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REMARKS ON DIAGONALIZABLE EMBEDDINGS OF GRAPHS

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In this paper we shall discuss the existence of diagonalizable quadrilateral embeddings of graphs. The concept was introduced by Železník [7] for bipartite graphs. First of all we note that it is possible to define diagonalizable quadrilateral embeddings for non-bipartite graphs also.

Definition. Let $i\colon G \hookrightarrow S$ be a quadrilateral embedding of a graph G in a closed surface S. Assume there exists an embedding $j\colon H \hookrightarrow S$ such that

(1) G is a spanning subgraph of H with $(j|G) = i$, and

(2) $M = E(H) - E(G)$ is a 1-factor in H each edge of which joins opposite vertices of a quadrilateral in i.

Then M will be called a *diagonal set* for i, and the embedding i itself will be called *diagonalizable.*

For bipartite graphs this definition agrees with the one given in [7]. An example of a non-bipartite graph having a diagonalizable embedding is the Cartesian product $C_3 \times C_4$ in the torus as shown in Fig. 1.

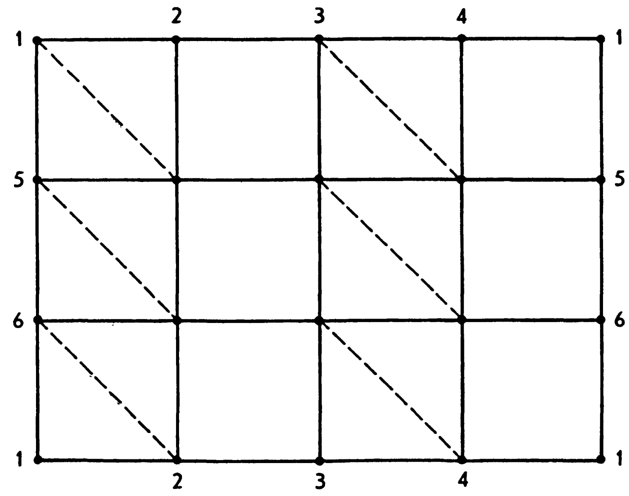

Fig. 1

It is assumed throughout that the reader is familiar with the basic concepts of topological graph theory. In addition, we adopt the convention that if a graph to be embedded is disconnected, then we embed each component in a separate surface. For terms not defined here see [1, 6].

The main result of [7] is the following:

Let G and H be graphs, G bipartite. If G has a diagonalizable embedding in some surface, so has $G \otimes H$, the tensor product of G and H. Moreover, the orientability characteristics of both embeddings can be chosen to be identical.

This result can be used to compute the minimum genus for a number of classes of graphs, see [7].

Before proceeding further we first observe that there is another way of introducing diagonalizable embeddings, equivalent to that above.

Let i: $G \hookrightarrow S$ be a quadrilateral embedding. Define the *diagonal graph* $D(G)$ for i as the graph with vertex-set $V(G)$ in which two vertices are adjacent if they are opposite in some quadrilateral of the embedding i; the multiple or the self-adjacencies in $D(G)$ correspond to the multiple or the self-adjacencies of quadrilaterals. Each quadrilateral f of the embedding i gives rise to two edges which will be called *diagonals* of f. We shall say that the two diagonals are *dual* to each other. It is easily seen that an embedding of G is diagonalizable if and only if $D(G)$ has a 1-factor M which contains at most one edge of each pair of dual edges. Moreover, M is a diagonal set for i.

If the graph in question is bipartite, a little more can be said.

Proposition 1. *Let i: $G \hookrightarrow S$ be a quadrilateral embedding of a graph G in a closed surface S. Then G is bipartite with bipartition $V(G) = A \cup B$ if and only if $D(G)$ is the disjoint union of two graphs D and D^* such that $V(D) = A$, $V(D^*) = B$ and D and D^* are dually embedded in S.*

Proof. "$\Leftarrow$" The definition of $D(G)$ implies that each edge of G has one end in A and the other end in B. Thus G is bipartite.

"$\Rightarrow$" From the assumption it immediately follows that the opposite vertices of each quadrilateral belong to the same part of the bipartition. Hence D and D^* are vertex disjoint. To prove that D and D^* have mutually dual embeddings in S observe that the embedding of D (and, similarly, that of D^*) is cellular. Indeed, each edge of D divides the quadrilateral in which it is contained into two triangles. Each face f of D in S is obviously the union of the set of all such triangles having a vertex of D^* in common. Consequently, the interior of f is a 2-cell. Moreover, it contains exactly one vertex of D^*. Thus there is a bijection between the faces of D and the vertices of D^* and this bijection obviously extends to the duality between the embeddings of D and D^*. ∎

Note that the construction of the diagonal graph $D(G) = D \cup D^*$ from a quadrilateral embedding of a bipartite graph G can be reversed. Given the embedding of D or D^* one can reconstruct G and its quadrilateral embedding. However, there is another way of obtaining the embedding of G from that of D. Start from the embedding of D, construct its medial graph $M(D)$ and take the dual of $M(D)$. The result is the required embedding of G. We recall that the *medial graph* [2, 3, 5] of a 2-cell embedded graph H is a 2-cell embedded graph $M(H)$ defined as follows. To form the vertex-set of $M(H)$ choose a midpoint in each edge of H. Then, for each face f of H, join two midpoints x and y by an edge in the interior of f if x and y are consecutive on the boundary of f. Observe that the medial graph for H is at the same time the medial graph for H^* and vice versa.

From the above mentioned facts it follows that the problem of the existence of a diagonal set for a quadrilateral embedding of a bipartite graph G can be transformed into the problem of the existence of some special 1-factors in the embeddings of D and D^*. To be more exact, we have:

Proposition 2. *An embedding i: $G \hookrightarrow S$ of a bipartite graph G is diagonalizable if and only if the diagonal graph $D(G) = D \cup D^*$ has 1-factors F in D and F^* in D^* such that $F \cup F^*$ contains at most one edge of each pair of dual edges.* ■

Corollary 3. *If i: $G \hookrightarrow S$ is a diagonalizable embedding of a bipartite graph G, then each of the two graphs D and D^* constituting $D(G)$ has a 1-factor.* ■

Thus, Proposition 2 yields a necessary and sufficient condition for an embedding of a bipartite graph to be diagonalizable. Its main disadvantage is that it is not free of topology. This disadvantage is removed in Corollary 3, however, the resulting combinatorial necessary condition is not sufficient, not even in the case of planar bipartite graphs.

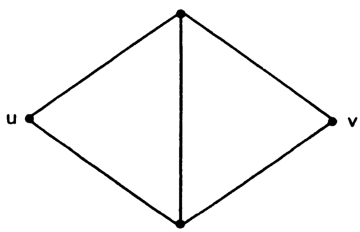

Fig. 2

Example. Let D_k ($k \geq 2$) be a graph obtained from a cycle of length $2k$ by replacing each edge uv with the graph in Fig. 2. Consider the unique embedding of D_k in the sphere (the example of D_2 and its dual is shown in Fig. 3). The edges of D_k joining 3-valent vertices will be called *rungs*. It is not difficult to show that each 1-factor of D_k contains exactly k rungs, and each 1-factor of D_k^* contains exactly $2k - 1$ dual rungs. Let G_k be the bipartite graph with the quadrilateral embedding for which $D(G_k) = D_k \cup D_k^*$. By Proposition 2, if there were a diagonal set $M = F \cup F^*$ for this embedding ($F \subseteq D_k$, $F^* \subseteq D_k^*$) then F would contain k rungs while F^* $2k - 1$ dual rungs corresponding to rungs not in F. Hence, D_k would have at least $3k - 1$ rungs. But there are just $2k$ rungs in D_k, a contradiction. This shows that the embedding of G_k is not diagonalizable, although the necessary condition of Corollary 3 is satisfied.

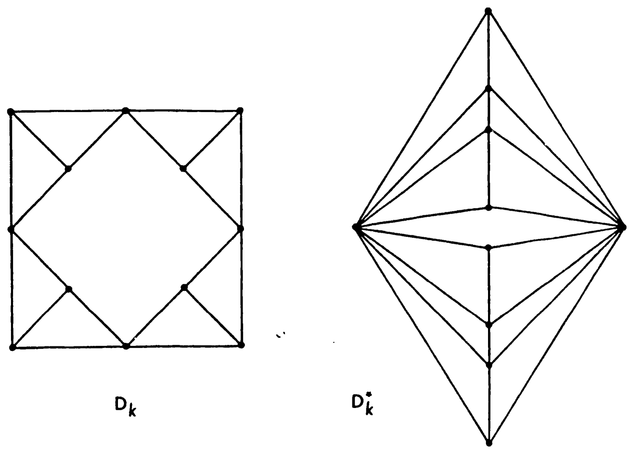

Fig. 3

The lack of a simple combinatorial condition for diagonalizability suggests that it makes sense to look for various classes of graphs admitting diagonalizable embeddings. Below we present two simple results which allow one to construct diagonalizable embeddings for new graphs from old ones.

Let $p: \tilde{S} \to S$ be an unbranched covering projection of a closed surface $\tilde{S}$ onto a surface S, that is, a surjective continuous mapping satisfying the following condition: every point $x \in \tilde{S}$ has an open neighbourhood U such that $p|U$ is a homeomorphism. It is well known (and easy to see) that for any 2-cell embedding $i: G \hookrightarrow S$ of a graph G in a surface S the insertion $j: \tilde{G} = p^{-1} \circ i(G) \hookrightarrow \tilde{S}$ is a 2-cell embedding of G. We say that the embedding j is an *unbranched covering over i*. As usual, such embeddings are constructed combinatorially using voltage graphs [1, 6].

Proposition 4. *Let i: $G \hookrightarrow S$ be a diagonalizable embedding of a graph* G *and let an embedding j: $\tilde{G} \hookrightarrow \tilde{S}$ be an unbranched covering over i. Then j is diagonalizable.*

Proof. Since the covering has no branch-points, the embedding j is quadrilateral. If M is a diagonal set for the embedding i, then the lifting $\tilde{M}$ of M to j is the required diagonal set for j. ∎

To state our final result denote by $G(m)$ the graph obtained from a graph G by replacing each vertex x of G with m vertices $x_1, x_2, \ldots, x_m$ and joining two vertices u_k and v_l in $G(m)$ if u and v are adjacent in G. In other words, $G(m) = G[\bar{K}_m]$, the lexicographic product with the graph $\bar{K}_m$.

Proposition 5. *If a graph G has a diagonalizable embedding in an orientable surface, so has the graph $G(2^n)$ for each $n \geq 1$.*

Proof. We first show that $G(2)$ has an orientable quadrilateral embedding. Let $e = uv$ be an arc (= oriented edge) in G. Then e gives rise to four arcs $e_{k,l}$, $k, l \in \{1, 2\}$, in $G(2)$. To obtain an appropriate quadrilateral embedding of $G(2)$ we start with the diagonalizable embedding i of G given by a rotation P. Let $d = P^{-1}(e)$ and $f = P(e)$. Define a rotation Q for $G(2)$ as follows: $Q(e_{1,1}) = f_{1,2}$, $Q(e_{1,2}) = e_{1,1}$, $Q(e_{2,1}) = e_{2,2}$ and $Q(e_{2,2}) = d_{2,1}$. Routine calculations show that Q determines a quadrilateral embedding j of $G(2)$. We now verify that this embedding is diagonalizable as well. Let $f = uvwx$ be a quadrilateral in the embedding i. Then there are four quadrilaterals in j corresponding to f: $u_1v_2u_2x_1$, $v_1w_2v_2u_1$, $w_1x_2w_2v_1$ and $x_1u_2x_2w_1$. Let M be a diagonal set for the embedding i of G. To define a diagonal set M' for j, assume without loss of generality that the diagonal uv of f is in M. Then include to into M' the diagonals u_1u_2 and w_1w_2 of $u_1v_2u_2x_1$ and $w_1x_2w_2v_1$, respectively. It is easy to see that the set M' thus defined is a diagonal set for j, implying that $G(2)$ has an orientable diagonalizable embedding. Since $(G(2^k))(2) = G(2^{k+1})$, the proof is easily completed by induction. ■

The construction of the embedding of $G(2)$ is taken from Pisanski [4].

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ЗАМЕТКА О ДИАГОНАЛИЗИРУЕМЫХ ЧЕТЫРЕХУГОЛЬНЫХ ВЛОЖЕНИЯХ ГРАФОВ

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Резюме

В статье изучается проблема существования диагонализируемых четырехугольных вложений графов в поверхности. Даны две конструкции таких вложений.