fixed point of U is also a continuous function. If we put

$$\varphi(t) = \begin{cases} u(t)\,, & a \le t \le a + m_0\,,\\ \psi_1(t) + \psi_2(t)\,C(u)\,, & a + m_0 < t < \infty\,, \end{cases} \tag{2.15}$$

then φ is a non-negative continuous solution of equation (2.12).

At the third step we prove that the operator $S_k =: I - Q_k$ is injective. □

LEMMA 2.7. *Let the following hypotheses be satisfied:*

$$\big(\forall t \in [a, a+m_0]\big)\big(\forall u, v \in \mathbb{R}^n\big)\Big(\big\|X^{-1}(t)\,[f(t,u) - f(t,v)]\big\| \le p(t)\|u - v\|\Big)\,. \tag{2.4}$$

Then the operator $S_k =: I - Q_k$ is injective.

Proof.

1. Let $c(x) = c(y)$.
Suppose that $x \ne y$ and $x - Q_k(x) = y - Q_k(y)$.

a) Let there exist $t_1 \in \big[a, a+\frac{1}{k}\big]$ such that $x(t_1) \ne y(t_1)$ and $x - Q_k(x) = y - Q_k(y)$. Then $x(t_1) - X(a)c(x) = y(t_1) - X(a)c(x) \implies x(t_1) = y(t_1)$, which is a contradiction.

b) Let $x(t) = y(t)$ for all $t \in \big[a, a+\frac{1}{k}\big]$ and let

$$t_1 = \sup\big\{\tau > a :\ \big(\forall t \in [a,\tau)\big)\big(x(t) = y(t)\big)\big\}\,.$$

Then there exists $t_0 \in \left(t_1, t_1+\frac{1}{k}\right)$ such that $x(t_0) \neq y(t_0)$ and

$$x(t_0) - X\left(t_0 - \tfrac{1}{k}\right)c(x) - \int_a^{t_0-\frac{1}{k}} X(t_0)X^{-1}(s)f\big(s, x(s)\big)\, \mathrm{d}s$$

$$= y(t_0) - X\left(t_0 - \tfrac{1}{k}\right)c(x) - \int_a^{t_0-\frac{1}{k}} X(t_0)X^{-1}(s)f\big(s, y(s)\big)\, \mathrm{d}s\,.$$

As $x(t) = y(t)$ for all $t \in \left[a, t_0-\frac{1}{k}\right]$, we have $x(t_0) = y(t_0)$, which is a contradiction.

2. Let $c(x) \neq c(y)$, $x \neq y$ and $x - Q_k(x) = y - Q_k(y)$.

a) Let there exist $t_1 \in \left[a, a+\frac{1}{k}\right]$ such that $x(t_1) \neq y(t_1)$ and

$$x - Q_k(x) = y - Q_k(y) \implies x(t_1) - y(t_1) = X(a)\big(c(x) - c(y)\big)\,.$$

Using (2.4) we have:

$$\|c(x) - c(y)\| = \left\| JT_0^{-1}T\left[\int_a^t X(t)X^{-1}(s)\big(f(s, x(s)) - f\big(s, y(s)\big)\, \mathrm{d}s\right]\right\|$$

$$\leq \|JT_0^{-1}\|\gamma \sup_{[a,a+m_0]}\left(\|X(t)\| \int_a^t p(s)\|x(s) - y(s)\|\, \mathrm{d}s\right)$$

and

$$\|x(t) - y(t)\| \leq H_{m_0}\|c(x) - c(y)\| + H_{m_0}\int_a^{t-\frac{1}{k}} p(s)\|x(s) - y(s)\|\, \mathrm{d}s\,,$$

$$t \in \left[a+\tfrac{1}{k}, a+m_0\right].$$

It is easy to verify that if (2.5) holds, then

$$\|JT_0^{-1}\|\gamma H_{m_0}^2\Gamma_{m_0}\exp\big(H_{m_0}\Gamma_{m_0}\big) < 1\,.$$

Using Gronwall's lemma we have:

$$\|x(t) - y(t)\| \leq H_{m_0}\|c(x) - c(y)\|\exp\left(H_{m_0}\int_a^{t-\frac{1}{k}} p(s)\, \mathrm{d}s\right),$$

$$t \in \left[a+\tfrac{1}{k}, a+m_0\right],$$

$$\sup_{[a,a+m_0]}\|x(t) - y(t)\| = \max\left\{\sup_{[a,a+\frac{1}{k}]}\|x(t) - y(t)\|\,, \sup_{[a+\frac{1}{k},a+m_0]}\|x(t) - y(t)\|\right\},$$

$$\begin{aligned}\|c(x) - c(y)\| &\leq \|JT_0^{-1}\|\gamma H_{m_0}^2\Gamma_{m_0}\exp\big(H_{m_0}\Gamma_{m_0}\big)\|c(x) - c(y)\| \\ &< \|c(x) - c(y)\|\,,\end{aligned}$$

and this is a contradiction.

b) If $x(t) = y(t)$ for all $t \in \left[a, a+\frac{1}{k}\right]$, the proof is trivial. □

Proof of Theorem 2.1. To prove this theorem we must verify the assumptions of Proposition 1.1. If we define φ_k, φ_{*k}, φ by (2.8), (2.10), (2.15), then from Lemmas 2.1, 2.2, 2.3, 2.4 and 2.7 it follows that the hypotheses of Proposition 1.1 are satisfied, so the set of solutions for (1.1)–(1.2) in C_{I-P} is an R_δ-set. □

EXAMPLE. Consider the nonlinear boundary value problem

$$\dot{x}(t) - \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} x(t) = \begin{pmatrix} 0 \\ f(t,x) \end{pmatrix}, \qquad t \in [0,\infty),$$
$$Tx = 0$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Tx = x_1(0) - x_1(1)$. Then

$$X(t) = \big(\omega_1(t), \omega_2(t)\big) = \begin{pmatrix} \mathrm{e}^{2t} - (\mathrm{e}+1)\,\mathrm{e}^{t} & \mathrm{e}^{2t} \\ 2\,\mathrm{e}^{2t} - (\mathrm{e}+1)\,\mathrm{e}^{t} & 2\,\mathrm{e}^{2t} \end{pmatrix}$$

is a fundamental matrix for the linear system, where ω_1 is a basis for $\operatorname{Ker} L$. Let $Px = \frac{1}{(\mathrm{e}+1)}\big(x_2(0) - 2x_1(0)\big)\omega_1(t)$, then

$$C_{I-P} = \operatorname{Ker} P = \left\{ x \in C\big([0,\infty), \mathbb{R}^2\big) :\ x_2(0) - 2x_1(0) = 0 \right\}.$$

After some calculations we obtain

$$X^{-1}(t) = \begin{pmatrix} \frac{-2}{(\mathrm{e}+1)}\,\mathrm{e}^{-t} & \frac{1}{(\mathrm{e}+1)}\,\mathrm{e}^{-t} \\ \frac{2}{(\mathrm{e}+1)}\,\mathrm{e}^{-t} - \mathrm{e}^{-2t} & \frac{-1}{(\mathrm{e}+1)}\,\mathrm{e}^{-t} + \mathrm{e}^{-2t} \end{pmatrix}, \qquad JT_0^{-1} = \begin{pmatrix} 0 \\ \frac{1}{(1-\mathrm{e}^2)} \end{pmatrix},$$

where $T_0 = T(\omega_2) = 1 - \mathrm{e}^2$, and by (1.7),

$$K_P N(x) = \begin{pmatrix} \int_0^t \Big[\big(\mathrm{e}^{2(t-s)} - \mathrm{e}^{t-s}\big) f\big(s, x(s)\big)\Big]\,\mathrm{d}s \\ \int_0^t \Big[\big(2\,\mathrm{e}^{2(t-s)} - \mathrm{e}^{t-s}\big) f\big(s, x(s)\big)\Big]\,\mathrm{d}s \end{pmatrix}$$
$$- \frac{1}{\mathrm{e}^2 - 1}\,\mathrm{e}^{2t} \int_0^1 \Big[\big(\mathrm{e}^{2(1-s)} - \mathrm{e}^{1-s}\big) f\big(s, x(s)\big)\Big]\,\mathrm{d}s \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

If we denote $x_1(t) = \mathrm{e}^{2t} - (\mathrm{e}+1)\,\mathrm{e}^{t}$, $x_2(t) = \mathrm{e}^{2t}$, we get

$$\begin{aligned} \|X(\tau) - X(t)\|^2 &= \big(x_1(\tau) - x_1(t)\big)^2 + \big(x_2(\tau) - x_2(t)\big)^2 \\ &\quad + \big(\dot{x}_1(\tau) - \dot{x}_1(t)\big)^2 + \big(\dot{x}_2(\tau) - \dot{x}_2(t)\big)^2 \\ &=: g(\tau, t). \end{aligned}$$

Then

$$\begin{aligned}\frac{\partial g}{\partial \tau}(\tau,t) &= 2\big(x_1(\tau)-x_1(t)\big)\dot{x}_1(\tau)+2\big(x_2(\tau)-x_2(t)\big)\dot{x}_2(\tau)\\ &\quad + 2\big(\dot{x}_1(\tau)-\dot{x}_1(t)\big)\ddot{x}_1(\tau)+2\big(\dot{x}_2(\tau)-\dot{x}_2(t)\big)\ddot{x}_2(\tau)\\ &= (\mathrm{e}^{\tau}-\mathrm{e}^{t})\,\mathrm{e}^{\tau}\Big[40\big(\mathrm{e}^{\tau}-\tfrac{9}{40}(\mathrm{e}+1)\big)^2+\tfrac{79}{40}(\mathrm{e}+1)^2+\mathrm{e}^{t}\big(40\,\mathrm{e}^{\tau}-6(\mathrm{e}+1)\big)\Big]<0\,,\end{aligned}$$

and hence for all $t \ge 0$, $\|X(\tau)-X(t)\|$ is non-increasing function of $\tau \in [0,t]$.

Let $f \in C\big([0,\infty)\times\mathbb{R}^2,\mathbb{R}\big)$ satisfy the following hypotheses:

(i) $|f(t,u)| \le \frac{(\mathrm{e}+1)\,\mathrm{e}^{t}}{\sqrt{1+(1-(\mathrm{e}+1)\,\mathrm{e}^{-t})^2}}\big(p(t)\|u\|+q(t)\big)$ for all $u\in\mathbb{R}^2$, $0\le t<\infty$, where p, q are locally integrable functions in $[0,\infty)$, $\int_0^m p(s)\,\mathrm{d}s=\Gamma_m<\infty$, $\int_0^m q(s)\,\mathrm{d}s=\Lambda_m<\infty$ for all $m\in\mathbb{N}$.

(ii) $|f(t,u)-f(t,v)| \le \frac{(\mathrm{e}+1)\,\mathrm{e}^{t}}{\sqrt{1+[1-(\mathrm{e}+1)\,\mathrm{e}^{-t}]^2}}p(t)\|u-v\|$ for all $u,v\in\mathbb{R}^2$, $0\le t\le 1$.

(iii) $\frac{12}{\mathrm{e}^2-1}\,\mathrm{e}^2(3\,\mathrm{e}^2-\mathrm{e}+1)\Gamma_1\Big(1+\mathrm{e}\sqrt{2}\sqrt{3\,\mathrm{e}^2-\mathrm{e}+1}\,\mathrm{e}^{\sqrt{2}\,\mathrm{e}\sqrt{3\,\mathrm{e}^2-\mathrm{e}+1}\Gamma_1}\Big)<1$.

(iv) $\frac{\mathrm{e}\sqrt{2}}{\mathrm{e}^2-1}\sqrt{3\,\mathrm{e}^2-\mathrm{e}+1}\,\Gamma_1 H_m+\Gamma_m H_m<1$, $m\ge 1$.

Then by [11; p. 53] and Theorem 2.1 the set of all solutions for BVP satisfying $x_2(0)-2x_1(0)=0$ is nonempty and it is an R_δ-set.

Remark 2.2. (1.2) may include multi-point boundary valued condition. However, if we assume $\lim_{t\to\infty} x(t)=x_\infty$, then the operator T is not continuous in Fréchet space C.

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