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NONSENSITIVENESS REGIONS IN UNIVERSAL MODELS

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ABSTRACT. Within universal mixed linear models, including the models with constraints, the algorithms are developed which enable to determine boundaries of nonsensitiveness regions. These are defined in the space of parameters of the covariance matrix of the observation vector; a shift of these parameters inside the nonsensitiveness regions does not cause any essential damage of the estimators of the parameters of the mean value of the observation vector.

Introduction

Let us consider a linear statistical model, i.e.

$$\boldsymbol{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \Sigma(\boldsymbol{\vartheta})),$$

where $\boldsymbol{Y}$ denotes an n-dimensional observation vector, $\mathbf{X}\boldsymbol{\beta}$ the mean value of the observation vector, $\mathbf{X}$ a design matrix, $\boldsymbol{\beta}$ an unknown vector parameter of the mean value of the observation vector $\boldsymbol{Y}$, $\Sigma(\boldsymbol{\vartheta})$ the covariance matrix of $\boldsymbol{Y}$ depending on a vector parameter $\boldsymbol{\vartheta}$.

If $\boldsymbol{\vartheta}^*$ is an actual value of the vector $\boldsymbol{\vartheta}$, then the $\boldsymbol{\vartheta}^*$-LBLUE (locally best linear unbiased estimator; in more detail cf. [7; p. 180]) of an unbiasedly estimable function of the parameter $\boldsymbol{\beta}$ depends more or less on the value of the vector $\boldsymbol{\vartheta}$. In many cases its actual value is not known, it must be estimated, or it is known only approximately. In such cases it is of some interest to know whether an uncertainty in $\boldsymbol{\vartheta}$ can or cannot destroy the optimum property of the $\boldsymbol{\vartheta}^*$-LBLUE of an unbiasedly estimable function of $\boldsymbol{\beta}$.

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In [2], [3], [4], [5], [6], this problem and analogous problems connected with confidence ellipsoids and test of linear hypotheses were studied in the case of regularity of the model.

Another type of sensitivity is studied in [1].

The aim of the paper is to find a solution in universal model (i.e. without conditions of regularity), in universal model with constraints of the type I and in the universal model with constraints of the type II.

1. Notations and auxiliary statements

Let $\boldsymbol{Y}$ be an n-dimensional random vector (observation vector) which realization $\boldsymbol{y}$ is the vector of results of measurements. The class of distribution functions assigned to the observation vector $\boldsymbol{Y}$ is assumed to have the properties

$$\forall\{\boldsymbol{\beta} \in \mathcal{V}\}\forall\{\boldsymbol{\vartheta} \in \underline{\vartheta}\}\big(E(\boldsymbol{Y} \mid \boldsymbol{\beta}, \boldsymbol{\vartheta}) = \mathbf{X}\boldsymbol{\beta}\big)$$

and

$$\forall\{\boldsymbol{\beta} \in \mathcal{V}\}\forall\{\boldsymbol{\vartheta} \in \underline{\vartheta}\}\left(\operatorname{Var}(\boldsymbol{Y} \mid \boldsymbol{\beta}, \boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right).$$

Here $\mathcal{V}$ means either $\mathbb{R}^k$ (the case of the universal model without constraints), or $\{\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^k, \ \boldsymbol{b} + \mathbf{B}\boldsymbol{u} = \boldsymbol{0}\}$, $\boldsymbol{b} \in \mathcal{M}(\mathbf{B}) = \{\mathbf{B}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^k\}$ a known vector, $\mathbf{B}$ a known $q \times k$ dimensional matrix (the case of the constraints of the type I) or $\left\{\binom{\boldsymbol{v}}{\boldsymbol{z}} : \boldsymbol{b} + (\mathbf{B}_1, \mathbf{B}_2)\binom{\boldsymbol{v}}{\boldsymbol{z}} = \boldsymbol{0}\right\}$, $\boldsymbol{b} \in \mathcal{M}(\mathbf{B}_1, \mathbf{B}_2)$ is a known vector, $\mathbf{B}_1$ and $\mathbf{B}_2$ known $q \times (k - l)$ and $k \times l$ dimensional matrices, respectively (the case of the constraints of the type II); the symmetric matrices $\mathbf{V}_1, \dots, \mathbf{V}_p$ are known and $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta} \subset \mathbb{R}^p$ is the vector of parameters known only approximately.

Let $\boldsymbol{\vartheta}^*$ be the true value of the vector parameter $\boldsymbol{\vartheta}$. A small change of $\boldsymbol{\vartheta}^*$ to $\boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}$ causes a small change of the $\boldsymbol{\vartheta}^*$-LBLUE $\widehat{\boldsymbol{h}'\boldsymbol{\beta}}$ of the function $h(\boldsymbol{\beta}) = \boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}$. Since $\boldsymbol{\vartheta}^*$-LBLUE is unbiased for all $\boldsymbol{\vartheta} \in \underline{\vartheta}$, i.e.

$$\forall\{\boldsymbol{\beta} \in \mathcal{V}\}\forall\{\boldsymbol{\vartheta} \in \underline{\vartheta}\}\left(E\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) \mid \boldsymbol{\beta}, \boldsymbol{\vartheta}\right] = \boldsymbol{h}'\boldsymbol{\beta}\right),$$

the effect of the change $\delta\boldsymbol{\vartheta}$ results in an enlargement of the variance of the estimator only. The enlarged variance can be tolerated with respect to the opinion of users. Generally, the ratio

$$\sqrt{\operatorname{Var}\big(\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \mid \boldsymbol{\vartheta}^*\big) / \operatorname{Var}\big(\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) \mid \boldsymbol{\vartheta}^*\big)},$$

or the difference

$$\sqrt{\operatorname{Var}\big(\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \mid \boldsymbol{\vartheta}^*\big)} - \sqrt{\operatorname{Var}\big(\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) \mid \boldsymbol{\vartheta}^*\big)},$$

or similar quantities are used for deciding to what extend the change of $\boldsymbol{\vartheta}$ can be tolerated. The problem is to determine the domain of those changes $\delta\boldsymbol{\vartheta}$ which do not cause a greater change in $\operatorname{Var}\big(\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y},\boldsymbol{\vartheta}^*)\mid\boldsymbol{\vartheta}^*\big)$ than the tolerable one.

If the observation vector is normally distributed, $\boldsymbol{Y}\sim N_n\big[\mathbf{X}\boldsymbol{\beta},\boldsymbol{\Sigma}(\boldsymbol{\vartheta})\big]$, the confidence regions for estimable functions can be easily determined. Also in this case an uncertainty in the value of the vector $\boldsymbol{\vartheta}$ causes a change in a shape of the confidence ellipsoid and in the level of confidence. Similarly in testing linear hypotheses on $\boldsymbol{\beta}$ the small change in $\boldsymbol{\vartheta}$ can cause an enlargement of the risk and a decrease of values of the power function of the test.

Let $\mathbf{A}$ be an $m\times n$ matrix, $\mathbf{W}$ an $m\times m$ p.s.d. (positive semidefinite) matrix and let $\mathcal{M}(\mathbf{A})=\{\mathbf{A}\boldsymbol{u}:\ \boldsymbol{u}\in\mathbb{R}^n\}\subset\mathcal{M}(\mathbf{W})$. Then $\mathbf{P}_{\mathbf{A}}^{\mathbf{W}^+}$ means the matrix $\mathbf{A}(\mathbf{A}'\mathbf{W}^-\mathbf{A})^-\mathbf{A}'\mathbf{W}^+$ ($^-$ means the g-inverse and $^+$ the Moore-Penrose g-inverse of the matrix; in more detail cf. [8]). If $\mathbf{W}$ is p.d. (positive definite), then $\mathbf{P}_{\mathbf{A}}^{\mathbf{W}^{-1}}$ is the $\mathbf{W}^{-1}$-projection matrix on $\mathcal{M}(\mathbf{A})$. The symbol $\mathbf{M}_{\mathbf{A}}^{\mathbf{W}^{-1}}$ means $\mathbf{I}-\mathbf{P}_{\mathbf{A}}^{\mathbf{W}^{-1}}$. If $\mathbf{W}=\mathbf{I}$ (identical matrix), then the symbol $\mathbf{P}_{\mathbf{A}}$ is used instead of $\mathbf{P}_{\mathbf{A}}^{\mathbf{I}}$; analogously $\mathbf{M}_{\mathbf{A}}^{\mathbf{I}}$ is substituted by $\mathbf{M}_{\mathbf{A}}$.

There are several kind of nonsensitiveness regions. One of them is defined here. The others will be defined in the following.

DEFINITION 1.1. The *nonsensitiveness region* for the unbiasedly estimable function $h(\boldsymbol{\beta})=\boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta}\in\mathbb{R}^k$, is

$$\Big\{\delta\boldsymbol{\vartheta}:\ \operatorname{Var}\big[\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y},\boldsymbol{\vartheta}^*+\delta\boldsymbol{\vartheta})\mid\boldsymbol{\vartheta}^*\big]\le(1+\varepsilon^2)\operatorname{Var}\big[\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y},\boldsymbol{\vartheta}^*)\mid\boldsymbol{\vartheta}^*\big]\Big\}.$$

Here $\varepsilon>0$ is chosen by a statistician.

2. Universal model without constraints

DEFINITION 2.1. The *universal model without constraints* is

$$\boldsymbol{Y}\sim_n\big(\mathbf{X}\boldsymbol{\beta},\boldsymbol{\Sigma}(\boldsymbol{\vartheta})\big),\qquad \boldsymbol{\beta}\in\mathbb{R}^k,\ \ \boldsymbol{\vartheta}\in\underline{\vartheta}\ \text{(an open set in } \mathbb{R}^p),$$

where $\boldsymbol{\Sigma}(\boldsymbol{\vartheta})=\sum_{i=1}^p\vartheta_i\mathbf{V}_i$, $\mathbf{V}_i=\mathbf{V}_i'$, $i=1,\dots,p$.

If $\mathbf{V}_1,\dots,\mathbf{V}_p$ are p.s.d. and $\vartheta_1,\dots,\vartheta_p$, are positive, then the model considered is a *mixed linear model.* The mixed linear model is a special case of a general linear model with covariance components in which the matrices $\mathbf{V}_1,\dots,\mathbf{V}_p$ are symmetric (they need not be p.s.d.) and $\vartheta_1,\dots,\vartheta_p\in\underline{\vartheta}\subset\mathbb{R}^p$ (they need not be positive), however $\boldsymbol{\Sigma}(\boldsymbol{\vartheta})=\sum_{i=1}^p\vartheta_i\mathbf{V}_i$ is p.s.d.

In the following the mixed linear model will be under consideration. (Authors have not been able to solve the mentioned problems in models with variance components yet.)

DEFINITION 2.2. Let $\mathbf{A}$ be an $m \times n$-matrix and $\boldsymbol{y} \in \mathcal{M}(\mathbf{A})$ (the subspace generated by the columns of the matrix $\mathbf{A}$). Let $\mathbf{N}$ be an $n \times n$ p.s.d. matrix. Then the matrix $\mathbf{G}$ with the property

$$\forall\{\boldsymbol{y} \in \mathcal{M}(\mathbf{A})\}\forall\{\boldsymbol{x} : \mathbf{A}\boldsymbol{x} = \boldsymbol{y}\}\Big(\mathbf{AG}\boldsymbol{y} = \boldsymbol{y} \ \ \& \ \ \|\mathbf{G}\boldsymbol{y}\|_{\mathbf{N}} \le \|\boldsymbol{x}\|_{\mathbf{N}}\Big),$$

where $\|\boldsymbol{x}\|_{\mathbf{N}} = \sqrt{\boldsymbol{x}'\mathbf{N}\boldsymbol{x}}$, is called the *minimum* $\mathbf{N}$*-seminorm* g*-inverse of the matrix* $\mathbf{A}$ and it is indicated as $\mathbf{A}^-_{m(\mathbf{N})}$ (in more detail cf. [8]).

LEMMA 2.3. *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix. Then the class of all matrices* $\mathbf{G}$ *from Definition* 2.2 *is given by solutions of the equations*

$$\mathbf{AGA} = \mathbf{A} \quad \& \quad \mathbf{NGA} = \mathbf{A}'\mathbf{G}'\mathbf{N}.$$

One of the reprezentations of the matrix $\mathbf{A}^-_{m(\mathbf{N})}$ *is*

$$\mathbf{A}^-_{m(\mathbf{N})} = \begin{cases} \mathbf{N}^-\mathbf{A}'(\mathbf{A}\mathbf{N}^-\mathbf{A}')^- & \text{if } \mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{N}), \\ (\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}'\big[\mathbf{A}(\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}'\big]^- & \text{otherwise.} \end{cases}$$

P r o o f . Cf. [8; p. 44]. □

In the following the abbreviate notations $\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ and $\boldsymbol{\Sigma}^* = \sum_{i=1}^{p} \vartheta_i^* \mathbf{V}_i$ will be used. Here ϑ_i^*, $i = 1, \dots, p$, are the actual values of the covariance parameters.

LEMMA 2.4. *In the universal model the class of all unbiasedly estimable linear functions of* $\boldsymbol{\beta}$ *is*

$$\{h(\cdot) : \ h(\boldsymbol{\beta}) = \boldsymbol{u}'\mathbf{X}\boldsymbol{\beta}, \ \ \boldsymbol{u} \in \mathbb{R}^n\}$$

and the $\boldsymbol{\vartheta}$*-LBLUE of* $h(\cdot)$ *is* $\boldsymbol{u}'\mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\boldsymbol{Y}$.

P r o o f . Cf. [8; p. 140]. □

THEOREM 2.5. *In the universal model it is valid*

(i)

$$\partial\widehat{\mathbf{X}\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta})/\partial\vartheta_i = \mathbf{X}\mathbf{Z}_i'\boldsymbol{v}$$

where $\mathbf{Z}_i$ *is any matrix satisfying the condition*

$$\begin{aligned} &\mathbf{V}_i(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}' + \boldsymbol{\Sigma}\mathbf{Z}_i\mathbf{X}' - \boldsymbol{\Sigma}(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}'\mathbf{Z}_i\mathbf{X}' \\ &\qquad = \mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\mathbf{V}_i + \mathbf{X}\mathbf{Z}_i'\boldsymbol{\Sigma} - \mathbf{X}\mathbf{Z}_i'\mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\boldsymbol{\Sigma} \end{aligned} \qquad (*)$$

and

$$\mathbf{v} = \left\{\mathbf{I} - \mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\right\} \mathbf{Y};$$

(ii)

$$\operatorname{cov}\left(\mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\mathbf{Y}, \partial\mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\mathbf{Y}/\partial\vartheta_i \,\middle|\, \vartheta\right) = \mathbf{0}$$

and

(iii) *the expression* $\mathbf{X}\mathbf{Z}_i'\mathbf{v}$ *is invariant of the solution of* $(*)$. *One solution of* $(*)$ *is*

$$\mathbf{Z}_i = -\boldsymbol{\Sigma}^-\mathbf{V}_i(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}.$$

Proof.

(i) With respect to Lemma 2.3

$$\mathbf{X}'(\mathbf{X}')^-_{m(\mathbf{N})}\mathbf{X}' = \mathbf{X}' \quad \& \quad \boldsymbol{\Sigma}(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}' = \mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\boldsymbol{\Sigma},$$

thus

$$\mathbf{X}'\left(\partial(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}/\partial\vartheta_i\right)\mathbf{X}' = \mathbf{0},$$

and

$$\begin{aligned}\mathbf{V}_i(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}' &+ \boldsymbol{\Sigma}\left(\partial(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}/\partial\vartheta_i\right)\mathbf{X}' \\ &= \mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\mathbf{V}_i + \mathbf{X}\left(\partial(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}/\partial\vartheta_i\right)'\boldsymbol{\Sigma}\end{aligned}$$

what implies

$$\left(\partial(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}/\partial\vartheta_i\right) \in \left\{\mathbf{Z}_i - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}'\mathbf{Z}_i\mathbf{X}'(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})} : \ \mathbf{Z}_i \text{ satisfies } (*)\right\}.$$

Further

$$\begin{aligned}\partial\mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\mathbf{Y}/\partial\vartheta_i &= \mathbf{X}\left\{\mathbf{Z}_i - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}'\mathbf{Z}_i\mathbf{X}'(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right\}'\mathbf{Y} \\ &= \mathbf{X}\mathbf{Z}_i'\left\{\mathbf{I} - \mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\right\}\mathbf{Y} \\ &= \mathbf{X}\mathbf{Z}_i'\mathbf{v}.\end{aligned}$$

(ii) With respect to (i)

$$\begin{aligned}&\operatorname{cov}\left(\mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\mathbf{Y}, \partial\mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\mathbf{Y}/\partial\vartheta_i \,\middle|\, \vartheta\right) \\ &\quad= \mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\operatorname{cov}(\mathbf{Y}, \mathbf{v} \mid \vartheta)\mathbf{Z}_i\mathbf{X}' \\ &\quad= \mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\operatorname{cov}\left(\mathbf{Y}, \left\{\mathbf{I} - \mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\right\}\mathbf{Y} \,\middle|\, \vartheta\right)\mathbf{Z}_i\mathbf{X}' \\ &\quad= \mathbf{X}[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}]'\boldsymbol{\Sigma}[\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}']\mathbf{Z}_i\mathbf{X}' \\ &\quad= \boldsymbol{\Sigma}(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}'[\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\mathbf{X}']\mathbf{Z}_i\mathbf{X}' = \mathbf{0}.\end{aligned}$$

(iii) One solution of the equation $(*)$ is

$$\mathbf{Z}_i = -\Sigma^-(\vartheta)\mathbf{V}_i(\mathbf{X}')^-_{m(\Sigma)},$$

which can be checked by substitution.

By multiplying both sides of $(*)$ from the right hand side by the matrix $\mathbf{T}^-\mathbf{X}'$, where $\mathbf{T} = \Sigma + \mathbf{X}\mathbf{X}'$, we obtain the equation

$$\mathrm{Var}(\boldsymbol{v})\mathbf{Z}_i\mathbf{X}'\mathbf{T}^-\mathbf{X} = \mathbf{V}_i(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}'\mathbf{T}^-\mathbf{X} - \mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{V}_i\mathbf{T}^-\mathbf{X}. \qquad (**)$$

Each of the solutions of the equation $(*)$ is also a solution of the equation $(**)$. The full class of all solutions of the new equation $(**)$ is

$$\mathcal{Z}_i = \Big\{ -\Sigma^-\mathbf{V}_i(\mathbf{X}')^-_{m(\Sigma)} + \mathbf{U}_i - \big(\mathrm{Var}(\boldsymbol{v})\big)^+\mathrm{Var}(\boldsymbol{v})\mathbf{U}_i(\mathbf{X}'\mathbf{T}^-\mathbf{X})(\mathbf{X}'\mathbf{T}^-\mathbf{X})^- :\ \mathbf{U}_i \ \text{arbitrary}\Big\}.$$

For any $\mathbf{Z}_i \in \mathcal{Z}_i$ we have

$$\begin{aligned}\mathbf{X}\mathbf{Z}_i'\boldsymbol{v} &= -\mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{V}_i\Sigma^-(\vartheta)\boldsymbol{v}\\ &\quad + \mathbf{X}\mathbf{U}_i'\boldsymbol{v} - \mathbf{X}(\mathbf{X}'\mathbf{T}^-\mathbf{X})^-(\mathbf{X}'\mathbf{T}^-\mathbf{X})\mathbf{U}_i'\,\mathrm{Var}(\boldsymbol{v})\big[\mathrm{Var}(\boldsymbol{v})\big]^+\boldsymbol{v}\\ &= -\mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{V}_i\Sigma^-(\vartheta)\boldsymbol{v},\end{aligned}$$

since $\mathrm{Var}(\boldsymbol{v})\big[\mathrm{Var}(\boldsymbol{v})\big]^+\boldsymbol{v} = \boldsymbol{v}$ and $\mathbf{X}(\mathbf{X}'\mathbf{T}^-\mathbf{X})^-(\mathbf{X}'\mathbf{T}^-\mathbf{X}) = \mathbf{X}$. □

COROLLARY 2.6. *With respect to Theorem* 2.5(ii)

$$\begin{aligned}&\forall\{\boldsymbol{u} \in \mathbb{R}^n\}\\ &\mathrm{Var}\Big[\boldsymbol{u}'\mathbf{X}\widehat{\beta}(\boldsymbol{Y}, \vartheta^* + \delta\vartheta) \mid \vartheta^*\Big]\\ &\approx \mathrm{Var}\Big[\boldsymbol{u}'\mathbf{X}\widehat{\beta}(\boldsymbol{Y}, \vartheta^*) \mid \vartheta^*\Big] + \mathrm{Var}\bigg(\Big\{\boldsymbol{u}'\partial\mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma)}\big]'\boldsymbol{Y}/\partial\vartheta'\Big\}\Big|_{\vartheta=\vartheta^*}\delta\vartheta \,\Big|\, \vartheta^*\bigg).\end{aligned}$$

This corollary is suitable for determining boundaries of the nonsensitiveness for the ϑ^*-LBLUE of an unbiasedly estimable function of the parameter β.

If it is required

$$\sqrt{\mathrm{Var}\Big(\big[\partial\widehat{\boldsymbol{h}'\beta}(\boldsymbol{Y}, \vartheta^*)/\partial(\vartheta^*)'\big]\delta\vartheta) \mid \vartheta^*\Big)} < \varepsilon_{\boldsymbol{h}}\sigma_{\boldsymbol{h}},$$

where $\sigma_{\boldsymbol{h}} = \sqrt{\boldsymbol{h}'\big[(\mathbf{X}')^-_{m(\Sigma^*)}\big]'\Sigma^*(\mathbf{X}')^-_{m(\Sigma^*)}\boldsymbol{h}}$ and $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}')$, then the followin theorem gives one of the possible solutions of the mentioned problem.

THEOREM 2.7. *Let* $\mathbf{Y} \sim_n \left(\mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right)$, $\boldsymbol{\beta} \in \mathbb{R}^k$. *Let* $h(\boldsymbol{\beta}) = \mathbf{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, *be an unbiasedly estimable function of the parameter* $\boldsymbol{\beta}$; *i.e.* $\mathbf{h} \in \mathcal{M}(\mathbf{X}')$. *Then*

$$\delta\boldsymbol{\vartheta} \in \left\{\mathbf{u} : \ \mathbf{u}'\mathbf{W}_{\mathbf{h}}\mathbf{u} \le \varepsilon_{\mathbf{h}}^2 \sigma_{\mathbf{h}}^2\right\} \quad (\textit{nonsensitiveness region})$$

$$\Longrightarrow \sqrt{\mathrm{Var}\Big(\big[\partial\widehat{\mathbf{h}'\boldsymbol{\beta}}(\mathbf{Y},\boldsymbol{\vartheta}^*)/\partial(\boldsymbol{\vartheta}^*)'\big]\delta\boldsymbol{\vartheta}) \mid \boldsymbol{\vartheta}^*\Big)} \le \varepsilon_{\mathbf{h}}\sigma_{\mathbf{h}},$$

where

$$\{\mathbf{W}_{\mathbf{h}}\}_{i,j} = \mathbf{L}_{\mathbf{h}}'\mathbf{V}_i\Sigma^{*-}\,\mathrm{Var}(\mathbf{v}^* \mid \boldsymbol{\vartheta}^*)\Sigma^{*-}\mathbf{V}_j\mathbf{L}_{\mathbf{h}}, \qquad i,j = 1,\dots,p$$

$$\mathbf{L}_{\mathbf{h}}' = \begin{cases} \mathbf{h}'\mathbf{C}^{-1}\mathbf{X}'\Sigma^{*-} & \textit{if } r(\mathbf{X}) = k < n, \ \ \Sigma^* \textit{ is p.d.},\\ \mathbf{h}'\mathbf{C}^{-}\mathbf{X}'\Sigma^{*-} & \textit{if } \mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma^*),\\ \mathbf{h}'(\mathbf{X}'\mathbf{T}^{*-}\mathbf{X})^{-}\mathbf{X}'\mathbf{T}^{*-} & \textit{otherwise}, \end{cases}$$

$$\mathbf{C} = \mathbf{X}'\Sigma^{*-}\mathbf{X},$$

$$\mathbf{v}^* = \left\{\mathbf{I} - \mathbf{X}\big[(\mathbf{X}')^{-}_{m(\Sigma^*)}\big]'\right\}\mathbf{Y},$$

$$\mathbf{T}^* = \Sigma^* + \mathbf{X}\mathbf{X}',$$

$$\mathrm{Var}(\mathbf{v}^* \mid \boldsymbol{\vartheta}^*) = \Sigma^* - \mathbf{X}\big[(\mathbf{X}')^{-}_{m(\Sigma^*)}\big]'\Sigma^* = \Sigma^*\big[\mathbf{M}_{\mathbf{X}}\Sigma^*\mathbf{M}_{\mathbf{X}}\big]^{+}\Sigma^*,$$

$$\mathbf{M}_{\mathbf{X}} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' = \mathbf{I} - \mathbf{X}\mathbf{X}^{+}$$

(*notice* $\widehat{\mathbf{h}'\boldsymbol{\beta}}(\mathbf{Y},\boldsymbol{\vartheta}^*) = \mathbf{L}_{\mathbf{h}}'\mathbf{Y}$).

Proof. Since

$$\partial\widehat{\mathbf{h}'\boldsymbol{\beta}}(\mathbf{Y},\boldsymbol{\vartheta})/\partial\vartheta_i = \mathbf{h}'\mathbf{Z}_i'\mathbf{v},$$

(see Theorem 2.5(i)) we can write

$$\mathrm{Var}\left[\sum_{i=1}^{p}\big(\partial\widehat{\mathbf{h}'\boldsymbol{\beta}}(\mathbf{Y},\boldsymbol{\vartheta})/\partial\vartheta_i\big)\delta\vartheta_i \,\Big|\, \boldsymbol{\vartheta}\right] = \sum_{i=1}^{p}\sum_{j=1}^{p}\delta\vartheta_i\delta\vartheta_j\,\mathrm{cov}\big(\mathbf{h}'\mathbf{Z}_i'\mathbf{v}, \mathbf{h}'\mathbf{Z}_j'\mathbf{v} \mid \boldsymbol{\vartheta}\big)$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\delta\vartheta_i\delta\vartheta_j\mathbf{h}'\mathbf{Z}_i'\,\mathrm{Var}(\mathbf{v} \mid \boldsymbol{\vartheta})\mathbf{Z}_j\mathbf{h}.$$

Since

$$\mathrm{Var}(\mathbf{v} \mid \boldsymbol{\vartheta}) = \Sigma(\boldsymbol{\vartheta}) - \Sigma(\boldsymbol{\vartheta})(\mathbf{X}')^{-}_{m(\Sigma)}\mathbf{X}' = \Sigma(\boldsymbol{\vartheta}) - \mathbf{X}\big[(\mathbf{X}')^{-}_{m(\Sigma)}\big]'\Sigma(\boldsymbol{\vartheta})$$

the expression $\mathbf{h}'\mathbf{Z}_i'\,\mathrm{Var}(\mathbf{v} \mid \boldsymbol{\vartheta})\mathbf{Z}_j\mathbf{h}$ can be written in the form

$$\mathbf{h}'\mathbf{Z}_i'\Sigma(\boldsymbol{\vartheta})\big[\mathbf{I} - (\mathbf{X}')^{-}_{m(\Sigma)}\mathbf{X}'\big]\mathbf{Z}_j\mathbf{h}.$$

The matrix $\mathbf{Z}_i$ satisfies the condition $(*)$ what implies

$$\mathbf{X}\mathbf{Z}_k'\Sigma(\boldsymbol{\vartheta})\big[\mathbf{I} - (\mathbf{X}')^{-}_{m(\Sigma)}\mathbf{X}'\big] = \mathbf{X}\big[(\mathbf{X}')^{-}_{m(\Sigma)}\big]'\mathbf{V}_k - \mathbf{X}\big[(\mathbf{X}')^{-}_{m(\Sigma)}\big]'\mathbf{V}_k(\mathbf{X}')^{-}_{m(\Sigma)}\mathbf{X}',$$

$k = i, j$. Thus we can write

$$\begin{aligned} & \mathbf{h}'\mathbf{Z}_i'\Sigma[\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\mathbf{Z}_j\mathbf{h} \\ & \quad = \mathbf{h}'\left\{[(\mathbf{X}')^-_{m(\Sigma)}]'\mathbf{V}_i - [(\mathbf{X}')^-_{m(\Sigma)}]'\mathbf{V}_i(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}'\right\}\Sigma^- \times \\ & \qquad \times \left\{\mathbf{V}_j(\mathbf{X}')^-_{m(\Sigma)} - \mathbf{X}[(\mathbf{X}')^-_{m(\Sigma)}]'\mathbf{V}_j(\mathbf{X}')^-_{m(\Sigma)}\right\}\mathbf{h}. \end{aligned}$$

Since $\mathbf{L}_\mathbf{h}' = \mathbf{h}'[(\mathbf{X}')^-_{m(\Sigma)}]'$, and $\mathbf{V}_i = \mathbf{V}_i\Sigma^-\Sigma = \Sigma\Sigma^-\mathbf{V}_i$,

$$\begin{aligned} & \mathbf{h}'\mathbf{Z}_i'\Sigma[\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\mathbf{Z}_j\mathbf{h} \\ & \quad = \mathbf{L}_\mathbf{h}'\mathbf{V}_i\Sigma^-\Sigma[\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\{\mathbf{I} - \mathbf{X}[(\mathbf{X}')^-_{m(\Sigma)}]'\}\Sigma\Sigma^-\mathbf{V}_j\mathbf{L}_\mathbf{h} \\ & \quad = \mathbf{L}_\mathbf{h}'\mathbf{V}_i\Sigma^-\{\mathbf{I} - \mathbf{X}[(\mathbf{X}')^-_{m(\Sigma)}]'\}\Sigma\Sigma^-\Sigma[\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{V}_j\mathbf{L}_\mathbf{h} \\ & \quad = \mathbf{L}_\mathbf{h}'\mathbf{V}_i\Sigma^-[\Sigma - \Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{V}_j\mathbf{L}_\mathbf{h}. \end{aligned}$$

Thus we obtain $\{\mathbf{W}\}_{i,j}$ in the general form

$$\mathbf{L}_\mathbf{h}'\mathbf{V}_i\Sigma^{*-}\operatorname{Var}(\mathbf{v}^* \mid \boldsymbol{\vartheta}^*)\Sigma^{*-}\mathbf{V}_j\mathbf{L}_\mathbf{h}, \qquad i, j = 1, \dots, p.$$

Now Lemma 2.3 can be used and it is obvious how to finish the proof. □

Remark 2.8. It is to be noticed that the shift $\delta\boldsymbol{\vartheta}$ in the direction of $\boldsymbol{\vartheta}^*$, i.e. $\delta\boldsymbol{\vartheta} = t\boldsymbol{\vartheta}^*$, does not cause a change of the estimator. It is implied by the following relation

$$\begin{aligned} & t^2\sum_{i=1}^{p}\sum_{j=1}^{p}\vartheta_i^*\vartheta_j^*\mathbf{L}_\mathbf{h}'\mathbf{V}_i\Sigma^{*-}\operatorname{Var}(\mathbf{v}^* \mid \boldsymbol{\vartheta}^*)\Sigma^{*-}\mathbf{V}_j\mathbf{L}_\mathbf{h} \\ & \quad = t^2\mathbf{L}_\mathbf{h}'\Sigma^*\Sigma^{*-}[\Sigma^* - \Sigma^*(\mathbf{X}')^-_{m(\Sigma^*)}\mathbf{X}']\Sigma^{*-}\Sigma^*\mathbf{L}_\mathbf{h} \\ & \quad = t^2\mathbf{u}'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma^*)}]'\left\{\Sigma^* - \mathbf{X}[(\mathbf{X}')^-_{m(\Sigma^*)}]'\Sigma^*\right\}\mathbf{L}_\mathbf{h} = \mathbf{0}, \end{aligned}$$

since there exists $\mathbf{u} \in \mathbb{R}^n$ such that $\mathbf{h}' = \mathbf{u}'\mathbf{X}$.

If we want to know the nonsensitiveness region for all linear unbiasedly estimable functions simultaneously, we can proceed according to the following lemma.

LEMMA 2.9. *Let $\mathbf{E}_{i,j}$ be the $p \times p$ matrix with the (i,j)-th entry equal to 1 and with other entries equal to 0, $\mathbf{s} \in \mathbb{R}^k$, such that $\mathbf{s}'\mathbf{s} = 1$ and $\mathbf{A}_{i,j}$ a $k \times k$ matrix given by the relation*

$$\mathbf{A}_{i,j} = [(\mathbf{X}')^-_{m(\Sigma^*)}]'\mathbf{V}_i\Sigma^{*-}\operatorname{Var}(\mathbf{v}^* \mid \boldsymbol{\vartheta}^*)\Sigma^{*-}\mathbf{V}_j\Sigma^{*-}(\mathbf{X}')^-_{m(\Sigma^*)},$$

$i, j = 1, \dots, p$, *then in the universal model from Theorem* 2.7 *we can write*

$$\forall\{\boldsymbol{h} : \boldsymbol{h}'\boldsymbol{h} = 1\}\forall\{\delta\boldsymbol{\vartheta} \in \mathbb{R}^p\}\left(\delta\boldsymbol{\vartheta}'\mathbf{W}_{\boldsymbol{h}}\delta\boldsymbol{\vartheta} \leq \delta\boldsymbol{\vartheta}' \sum_{i=1}^{p\times k} \gamma_i \mathbf{G}_i \mathbf{G}_i' \delta\boldsymbol{\vartheta}\right),$$

where

$$\mathbf{G}_i = \begin{pmatrix} \boldsymbol{g}_{i,1}' \\ \vdots \\ \boldsymbol{g}_{i,p}' \end{pmatrix}, \qquad i = 1, \dots, p \times k,$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p} \mathbf{E}_{i,j} \otimes \mathbf{A}_{i,j} = \sum_{r=1}^{p\times k} \gamma_r \boldsymbol{g}_r \boldsymbol{g}_r' \qquad (\textit{the spectral decomposition}),$$

$$\boldsymbol{g}_r = (\boldsymbol{g}_{r,1}', \dots, \boldsymbol{g}_{r,p}')', \qquad r = 1, \dots, p \times k.$$

The vectors $\boldsymbol{g}_r$, $r = 1, \dots, p \times k$ *are* $p \times k$*-dimensional and* $\boldsymbol{g}_r'\boldsymbol{g}_s = \delta_{r,s}$ (*the Kronecker delta*).

Proof The expression $\delta\boldsymbol{\vartheta}'\mathbf{W}_{\boldsymbol{h}}\delta\boldsymbol{\vartheta}$ can be rewritten in the form

$$(\delta\boldsymbol{\vartheta}' \otimes \boldsymbol{h}')\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{E}_{i,j} \otimes \mathbf{A}_{i,j})(\delta\boldsymbol{\vartheta} \otimes \boldsymbol{h}),$$

since

$$(\delta\boldsymbol{\vartheta}' \otimes \boldsymbol{h}')\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{E}_{i,j} \otimes \mathbf{A}_{i,j})(\delta\boldsymbol{\vartheta} \otimes \boldsymbol{h}) = \sum_{i=1}^{p}\sum_{j=1}^{p} \delta\boldsymbol{\vartheta}'\mathbf{E}_{i,j}\delta\boldsymbol{\vartheta}\boldsymbol{h}'\mathbf{A}_{i,j}\boldsymbol{h}$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{p} \delta\vartheta_i \delta\vartheta_j \{\mathbf{W}_{\boldsymbol{h}}\}_{i,j}.$$

Let $\sum_{r=1}^{p\times k} \gamma_r \boldsymbol{g}_r \boldsymbol{g}_r'$ be the spectral decomposition of the matrix $\sum_{i=1}^{p}\sum_{j=1}^{p} \mathbf{E}_{i,j} \otimes \mathbf{A}_{i,j}$. Then we can write

$$\delta'\boldsymbol{\vartheta}\mathbf{W}_{\boldsymbol{h}}\delta\boldsymbol{\vartheta} = (\delta\boldsymbol{\vartheta}' \otimes \boldsymbol{h}')\sum_{r=1}^{p\times k} \gamma_r \boldsymbol{g}_r \boldsymbol{g}_r'(\delta\boldsymbol{\vartheta} \otimes \boldsymbol{h}) = \sum_{r=1}^{p\times k} \gamma_r (\delta\boldsymbol{\vartheta}'\mathbf{G}_r\boldsymbol{h})^2$$
$$\leq \sum_{r=1}^{p\times k} \gamma_r \left(\delta\boldsymbol{\vartheta}'\mathbf{G}_r\mathbf{G}_r'\delta\boldsymbol{\vartheta}/\sqrt{\delta\boldsymbol{\vartheta}'\mathbf{G}_r\mathbf{G}_r'\delta\boldsymbol{\vartheta}}\right)^2 = \delta\boldsymbol{\vartheta}' \sum_{r=1}^{p\times k} \gamma_r \mathbf{G}_r\mathbf{G}_r'\delta\boldsymbol{\vartheta},$$

since

$$\forall\{\boldsymbol{h} : \boldsymbol{h}'\boldsymbol{h} = 1\}\left((\delta\boldsymbol{\vartheta}'\mathbf{G}_r\boldsymbol{h})^2 \leq \left(\delta\boldsymbol{\vartheta}'\mathbf{G}_r \frac{\mathbf{G}_r'\delta\boldsymbol{\vartheta}}{\sqrt{\delta\boldsymbol{\vartheta}'\mathbf{G}_r\mathbf{G}_r'\delta\boldsymbol{\vartheta}}}\right)^2\right). \qquad \square$$

Let $\operatorname{Ker}(\Sigma^*) = \{\boldsymbol{u} : \Sigma^* \boldsymbol{u} = \boldsymbol{0}\}$ and $\mathbf{K}$ be the matrix with the property $\mathcal{M}(\mathbf{K}) = \operatorname{Ker}(\Sigma^*)$.

If $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}'\mathbf{K})$, then $\operatorname{Var}\big[\widehat{\boldsymbol{h}'\beta}(\boldsymbol{Y}, \vartheta^*) \mid \vartheta^*\big] = 0$, which means that this function is nonsensitive. If $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma^*)$, then $\mathcal{M}(\mathbf{X}'\mathbf{K}) = \{\boldsymbol{0}\}$ and $\mathcal{M}(\mathbf{X}'\Sigma^*) = \mathcal{M}(\mathbf{X}')$. Therefore in the following theorem the inclusion

$$\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma^*)$$

is assumed.

THEOREM 2.10. *Let $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma^*)$, $\sum_{i=1}^{p} \kappa_i \boldsymbol{l}_i \boldsymbol{l}_i'$ be the spectral decomposition of the matrix $\sum_{i=1}^{p\times k} \gamma_i \mathbf{G}_i \mathbf{G}_i'$ from Lemma 2.9 and let $\kappa_1 \geq \kappa_2 \geq \dots$. Let*

$$\omega_1 = 1\Big/\Big(\min\Big\{\boldsymbol{h}'\big[(\mathbf{X}')^-_{m(\Sigma^*)}\big]'\Sigma^*(\mathbf{X}')^-_{m(\Sigma^*)}\boldsymbol{h} :\ \boldsymbol{h} \in \mathcal{M}(\mathbf{X}'),\ \boldsymbol{h}'\boldsymbol{h} = 1\Big\}\Big).$$

(In the regular case ω_1 is the maximum eigenvalue of the matrix $\mathbf{C}^{-1}$.)

Then

$$\forall\{\delta\vartheta \in \mathcal{E}\}\forall\{\boldsymbol{h} \in \mathcal{M}(\mathbf{X}')\}\ \left(\sqrt{\operatorname{Var}\left[\left(\frac{\partial \boldsymbol{h}'\hat{\beta}(\boldsymbol{Y}, \vartheta^*)}{\partial \vartheta'}\right)\delta\vartheta \,\Big|\, \vartheta^*\right]}\Big/\sigma_{\boldsymbol{h}} \leq \varepsilon\right),$$

where

$$\mathcal{E} = \left\{\delta\vartheta :\ \|\delta\vartheta\| \leq \frac{\varepsilon}{\sqrt{\kappa_1\omega_1}}\right\} \qquad (\textit{nonsensitiveness region}).$$

P r o o f. In the first step the inequality $\infty > \omega_1 > 0$ must be proved. Let $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}')$. Since $\mathcal{M}(\mathbf{X}') = \mathcal{M}(\mathbf{X}'\Sigma^*) = \mathcal{M}(\mathbf{X}'\Sigma^{*+}\mathbf{X})$ and $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma^*)$,

$$\begin{aligned}
&\min\Big\{\boldsymbol{h}'\big[(\mathbf{X}')^-_{m(\Sigma^*)}\big]'\Sigma^*(\mathbf{X}')^-_{m(\Sigma^*)}\boldsymbol{h} :\ \boldsymbol{h} \in \mathcal{M}(\mathbf{X}'),\ \boldsymbol{h}'\boldsymbol{h} = 1\Big\}\\
&= \min\big\{\boldsymbol{t}'\mathbf{X}'\Sigma^{*+}\mathbf{X}(\mathbf{X}'\Sigma^{*+}\mathbf{X})^+\mathbf{X}'\Sigma^{*+}\mathbf{X}\boldsymbol{t} :\ \boldsymbol{t}'(\mathbf{X}'\Sigma^{*+}\mathbf{X})^2\boldsymbol{t} = 1\big\}\\
&= \min\big\{\boldsymbol{t}'\mathbf{X}'\Sigma^{*+}\mathbf{X}\boldsymbol{t} :\ \boldsymbol{t}'(\mathbf{X}'\Sigma^{*+}\mathbf{X})^2\boldsymbol{t} = 1\big\} > 0
\end{aligned}$$

and simultaneously

$$\min\Big\{\boldsymbol{h}'\big[(\mathbf{X}')^-_{m(\Sigma^*)}\big]'\Sigma^*(\mathbf{X}')^-_{m(\Sigma^*)}\boldsymbol{h} :\ \boldsymbol{h} \in \mathcal{M}(\mathbf{X}'),\ \boldsymbol{h}'\boldsymbol{h} = 1\Big\} < \infty.$$

Here the relations

$$\begin{aligned}
\boldsymbol{h} &= \mathbf{X}'\Sigma^{*+}\mathbf{X}\boldsymbol{t},\\
\Sigma^*(\mathbf{X}')^-_{m(\Sigma^*)}\mathbf{X}' &= \mathbf{X}(\mathbf{X}'\Sigma^{*+}\mathbf{X})^+\mathbf{X}'
\end{aligned}$$

and

$$\boldsymbol{t}'\mathbf{X}'\Sigma^{*+}\mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma^*)}\big]'\Sigma^*(\mathbf{X}')^-_{m(\Sigma^*)}\mathbf{X}'\Sigma^{*+}\mathbf{X}\boldsymbol{t} = \boldsymbol{t}'\mathbf{X}'\Sigma^{*+}\mathbf{X}\boldsymbol{t}$$

are used (it is to be noticed that the expression $\Sigma^*(\mathbf{X}')^-_{m(\Sigma^*)}\mathbf{X}'$ is invariant with respect to the choice of the matrix $(\mathbf{X}')^-_{m(\Sigma^*)}$.

Let $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}')$ be an arbitrary vector; with respect to Lemma 2.9

$$\delta\vartheta'\mathbf{W}_{\boldsymbol{h}}\delta\vartheta \le \boldsymbol{h}'\boldsymbol{h}\delta\vartheta'\sum_{i=1}^{p\times k}\gamma_i\mathbf{G}_i\mathbf{G}_i'\delta\vartheta = \boldsymbol{h}'\boldsymbol{h}\sum_{i=1}^{p}\kappa_i(\boldsymbol{l}'_i\delta\vartheta)^2 .$$

Thus

$$\max\left\{\sum_{i=1}^{p}\kappa_i(\boldsymbol{l}_i'\delta\vartheta)^2 : \ \|\delta\vartheta\| = c\right\} = c^2\kappa_1$$

and

$$\delta\vartheta = t\boldsymbol{l}_1 \implies \delta\vartheta'\mathbf{W}_{\boldsymbol{h}}\delta\vartheta \le \boldsymbol{h}'\boldsymbol{h}t^2\kappa_1 .$$

It implies

$$\|\delta\vartheta\| \le t \implies$$
$$\implies \forall\{\boldsymbol{h} \in \mathcal{M}(\mathbf{X}')\}$$
$$\sqrt{\operatorname{Var}\left[\left(\frac{\partial\boldsymbol{h}'\hat{\beta}(\boldsymbol{Y},\vartheta^*)}{\partial\vartheta'}\right)\delta\vartheta \,\middle|\, \vartheta^*\right]}\Big/\sigma_{\boldsymbol{h}} \le t\sqrt{\frac{\kappa_1\boldsymbol{h}'\boldsymbol{h}}{\boldsymbol{h}'\big[(\mathbf{X}')_{m(\Sigma)}\big]'\Sigma(\vartheta^*)(\mathbf{X}')^-_{m(\Sigma)}\boldsymbol{h}}}$$
$$\le t\sqrt{\kappa_1\omega_1} .$$

□

Till now the problem was considered in an a priori version, i.e. before the realization of the observation vector $\boldsymbol{Y}$. If the experiment was already realized, i.e. we have a realization $\boldsymbol{y}$ of the observation vector $\boldsymbol{Y}$ at our disposal, then the quantity $\boldsymbol{h}'\mathbf{Z}_i'\boldsymbol{v}_{\text{real}}$ (cf. Theorem 2.5), where

$$\boldsymbol{v}_{\text{real}} = \left\{\mathbf{I} - \mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma^*)}\big]'\right\}\boldsymbol{y} ,$$

is a function of the vector ϑ only and it holds

$$\boldsymbol{h}'\hat{\beta}(\boldsymbol{y},\vartheta^* + \delta\vartheta) = \boldsymbol{h}'\hat{\beta}(\boldsymbol{y},\vartheta^*) + (\boldsymbol{h}'\mathbf{Z}_1'\boldsymbol{v}_{\text{real}},\dots,\boldsymbol{h}'\mathbf{Z}_p'\boldsymbol{v}_{\text{real}})\delta\vartheta .$$

If a shift $|\widehat{\boldsymbol{h}'\beta}(\boldsymbol{y},\vartheta^* + \delta\vartheta) - \widehat{\boldsymbol{h}'\beta}(\boldsymbol{y},\vartheta^*)|$ smaller than $\varepsilon\sigma_{\boldsymbol{h}}$ is tolerable, then the region

$$\{\delta\vartheta : \ |\boldsymbol{p}_{\boldsymbol{h}}'\delta\vartheta| \le \varepsilon_{\boldsymbol{h}}\sigma_{\boldsymbol{h}}\}$$

where $\boldsymbol{p}_{\boldsymbol{h}}' = (\boldsymbol{h}'\mathbf{Z}_1'\boldsymbol{v}_{\text{real}},\dots,\boldsymbol{h}'\mathbf{Z}_p'\boldsymbol{v}_{\text{real}})$, is an a posteriori nonsensitiveness region for the function $h(\beta) = \boldsymbol{h}'\beta$, $\beta \in \mathbb{R}^k$. This can be substantially greater than the a priori region in the case that the norm of the vector $\boldsymbol{v}_{\text{real}}$ is small.

COROLLARY 2.11. *The a posteriori nonsensitiveness region for the function* $h(\boldsymbol{\beta}) = \boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$ *(it is to be remind that* $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}')$*) is*

$$\{\delta\boldsymbol{\vartheta} : \ |\boldsymbol{p}_{\boldsymbol{h}}'\delta\boldsymbol{\vartheta}| \le \varepsilon\sigma_{\boldsymbol{h}}\},$$

where

$$\boldsymbol{p}_{\boldsymbol{h}}' = \left(-\boldsymbol{L}_{\boldsymbol{h}}'\mathbf{V}_1\Sigma^{*-}\boldsymbol{v}_{\text{real}}, \dots, -\boldsymbol{L}_{\boldsymbol{h}}'\mathbf{V}_p\Sigma^{*-}\boldsymbol{v}_{\text{real}}\right).$$

P r o o f . It is an obvious consequence of Theorem 2.5. □

The maximum sphere

$$\left\{\delta\boldsymbol{\vartheta} : \ \|\delta\boldsymbol{\vartheta}\| \le \varepsilon\sigma_{\boldsymbol{h}} \Big/ \sqrt{\sum_{i=1}^{p}\left(\boldsymbol{L}_{\boldsymbol{h}}'\mathbf{V}_i\Sigma^{*-}(\boldsymbol{\vartheta}^*)\boldsymbol{v}_{\text{real}}\right)^2}\ \right\}$$

included into $\{\delta\boldsymbol{\vartheta} : \ |\boldsymbol{p}_{\boldsymbol{h}}'\delta\boldsymbol{\vartheta}| \le \varepsilon\sigma_{\boldsymbol{h}}\boldsymbol{\beta}\}$, can be more suitable in practice.

Remark 2.12. If $\delta\boldsymbol{\vartheta} = t\boldsymbol{\vartheta}^*$, then

$$\begin{aligned}\boldsymbol{p}_{\boldsymbol{h}}'\delta\boldsymbol{\vartheta} &= -t\boldsymbol{h}'\left[(\mathbf{X}')^-_{m(\Sigma^*)}\right]'\Sigma^*\Sigma^{*-}\boldsymbol{v}_{\text{real}}\\ &= -t\boldsymbol{u}'\mathbf{X}\left[(\mathbf{X}')^-_{m(\Sigma^*)}\right]'\left\{\mathbf{I} - \mathbf{X}\left[(\mathbf{X}')^-_{m(\Sigma^*)}\right]'\right\}\boldsymbol{y} = \boldsymbol{0}.\end{aligned}$$

Remark 2.13. If $\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta})$ is used instead of $\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*)$, then

$$\begin{aligned}&\operatorname{Var}\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \mid \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}\right]\\ &\qquad = \operatorname{Var}\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) \mid \boldsymbol{\vartheta}^*\right] + (\boldsymbol{L}_{\boldsymbol{h}}'\mathbf{V}_1\boldsymbol{L}_{\boldsymbol{h}}, \dots, \boldsymbol{L}_{\boldsymbol{h}}'\mathbf{V}_p\boldsymbol{L}_{\boldsymbol{h}})\delta\boldsymbol{\vartheta}.\end{aligned}$$

If a difference

$$\left|\operatorname{Var}\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \mid \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}\right] - \operatorname{Var}\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) \mid \boldsymbol{\vartheta}^*\right]\right|$$

smaller than $\varepsilon^2\sigma^2_{\boldsymbol{h}}$ is tolerable, then the nonsensitiveness region for the dispersion $\operatorname{Var}\left[\widehat{\boldsymbol{h}'\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) \mid \boldsymbol{\vartheta}^*\right]$ is

$$\{\delta\boldsymbol{\vartheta} : \ |(\boldsymbol{L}_{\boldsymbol{h}}'\mathbf{V}_1\boldsymbol{L}_{\boldsymbol{h}}, \dots, \boldsymbol{L}_{\boldsymbol{h}}'\mathbf{V}_p\boldsymbol{L}_{\boldsymbol{h}})\delta\boldsymbol{\vartheta}| \le \varepsilon^2\sigma^2_{\boldsymbol{h}}\}.$$

The maximum sphere

$$\left\{\delta\boldsymbol{\vartheta} : \ \|\delta\boldsymbol{\vartheta}\| \le \varepsilon^2\sigma^2_{\boldsymbol{h}} \Big/ \sqrt{\sum_{i=1}^{p}(\boldsymbol{L}_{\boldsymbol{h}}'\mathbf{V}_i\boldsymbol{L}_{\boldsymbol{h}})^2}\ \right\},$$

included into it, seems to be more suitable for practice.

3. Universal models with constraints of the type I

DEFINITION 3.1. The *universal model with constraints of the type I* is

$$\boldsymbol{Y} \sim_n \left(\mathbf{X}\boldsymbol{\beta}, \Sigma(\boldsymbol{\vartheta})\right), \qquad \boldsymbol{\beta} \in \{\boldsymbol{u}: \ \boldsymbol{b} + \mathbf{B}\boldsymbol{u} = \boldsymbol{0}\} = \mathcal{V}_{\mathrm{I}}, \ \ \boldsymbol{\vartheta} \in \underline{\vartheta},$$

where $\mathbf{B}$ is a given $q \times k$ matrix and $\boldsymbol{b} \in \mathcal{M}(\mathbf{B})$.

There are two equivalent expressions of this model.

(i) *Model without restrictions*

$$\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0 \sim_n \left(\mathbf{X}\mathbf{K}_{\mathbf{B}}\gamma, \Sigma(\boldsymbol{\vartheta})\right), \qquad \gamma \in \mathbb{R}^{k-r(B)}, \ \ \boldsymbol{\vartheta} \in \underline{\vartheta},$$

where $\boldsymbol{\beta}_0$ is any vector satisfying the equality $\boldsymbol{b} + \mathbf{B}\boldsymbol{\beta}_0 = \boldsymbol{0}$ and $\mathbf{K}_{\mathbf{B}}$ is $k \times \left(k - r(\mathbf{B})\right)$ matrix with the property $\mathcal{M}(\mathbf{K}_{\mathbf{B}}) = \mathrm{Ker}(\mathbf{B}) = \{\boldsymbol{u}: \boldsymbol{u} \in \mathbb{R}^k, \ \mathbf{B}\boldsymbol{u} = \boldsymbol{0}\} = \mathcal{M}(\mathbf{M}_{\mathbf{B}'})$, where $\mathbf{M}_{\mathbf{B}'} = \mathbf{I} - \mathbf{P}_{\mathbf{B}'} = \mathbf{I} - \mathbf{B}'(\mathbf{B}\mathbf{B}')^{-}\mathbf{B}$.

(ii)

$$\begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix} \sim_{n+q} \left(\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \Sigma, & 0 \\ 0, & 0 \end{pmatrix} \right), \qquad \boldsymbol{\beta} \in \mathcal{V}_{\mathrm{I}}, \ \ \boldsymbol{\vartheta} \in \underline{\vartheta}.$$

In the model from Definition 3.1 a function $h(\boldsymbol{\beta}) = \boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}_{\mathrm{I}}$, is unbiasedly estimable if and only if $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$, thus there exist vectors $\boldsymbol{u}$ and $\boldsymbol{z}$ such that $\boldsymbol{h} = \boldsymbol{h}_1 + \boldsymbol{h}_2$, where $\boldsymbol{h}_1 = \mathbf{X}'\boldsymbol{u}$ and $\boldsymbol{h}_2 = \mathbf{B}'\boldsymbol{z}$.

LEMMA 3.2. *It holds that*

$$\mathbf{X}\mathbf{K}_{\mathbf{B}}\left[(\mathbf{K}_{\mathbf{B}}'\mathbf{X})^{-}_{m(\Sigma)}\right]' = \mathbf{X}\mathbf{M}_{\mathbf{B}'}\left[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{-}_{m(\Sigma)}\right]'.$$

P r o o f. With respect to our assumption on the matrix $\mathbf{K}_{\mathbf{B}}$,

$$\forall\{\boldsymbol{x} \in \mathcal{M}(\mathbf{X}')\}\left(\{\boldsymbol{u}: \ \mathbf{K}_{\mathbf{B}}'\mathbf{X}'\boldsymbol{u} = \mathbf{K}_{\mathbf{B}}'\boldsymbol{x}\} = \{\boldsymbol{u}: \ \mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{u} = \mathbf{M}_{\mathbf{B}'}\boldsymbol{x}\} = \mathcal{T}\right).$$

It is implied by the following

$$\begin{aligned} \mathbf{X}\mathcal{M}(\mathbf{K}_{\mathbf{B}}) = \mathbf{X}\mathcal{M}(\mathbf{M}_{\mathbf{B}'}) &\implies \mathcal{M}(\mathbf{X}\mathbf{K}_{\mathbf{B}}) = \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \\ &\iff \mathrm{Ker}(\mathbf{K}_{\mathbf{B}}'\mathbf{X}') = \mathrm{Ker}(\mathbf{M}_{\mathbf{B}'}\mathbf{X}'). \end{aligned}$$

Further, with respect do Definition 2.2

$$\begin{aligned} &\forall\{\boldsymbol{u} \in \mathbb{R}^n\}\Big(\boldsymbol{u}'\mathbf{X}\mathbf{K}_{\mathbf{B}}\left[(\mathbf{K}_{\mathbf{B}}'\mathbf{X}')^{-}_{m(\Sigma)}\right]'\Sigma(\mathbf{K}_{\mathbf{B}}'\mathbf{X}')^{-}_{m(\Sigma)}\mathbf{K}_{\mathbf{B}}'\mathbf{X}'\boldsymbol{u} \\ &\qquad = \boldsymbol{u}'\mathbf{X}\mathbf{M}_{\mathbf{B}'}\left[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{-}_{m(\Sigma)}\right]'\Sigma(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{-}_{m(\Sigma)}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{u}\Big) \\ &\implies \Sigma(\mathbf{K}_{\mathbf{B}}'\mathbf{X}')^{-}_{m(\Sigma)}\mathbf{K}_{\mathbf{B}}'\mathbf{X}' = \Sigma(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{-}_{m(\Sigma)}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'. \end{aligned}$$

Thus $\mathbf{X}\mathbf{M}_{\mathbf{B}'}\left[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{-}_{m(\Sigma)}\right]'$ is one version of the matrix $\mathbf{X}\mathbf{K}_{\mathbf{B}}\left[(\mathbf{K}_{\mathbf{B}}'\mathbf{X}')^{-}_{m(\Sigma)}\right]'$. □

LEMMA 3.3. *We have*

$$\mathcal{M}(\mathbf{X}', \mathbf{B}') = \mathcal{M}\big[\mathbf{X}'(\Sigma + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}\mathbf{X} + \mathbf{B}'\mathbf{B}\big].$$

Proof.

$$\forall\{\boldsymbol{u} \in \mathbb{R}^n\}\,\exists\{\boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{y} \in \mathbb{R}^q\}\ \mathbf{X}'\boldsymbol{u} = \mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{x} + \mathbf{B}'\boldsymbol{y}.$$

It suffices to choose

$$\boldsymbol{x} = \boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{y} = (\mathbf{B}\mathbf{B}')^{-}\mathbf{B}\mathbf{X}'\boldsymbol{u};$$

thus

$$\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{x} + \mathbf{B}'\boldsymbol{y} = \mathbf{M}_{\mathbf{B}'}\mathbf{X}'\boldsymbol{u} + \mathbf{B}'(\mathbf{B}\mathbf{B}')^{-}\mathbf{B}\mathbf{X}\boldsymbol{u} = \mathbf{X}'\boldsymbol{u},$$

i.e.,

$$\mathcal{M}(\mathbf{X}', \mathbf{B}') = \mathcal{M}(\mathbf{M}_{\mathbf{B}'}\mathbf{X}', \mathbf{B}').$$

Let $\mathbf{X}'(\Sigma + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}\mathbf{X} = \mathbf{J}\mathbf{J}'$; then

$$\begin{aligned}\mathcal{M}(\mathbf{X}', \mathbf{B}') &= \mathcal{M}(\mathbf{M}_{\mathbf{B}'}\mathbf{X}', \mathbf{B}') = \mathcal{M}\Big\{\mathbf{M}_{\mathbf{B}'}\big[\mathbf{X}'(\Sigma + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}\mathbf{X}\big]\mathbf{M}_{\mathbf{B}'}, \mathbf{B}'\mathbf{B}\Big\}\\ &= \mathcal{M}(\mathbf{M}_{\mathbf{B}'}\mathbf{J}, \mathbf{B}') = \mathcal{M}(\mathbf{J}, \mathbf{B}') = \mathcal{M}(\mathbf{J}\mathbf{J}' + \mathbf{B}'\mathbf{B})\\ &= \mathcal{M}\big[\mathbf{X}'(\Sigma + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}\mathbf{X} + \mathbf{B}'\mathbf{B}\big].\end{aligned}$$

□

To solve the problems given in the previous section, it is necessary to know the expressions for $\boldsymbol{L}_{\boldsymbol{h}}'$, $\operatorname{Var}(\boldsymbol{v}^* \mid \vartheta^*)$, $\operatorname{Var}\big[\widehat{\boldsymbol{h}'\beta}(\boldsymbol{Y}, \vartheta^*) \mid \vartheta^*\big]$ and for the matrix $\mathbf{A}_{i,j}$, $i,j = 1,\dots,p$, in the model from Definition 3.1.

These expressions are given in the following sequences of statements.

Here

$$\mathbf{C} = \mathbf{X}'\Sigma^{*+}\mathbf{X}, \qquad \mathbf{W} = \mathbf{X}'(\Sigma^* + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}\mathbf{X} + \mathbf{B}'\mathbf{B}.$$

LEMMA 3.4. *The ϑ^*-LBLUE of a function $h(\beta) = \boldsymbol{h}'\beta$, $\beta \in \mathcal{V}_{\mathbf{I}}$, where $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$, is*

$$\widehat{\boldsymbol{h}'\beta}(\boldsymbol{Y}, -\boldsymbol{b}, \vartheta^*) = \boldsymbol{L}_{\boldsymbol{h}}'\begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{b}\end{pmatrix} = \Big[(\boldsymbol{L}_{\boldsymbol{h}}^{(1)})', (\boldsymbol{L}_{\boldsymbol{h}}^{(2)})'\Big]\begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{b}\end{pmatrix},$$

where

$$\Big[(\boldsymbol{L}_{\boldsymbol{h}}^{(1)})', (\boldsymbol{L}_{\boldsymbol{h}}^{(2)})'\Big] =$$

$$= \begin{cases} \Big(\boldsymbol{h}'\mathbf{P}^{\mathbf{C}}_{\mathrm{Ker}(\mathbf{B})}\mathbf{C}^{+}\mathbf{X}'\Sigma^{*-}, \boldsymbol{h}'\mathbf{C}^{+}\mathbf{B}'(\mathbf{B}\mathbf{C}^{+}\mathbf{B}')^{+}\Big) \\ \textit{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\Sigma^*) \textit{ and } \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C}), \\ \Big(\boldsymbol{h}'\mathbf{P}^{\mathbf{C}+\mathbf{B}'\mathbf{B}}_{\mathrm{Ker}(\mathbf{B})}(\mathbf{C} + \mathbf{B}'\mathbf{B})^{+}\mathbf{X}'\Sigma^{*-}, \boldsymbol{h}'(\mathbf{C} + \mathbf{B}'\mathbf{B})^{+}\mathbf{B}'\big[\mathbf{B}(\mathbf{C} + \mathbf{B}'\mathbf{B})^{+}\mathbf{B}'\big]^{+}\Big) \\ \textit{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\Sigma^*), \\ \Big(\boldsymbol{h}'\mathbf{P}^{\mathbf{W}}_{\mathrm{Ker}(\mathbf{B})}\mathbf{W}^{+}\mathbf{X}'(\Sigma^* + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}, \boldsymbol{h}'\mathbf{W}^{+}\mathbf{B}'(\mathbf{B}\mathbf{W}^{+}\mathbf{B}')^{+}\Big) \quad \textit{otherwis}. \end{cases}$$

Here $\mathrm{Ker}(\mathbf{B}) = \{\boldsymbol{u} : \mathbf{B}\boldsymbol{u} = \boldsymbol{0}\}$ *and* $\mathbf{P}^{\mathbf{C}}_{\mathrm{Ker}(\mathbf{B})} = \mathbf{I} - \mathbf{C}^{+}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-}\mathbf{B}')^{-}\mathbf{B}$*; an analogous meaning has the symbol* $\mathbf{P}^{\mathbf{W}}_{\mathrm{Ker}(\mathbf{B})}$.

P r o o f . Let the last case be considered only. Then with respect to Lemma 3.2

$$
\begin{aligned}
\widehat{\boldsymbol{h}'\boldsymbol{\beta}} &= \boldsymbol{h}'\boldsymbol{\beta}_0 + \boldsymbol{h}'\mathbf{K}_{\mathbf{B}}\big[(\mathbf{K}'_{\mathbf{B}}\mathbf{X}')^{-}_{m(\boldsymbol{\Sigma}^*)}\big]'(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0) \\
&= \boldsymbol{h}'\boldsymbol{\beta}_0 + \boldsymbol{h}'\mathbf{M}_{\mathbf{B}'}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{X}'(\boldsymbol{\Sigma}^* + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{-}\mathbf{X}\mathbf{M}_{\mathbf{B}'}\big]^{-}\times \\
&\quad \times \mathbf{M}_{\mathbf{B}'}\mathbf{X}'(\boldsymbol{\Sigma}^* + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{-}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0) \\
&= \boldsymbol{h}'\boldsymbol{\beta}_0 + \boldsymbol{h}'(\mathbf{M}_{\mathbf{B}'}\mathbf{W}\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'(\boldsymbol{\Sigma}^* + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0) \\
&= \boldsymbol{h}'\boldsymbol{\beta}_0 + \boldsymbol{h}'\{\mathbf{W}^{+} - \mathbf{W}^{+}\mathbf{B}'(\mathbf{B}\mathbf{W}^{+}\mathbf{B}')^{+}\mathbf{B}\mathbf{W}^{+}\}\times \\
&\quad \times \mathbf{X}'(\boldsymbol{\Sigma}^* + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0) \\
&= \boldsymbol{h}'\mathbf{P}^{\mathbf{W}}_{\mathrm{Ker}(\mathbf{B})}\mathbf{W}^{+}\mathbf{X}'(\boldsymbol{\Sigma}^* + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}\boldsymbol{Y} - \boldsymbol{h}'\mathbf{W}^{+}\mathbf{B}'\big[\mathbf{B}\mathbf{W}^{+}\mathbf{B}'\big]^{+}\boldsymbol{b}.
\end{aligned}
$$

Analogously other cases can be proved. □

COROLLARY 3.5. *One version of the minimum* $\begin{pmatrix}\boldsymbol{\Sigma}^*, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}$*-seminorm g-inverse of the matrix* $(\mathbf{X}', \mathbf{B}')$ *is given by the relation*

$$
\left((\mathbf{X}', \mathbf{B}')^{-}_{m\begin{pmatrix}\boldsymbol{\Sigma}^*, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}}\right)' =
$$

$$
= \begin{cases}
\left(\mathbf{P}^{\mathbf{C}}_{\mathrm{Ker}(\mathbf{B})}\mathbf{C}^{+}\mathbf{X}'\boldsymbol{\Sigma}^{*-}, \mathbf{C}^{+}\mathbf{B}'(\mathbf{B}\mathbf{C}^{+}\mathbf{B}')^{+}\right) \\
\text{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\boldsymbol{\Sigma}^*) \text{ and } \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C}), \\
\left(\mathbf{P}^{\mathbf{C}+\mathbf{B}'\mathbf{B}}_{\mathrm{Ker}(\mathbf{B})}(\mathbf{C} + \mathbf{B}'\mathbf{B})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{*-}, (\mathbf{C} + \mathbf{B}'\mathbf{B})^{+}\mathbf{B}'\big[\mathbf{B}(\mathbf{C} + \mathbf{B}'\mathbf{B})^{+}\mathbf{B}'\big]^{+}\right) \\
\text{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\boldsymbol{\Sigma}^*), \\
\left(\mathbf{P}^{\mathbf{W}}_{\mathrm{Ker}(\mathbf{B})}\mathbf{W}^{+}\mathbf{X}'(\boldsymbol{\Sigma}^* + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^{+}, \mathbf{W}^{+}\mathbf{B}'(\mathbf{B}\mathbf{W}^{+}\mathbf{B}')^{+}\right) \quad \text{otherwise.}
\end{cases}
$$

LEMMA 3.6. *In the model from Definition* 3.1

$$
\boldsymbol{v}^* = \begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{b}\end{pmatrix} - \begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\left[(\mathbf{X}', \mathbf{B}')^{-}_{m\begin{pmatrix}\boldsymbol{\Sigma}^*, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}}\right]'\begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{b}\end{pmatrix} = \begin{pmatrix}\boldsymbol{v}_1^*\\ \boldsymbol{0}\end{pmatrix}.
$$

Further

$$\operatorname{Var}(\mathbf{v}_1^*|\boldsymbol{\vartheta}^*) = \begin{cases} \Sigma^* - \mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^+\mathbf{X}' = \Sigma^* - \mathbf{X}\big[\mathbf{C}^+ - \mathbf{C}^+\mathbf{B}'(\mathbf{B}\mathbf{C}^-\mathbf{B}')^-\mathbf{B}\mathbf{C}^+\big]\mathbf{X}' \\ \quad \textit{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\Sigma^*) \textit{ and } \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C}), \\ \Sigma^* - \mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^+\mathbf{X}' \\ \quad = \Sigma^* - \mathbf{X}\Big\{(\mathbf{C}+\mathbf{B}'\mathbf{B})^+ + (\mathbf{C}+\mathbf{B}'\mathbf{B})^+\mathbf{B}'\big[\mathbf{B}(\mathbf{C}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'\big]^- \times \\ \qquad \times \mathbf{B}(\mathbf{C}+\mathbf{B}'\mathbf{B})^+\Big\}\mathbf{X}' \quad \textit{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\Sigma^*), \\ \Sigma^* - \mathbf{X}\big[\mathbf{W}^+ - \mathbf{W}^+\mathbf{B}'(\mathbf{B}\mathbf{W}^+\mathbf{B})^+\mathbf{B}'\mathbf{W}^+\big]\mathbf{X}' + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}' \quad \textit{otherwise.} \end{cases}$$

P r o o f . It is obvious, however, in the general case, Lemma 3.3 must be taken into account. □

LEMMA 3.7. *Let $\mathbf{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$; then in the model from Definition* 3.1

$$\operatorname{Var}\Big[\widehat{\mathbf{h}'\boldsymbol{\beta}}(\mathbf{Y}, -\mathbf{b}, \boldsymbol{\vartheta}^*)|\boldsymbol{\vartheta}^*\Big] = \begin{cases} \mathbf{h}'(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^+\mathbf{h} \\ \textit{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\Sigma^*) \textit{ and } \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C}), \\ \mathbf{h}'\big[\mathbf{M}_{\mathbf{B}'}(\mathbf{C}+\mathbf{B}'\mathbf{B})\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{h} \quad \textit{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\Sigma^*), \\ \mathbf{h}'(\mathbf{M}_{\mathbf{B}'}\mathbf{W}\mathbf{M}_{\mathbf{B}'})^+\mathbf{h} - \mathbf{h}'\mathbf{M}_{\mathbf{B}'}\mathbf{h}, \quad \textit{otherwise.} \end{cases}$$

LEMMA 3.8. *Let $\mathbf{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$; then in the model from Definition* 3.1

$$\partial \mathbf{h}'\left[(\mathbf{X}', \mathbf{B}')^-_{m\begin{pmatrix}\Sigma, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}}\right]'\begin{pmatrix}\mathbf{Y}\\ -\mathbf{b}\end{pmatrix}\Big/\partial\vartheta_i\bigg|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}^*} = (\mathbf{L}_{\mathbf{h}}^{(1)})'\mathbf{V}_i\Sigma^{*-}\mathbf{v}_1^*\,.$$

P r o o f . It is implied by Theorem 2.7, Lemma 3.4, Corollary 3.5. and Lemma 3.6. □

LEMMA 3.9. *The matrix $\mathbf{A}_{i,j}$, $i, j = 1, \dots, p$, in the case of the model from Definition* 3.1 *is*

$$\mathbf{A}_{i,j} = \mathbf{P}^{(\cdot)}_{\mathrm{Ker}(\mathbf{B})}(\cdot)^+\mathbf{X}'\mathbf{V}_i\Sigma^{*-}\operatorname{Var}(\mathbf{v}_1^* \mid \boldsymbol{\vartheta}^*)\Sigma^{*-}\mathbf{V}_j\mathbf{X}\mathbf{P}^{(\cdot)}_{\mathrm{Ker}(\mathbf{B})}(\cdot)^+\,,$$

where

$$\mathbf{P}^{(\cdot)}_{\mathrm{Ker}(\mathbf{B})}(\cdot)^+ = \begin{cases} \mathbf{C}^+ - \mathbf{C}^+\mathbf{B}'(\mathbf{B}\mathbf{C}^-\mathbf{B}')^-\mathbf{B}\mathbf{C}^+ \\ \textit{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\Sigma^*) \textit{ and } \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C}), \\ (\mathbf{C}+\mathbf{B}'\mathbf{B})^+ - (\mathbf{C}+\mathbf{B}'\mathbf{B})^+\mathbf{B}'\big[\mathbf{B}(\mathbf{C}+\mathbf{B}'\mathbf{B})^+\mathbf{B}'\big]^+ \times \\ \qquad \times \mathbf{B}'(\mathbf{C}+\mathbf{B}'\mathbf{B})^+ \quad \textit{if } \mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'}) \subset \mathcal{M}(\Sigma^*) \\ \mathbf{W}^+ - \mathbf{W}^+\mathbf{B}'(\mathbf{B}\mathbf{W}^+\mathbf{B}')^+\mathbf{B}\mathbf{W}^+ \quad \textit{otherwise.} \end{cases}$$

This sequence of statements enables us to use Theorem 2.7 for the model with constraints of the type I.

Remark 3.10. Since it is easy to obtain the other nonsensitivity regions for this model which are analogous to the regions given in the preceding section, the formulae in this section are omitted.

4. Universal model with constraints of the type II

DEFINITION 4.1. The *universal model with constraints of the type II* is

$$\boldsymbol{Y} \sim_n \left(\mathbf{X}\boldsymbol{\beta}_1, \Sigma(\boldsymbol{\vartheta})\right), \qquad \boldsymbol{\vartheta} \in \underline{\vartheta},$$

$$\boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \left\{ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{z} \end{pmatrix} : \ \boldsymbol{u} \in \mathbb{R}^k, \ \boldsymbol{z} \in \mathbb{R}^l, \ \boldsymbol{b} + \mathbf{B}_1\boldsymbol{u} + \mathbf{B}_2\boldsymbol{z} = \boldsymbol{0} \right\} = \mathcal{V}_{\mathrm{II}}.$$

There are two equivalent expressions of this model

(i) *Model without restrictions*

$$\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_{1,0} \sim_n \left(\mathbf{X}\mathbf{K}_1\gamma, \Sigma(\boldsymbol{\vartheta})\right), \qquad \gamma \in \mathbb{R}^{k+l-r(\mathbf{B}_1, \mathbf{B}_2)}, \ \boldsymbol{\vartheta} \in \underline{\vartheta},$$

where $\mathrm{Ker}(\mathbf{B}_1, \mathbf{B}_2) = \mathcal{M}\left[\begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}\right]$, the matrix $\begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}$ is of the full rank in columns and $\begin{pmatrix} \beta_{1,0} \\ \beta_{2,0} \end{pmatrix}$ is any vector satisfying the equality $\boldsymbol{b} + \mathbf{B}_1\boldsymbol{\beta}_{1,0} + \mathbf{B}_2\boldsymbol{\beta}_{2,0} = \boldsymbol{0}$.

(ii)

$$\begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix} \sim_{n+q} \left(\begin{pmatrix} \mathbf{X}, & 0 \\ \mathbf{B}_1, & \mathbf{B}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \begin{pmatrix} \Sigma, & 0 \\ 0, & 0 \end{pmatrix} \right),$$
$$\begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \mathcal{V}_{\mathrm{II}}, \quad \boldsymbol{\vartheta} \in \underline{\vartheta}.$$

LEMMA 4.2. *We have*

$$\mathcal{M}(\mathbf{K}_1) = \mathcal{M}(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}).$$

P r o o f. Since

$$\mathbf{B}_1\mathbf{K}_1 + \mathbf{B}_2\mathbf{K}_2 = \mathbf{0} \implies \mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{K}_1 = \mathbf{0},$$

obviously $\mathcal{M}(\mathbf{K}_1) \subset \mathcal{M}(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}})$.

From the other side

$$(\mathbf{0} \neq)\ \mathbf{x} \in \mathcal{M}(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}) \iff \mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{x} = \mathbf{0} \iff \exists\{\mathbf{z} \in \mathbb{R}^k\}(\mathbf{B}_1\mathbf{x} = -\mathbf{B}_2\mathbf{z})$$

i.e.

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{z} \end{pmatrix} \in \mathcal{M}\left[\begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}\right] \qquad \text{thus} \quad \mathcal{M}(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}) \subset \mathcal{M}(\mathbf{K}_1).$$

□

LEMMA 4.3. *We have*

$$\mathcal{M}(\mathbf{B}_1, \mathbf{B}_2) = \mathcal{M}(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1, \mathbf{B}_2).$$

P r o o f . Obviously $\mathcal{M}(\mathbf{B}_1) \subset \mathcal{M}(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1, \mathbf{B}_2)$, since

$$\forall\{\mathbf{u} \in \mathbb{R}^k\}\Big(\mathbf{B}_1\mathbf{u} = (\mathbf{P}_{\mathbf{B}_2} + \mathbf{M}_{\mathbf{B}_2})\mathbf{B}_1\mathbf{u} = \mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{u} + \mathbf{B}_2(\mathbf{B}_2'\mathbf{B}_2)^-\mathbf{B}_2'\mathbf{B}_1\mathbf{u} = \mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{z}\Big),$$

where $\mathbf{z} = (\mathbf{B}_2'\mathbf{B}_2)^-\mathbf{B}_2'\mathbf{B}_1\mathbf{u}$.

Further $\mathcal{M}(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1) \subset \mathcal{M}(\mathbf{B}_1, \mathbf{B}_2)$, since

$$\forall\{\mathbf{u} \in \mathbb{R}^k\}(\mathbf{M}_{\mathbf{B}_2}\mathbf{u} = \mathbf{B}_1\mathbf{u} - \mathbf{P}_{\mathbf{B}_2}\mathbf{u} = \mathbf{B}_1\mathbf{u} + \mathbf{B}_2(\mathbf{B}_2'\mathbf{B}_2)^-\mathbf{B}_2'(-\mathbf{u})).$$

□

LEMMA 4.4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *p.s.d. matrices. Then* $\mathcal{M}(\mathbf{A}, \mathbf{B}) = \mathcal{M}(\mathbf{A} + \mathbf{B})$.

P r o o f . Cf. [8; p. 120]. □

LEMMA 4.5. *Let*

$$\mathbf{W} = \mathbf{X}'(\Sigma^* + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^+\mathbf{X} + \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1.$$

Then $\mathcal{M}(\mathbf{X}') \subset \mathcal{M}(\mathbf{W})$.

P r o o f . With respect to Lemmas 4.3. and 4.4 (cf. also Lemma 3.3)

$$\mathcal{M}(\mathbf{W}) = \mathcal{M}[\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'(\Sigma^* + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}} + \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1] = \mathcal{M}(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}', \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}).$$

Since

$$\mathbf{X}' = (\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}} + \mathbf{P}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}})\mathbf{X}' = \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}' + \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^-\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{X} = \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}' + \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{Z},$$

where $\mathbf{Z} = (\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^-\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{X}$, it is valid $\mathcal{M}(\mathbf{X}') \subset \mathcal{M}(\mathbf{W})$. □

THEOREM 4.6. *One version of the minimum* $\begin{pmatrix}\Sigma^*, & 0\\ 0, & 0\end{pmatrix}$*-seminorm g-inverse of the matrix* $\begin{pmatrix}X', & B_1'\\ 0, & B_2'\end{pmatrix}$ *is given by the relation*

$$\left[\begin{pmatrix}X', & B_1'\\ 0, & B_2'\end{pmatrix}^-_{m\begin{pmatrix}\Sigma^*, & 0\\ 0, & 0\end{pmatrix}}\right]' = \begin{pmatrix}\boxed{1}, & \boxed{2}\\ \boxed{3}, & \boxed{4}\end{pmatrix},$$

where

$$\boxed{1} = M_{B_1'M_{B_2}}\left[(M_{B_1'M_{B_2}}X')^-_{m(\Sigma^*)}\right]',$$
$$\boxed{2} = W^+B_1'(M_{B_2}B_1W^+B_1'M_{B_2})^+,$$
$$\boxed{3} = -\left[(B_2')^-_{m(B_1W^+B_1')}\right]'B_1W^+X'(\Sigma^* + XM_{B_1'M_{B_2}}X')^+,$$
$$\boxed{4} = \left[(B_2')^-_{m(B_1W^+B_1')}\right]'.$$

Proof. Let $\begin{pmatrix}\beta_{1,0}\\ \beta_{2,0}\end{pmatrix}$ be any vector satisfying the equality $b + B_1\beta_{1,0} + B_2\beta_{2,0} = 0$.

With respect to Lemmas 2.4, 4.2 and 4.5 we have

$$\begin{aligned}
&\widehat{X\beta}_1(Y,-b,\vartheta^*)\\
&= X\beta_{1,0} + XK_1\left[(K_1'X')^-_{m(\Sigma^*)}\right]'(Y - X\beta_{1,0})\\
&= XM_{B_1'M_{B_2}}\left[(M_{B_1'M_{B_2}}X')^-_{m(\Sigma^*)}\right]'Y\\
&\quad + X\beta_{1,0} - XM_{B_1'M_{B_2}}\left[(M_{B_1'M_{B_2}}X')^-_{m(\Sigma^*)}\right]'X\beta_{1,0}\\
&= X\boxed{1}Y + X\beta_{1,0}\\
&\quad - XM_{B_1'M_{B_2}}\left[M_{B_1'M_{B_2}}X'(\Sigma^* + XM_{B_1'M_{B_2}}X')^-XM_{B_1'M_{B_2}}\right]^+\times\\
&\quad \times M_{B_1'M_{B_2}}X'(\Sigma^* + XM_{B_1'M_{B_2}}X')^+X\beta_{1,0}\\
&= X\boxed{1}Y + X\beta_{1,0} - X\beta_{1,0}\\
&\quad + XW^+B_1'(M_{B_2}B_1W^+B_1'M_{B_2})^+M_{B_2}B_1\beta_{1,0}.
\end{aligned}$$

Since $B_1\beta_{1,0} = -B_2\beta_{2,0} - b$, we have

$$\widehat{X\beta}_1(Y,-b,\vartheta^*) = X\left(\boxed{1}, W^+B_1'(M_{B_2}B_1W^+B_1'M_{B_2})^+\right)\begin{pmatrix}Y\\ -b\end{pmatrix}.$$

Thus $\boxed{2} = W^+B_1'(M_{B_2}B_1W^+B_1'M_{B_2})^+$.

With respect to Lemma 2.3 the matrices $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ and $\boxed{4}$ must satisfy the equalities

$$\mathbf{X}\boxed{1}\mathbf{X} + \mathbf{X}\boxed{2}\mathbf{B}_1 = \mathbf{X}\,, \qquad \mathbf{X}\boxed{2}\mathbf{B}_2 = \mathbf{0}\,,$$

$$\mathbf{B}_1\boxed{1}\mathbf{X} + \mathbf{B}_2\boxed{3}\mathbf{X} + \mathbf{B}_1\boxed{2}\mathbf{B}_1 + \mathbf{B}_2\boxed{4}\mathbf{B}_1 = \mathbf{B}_1\,, \qquad \mathbf{B}_1\boxed{2}\mathbf{B}_2 + \mathbf{B}_2\boxed{4}\mathbf{B}_2 = \mathbf{B}_2\,,$$

$$\mathbf{X}\boxed{1}\Sigma = \Sigma\boxed{1}'\mathbf{X}'\,, \qquad (\mathbf{B}_1\boxed{1} + \mathbf{B}_2\boxed{3})\Sigma^* = \mathbf{0}\,.$$

It is obvious that $\boxed{4} = \mathbf{B}_2^-$. Let us choose $\boxed{4} = \big[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1')}\big]'$; then we can find out that a possible form of $\boxed{3}$ is $-\big[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1')}\big]'\mathbf{B}_1\mathbf{W}^+\mathbf{X}'(\Sigma^* + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^+$, however, it is necessary to check necessary and sufficient conditions. It is simple, however, tedious; thus as an example only the equality $\mathbf{B}_1\boxed{1}\mathbf{X} + \mathbf{B}_2\boxed{3}\mathbf{X} + \mathbf{B}_1\boxed{2}\mathbf{B}_1 + \mathbf{B}_2\boxed{4}\mathbf{B}_1 = \mathbf{B}_1$, is proved.

The notation $\mathbf{V} = \mathbf{B}_1\mathbf{W}^+\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'$ is used in the following.

$$\begin{aligned}
&\mathbf{B}_1\boxed{1}\mathbf{X} + \mathbf{B}_2\boxed{3}\mathbf{X} + \mathbf{B}_1\boxed{2}\mathbf{B}_1 + \mathbf{B}_2\boxed{4}\mathbf{B}_1\\
&= \mathbf{B}_1\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\big[(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X} - \mathbf{B}_2\big[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1')}\big]'\times\\
&\quad\times \mathbf{B}_1\mathbf{W}^+\mathbf{X}'(\Sigma^* + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^+\mathbf{X}\\
&\quad+ \mathbf{B}_1\mathbf{W}^+\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^+\mathbf{B}_1 + \mathbf{B}_2\big[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1')}\big]'\mathbf{B}_1\\
&= \mathbf{B}_1\big[\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'(\Sigma^* + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\big]^+\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\times\\
&\quad\times(\Sigma^* + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^+\mathbf{X} - \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1\mathbf{W}^+\mathbf{X}'\times\\
&\quad\times(\Sigma^* + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^+\mathbf{X}\\
&\quad+ \mathbf{B}_1\mathbf{W}^+\mathbf{B}_1'\big[\mathbf{V}^+ - \mathbf{V}^+\mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\big]\mathbf{B}_1\\
&\quad+ \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1\\
&= \mathbf{B}_1\big[\mathbf{W}^+ - \mathbf{W}^+\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^+\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{W}^+\big]\times\\
&\quad\times(\mathbf{W} - \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1) - \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1\mathbf{W}^+(\mathbf{W} - \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)\\
&\quad+ \mathbf{B}_1\mathbf{W}^+\mathbf{B}_1'\big[\mathbf{V}^+ - \mathbf{V}^+\mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\big]\mathbf{B}_1\\
&\quad+ \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1\\
&= \mathbf{B}_1\mathbf{W}^+\mathbf{W} - \mathbf{B}_1\mathbf{W}^+\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^+\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\\
&\quad- \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1\mathbf{W}^+\mathbf{W}\\
&\quad+ \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+(\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\\
&\quad+ \mathbf{V}\big[\mathbf{V}^+ - \mathbf{V}^+\mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\big]\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\\
&\quad+ \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1
\end{aligned}$$

$$= \mathbf{B}_1\mathbf{W}^+\mathbf{W} - \mathbf{B}_1\mathbf{W}^+\mathbf{B}_1'[\mathbf{V}^+ - \mathbf{V}^+\mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+]\mathbf{B}_1$$
$$- \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1\mathbf{W}^+\mathbf{W}$$
$$+ \mathbf{V}\mathbf{V}^+\mathbf{B}_1 - \mathbf{V}\mathbf{V}^+\mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1$$
$$+ \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1$$

$$= \mathbf{B}_1\mathbf{W}^+\mathbf{W} - \mathbf{V}[\mathbf{V}^+ - \mathbf{V}^+\mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+]\mathbf{B}_1$$
$$- \mathbf{B}_2(\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_2)^+\mathbf{B}_2'\mathbf{V}^+\mathbf{B}_1\mathbf{W}^+\mathbf{W} + \mathbf{V}\mathbf{V}^+\mathbf{B}_1$$

$$= \mathbf{B}_1\mathbf{W}^+\mathbf{W} + \mathbf{P}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{B}_1 - \mathbf{P}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{B}_1\mathbf{W}^+\mathbf{W} = \mathbf{P}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{B}_1 + \mathbf{M}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{B}_1\mathbf{W}^+\mathbf{W}$$

$$= \mathbf{P}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{B}_1 + \mathbf{M}_{\mathbf{B}_2}^{\mathbf{V}^+}(\mathbf{P}_{\mathbf{B}_2} + \mathbf{M}_{\mathbf{B}_2})\mathbf{B}_1\mathbf{W}^+\mathbf{W}$$

$$= \mathbf{P}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{B}_1 + \mathbf{M}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{W}^+\mathbf{W}$$

$$= \mathbf{P}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{B}_1 + \mathbf{M}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1 = \mathbf{P}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{B}_1 + \mathbf{M}_{\mathbf{B}_2}^{\mathbf{V}^+}\mathbf{B}_1 = \mathbf{B}_1 .$$

In an analogous way the other equalities can be proved. □

For any unbiasedly estimable function $h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \boldsymbol{h}_1'\boldsymbol{\beta}_1 + \boldsymbol{h}_2'\boldsymbol{\beta}_2$, $\binom{\boldsymbol{\beta}_1}{\boldsymbol{\beta}_2} \in \mathcal{V}_{\mathrm{II}}$, the nonsensitiveness region is given by Theorem 2.7, where the quantities $\mathbf{W}_{i,j}$, $i,j = 1,\dots,p$, $\boldsymbol{L}_{\boldsymbol{h}}'$, $\boldsymbol{v}^*$, $\operatorname{Var}(\boldsymbol{v}^* \mid \boldsymbol{\vartheta}^*)$ and $\operatorname{Var}[\widehat{h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)}(\boldsymbol{Y}, -\boldsymbol{b}, \boldsymbol{\vartheta}^*) \mid \boldsymbol{\vartheta}^*]$ must be expressed in terms of the model with constraints of the type II.

These expressions are

$$\boldsymbol{L}_{\boldsymbol{h}}' = \left((\boldsymbol{L}_{\boldsymbol{h}}^{(1)})', (\boldsymbol{L}_{\boldsymbol{h}}^{(2)})'\right),$$
$$(\boldsymbol{L}_{\boldsymbol{h}}^{(1)})' = \boldsymbol{h}_1'\boxed{1} + \boldsymbol{h}_2'\boxed{3},$$
$$(\boldsymbol{L}_{\boldsymbol{h}}^{(2)})' = \boldsymbol{h}_1'\boxed{2} + \boldsymbol{h}_2'\boxed{4},$$
$$\boldsymbol{v}^* = \begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix} - \begin{pmatrix} \mathbf{X}, & \boldsymbol{0} \\ \mathbf{B}_1, & \mathbf{B}_2 \end{pmatrix} \begin{pmatrix} \boxed{1}, & \boxed{2} \\ \boxed{3}, & \boxed{4} \end{pmatrix} \begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_1^* \\ \boldsymbol{0} \end{pmatrix},$$
$$\{\mathbf{W}_{\boldsymbol{h}}\}_{i,j} = (\boldsymbol{L}_{\boldsymbol{h}}^{(1)})'\mathbf{V}_i\Sigma^{*-}\operatorname{Var}(\boldsymbol{v}_1^* \mid \boldsymbol{\vartheta}^*)\Sigma^{*-}\mathbf{V}_j\boldsymbol{L}_{\boldsymbol{h}}^{(1)},$$
$$\boldsymbol{v}_1^* = \boldsymbol{Y} - \mathbf{X}\boxed{1}\boldsymbol{Y} - \mathbf{X}\boxed{2}(-\boldsymbol{b}),$$
$$\operatorname{Var}(\boldsymbol{v}_1^* \mid \boldsymbol{\vartheta}^*) = (\mathbf{I} - \mathbf{X}\boxed{1})\Sigma^*(\mathbf{I} - \mathbf{X}\boxed{1})'$$

and

$$\operatorname{Var}(\widehat{h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)}(\boldsymbol{Y}, -\boldsymbol{b}, \boldsymbol{\vartheta}^*) \mid \boldsymbol{\vartheta}^*)$$
$$= (\boldsymbol{h}_1', \boldsymbol{h}_2') \begin{pmatrix} \boxed{1}, & \boxed{2} \\ \boxed{3}, & \boxed{4} \end{pmatrix} \begin{pmatrix} \Sigma^*, & 0 \\ 0, & 0 \end{pmatrix} \begin{pmatrix} \boxed{1}', & \boxed{3}' \\ \boxed{2}', & \boxed{4}' \end{pmatrix} \begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \end{pmatrix}.$$

It is obvious how to proceed in the case of the other mentioned kinds of nonsensitiveness regions.

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