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ON CONNECTIVITY POINTS OF DARBOUX FUNCTIONS

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1. Introduction.

In articles [1] and [2] the notions of Darboux points and connectivity points of a real function of a real variable were introduced and considered. In those articles the following theorems were proved:

Theorem A. *A function $f\colon R \to R$ is a Darboux function if and only if it is Darboux at every point of the set of all real numbers.*

Theorem B. *A function $f\colon R \to$ is connected (i.e. it has a connected graph if and only if it is connected at evry point of its domain.*

It was proved in [4] that the set of all Darboux points (and also the set of all connectivity points) of a function $f\colon R \to R$ is of type G_δ. It follows immediately from the definitions that the set $\mathscr{C}\text{ted}(f)$ of all connectivity points of a function $f\colon R \to R$ is contained in the set $\mathscr{D}(f)$ of all Darboux points of the function f; moreover, both these sets contain the set $\mathscr{C}(f)$ of all points of continuity of the function f. J. S. Lipiński in [3] proved that for two arbitrary sets A, B of the type G_δ and such that $A \subset B$ there exists a function $f\colon R \to R$ for which $\mathscr{C}\text{ted}(f) = A$ and $\mathscr{D}(f) = B$. Next, L. Snoha in [5] constructed a function nowhere continuous, which is connected in a given set of type G_δ. Simultaneously, he posed the problem: characterize the set of all connectivity points of a function which has Darboux property. The present paper gives the answer to this problem.

2. Preliminaries.

In the work we shall use the following notions, denotations and theorems. Sometimes we shall identify a function with its graph (as a subset of R^2). $L(f, x)$, $L^+(f, x)$ and $L^-(f, x)$ will denote the set of all limit numbers of f at the point x, the set of all right-sided limit numbers and the set of all left-sided limit numbers, respectively. By card (A) we shall denote the cardinal of the set A. For a set $M \subset R^2$ $\text{proj}_x M$, $\text{proj}_y M$ will denote the projections of M onto the x-axis or the y-axis, respectively.

$$P(x_0) = \{(x_0, y) | y \in R\}, \qquad \operatorname{osc}(f, E) = \sup_{x,y \in E} |f(x) - f(y)| .$$

A set A is called $\mathfrak{c}$-dense in B if for any point x of the set B and an arbitrary neighbourhood U of x the set $A \cap U$ is of the power of continuum.

Definition 1 ([2]). If $f: R \to R$ *and* $x_0 \in R$, *then we say that* x_0 *is a right-sided connectivity point of the function* f (*or* f *is connected from the right side at* x_0) *if*

(1) $f(x_0) \in L^+(f, x_0)$,

(2) *if* $a, b \in L^+(f, x_0)$ *and* M *is an arbitrary continuum such that* $\operatorname{proj}_y M \subset \subset (a, b)$, $\operatorname{proj}_x M = [x_0, x_0 + \varepsilon]$ *for some* $\varepsilon > 0$, *then* $M \cap f \neq \emptyset$.

In an analogous way we define the left-sided connectivity points of a function. A point x is a connectivity point of a function if it is a left-sided and a right-sided connectivity point of the function.

By $\mathscr{C}\operatorname{ted}(f)$, $\mathscr{C}\operatorname{ted}^+(f)$, $\mathscr{C}\operatorname{ted}^-(f)$ we shall denote the set of all connectivity points of the function f, the set of all right-sided connectivity points of f and the set of all left-sided connectivity points of f.

Definition 2 ([1]). *Let* $f: R \to R$ *be an arbitrary function. A point* $x_0 \in R$ *is called a right-sided Darboux point of the function* f (*or* f *has the Darboux property from the right-side*) *if the condition* (1) *is sulfilled and*

(3) *if* $a, b \in L^+(f, x_0)$, $a < b$ *and* $c \in (a, b)$, *then for an arbitrary positive number* ε *there exists a point* $t \in (x_0, x_0 + \varepsilon)$ *such that* $f(t) = c$.

The set of all right-sided Darboux points of a function f will be denoted by $\mathscr{D}^+(f)$.

Analogously, $\mathscr{D}^-(f)$ denotes the set of all left-sided Darboux points (defined analogously) of a function f. Moreover, $\mathscr{D}(f) = \mathscr{D}^+(f) \cap \mathscr{D}^-(f)$.

Theorem C ([1]). *The function* $f: R \to R$ *has the Darboux property if and only if* $\mathscr{D}(f) = R$.

Theorem D ([2]). The function $f: R \to R$ is connected if and only if $\mathscr{C}\operatorname{ted}(f) = R$.

By A^c we shall denote the set of all condensation points of a set A (a point x belongs to A^c: if for every neighbourhood U of x, $\operatorname{card}(U \cap A) = \mathfrak{c}$). As usual, ϱ will denote the Euclidean metric in R or R^2.

3. Necessary condition.

Theorem 1. *For every function* $f: R \to R$ *with the Darboux property the set of all nonconnectivity points of* f *is empty or is dense in itself, a set of type* F_σ.

Proof. We know from [4] that the set of nonconnectivity points of an arbitrary function is of type F_σ. Now suppose that the set of nonconnectivity points of a Darboux function f is not dense in itself (and nonempty). Then there exists a point $x_0 \in R$ and a non-empty interval (a, b) such that

$$(a, b) \cap (R \setminus \mathscr{C}\,\mathrm{ted}\,(f)) = \{x_0\}.$$

Assume that x_0 is a right-sided nonconnectivity point of f. The function f has the Darboux property, hence $f(x_0) \in L^+(f, x_0)$. Thus there exist a number $\delta > 0$ and continuum $M \subset R^2$ such that

$$\mathrm{proj}_x M = [x_0, x_0 + \delta], \qquad x_0 + \delta \leqq b$$

and

$$\emptyset \neq M \cap P(x_0) \subset \{x_0\} \times \mathrm{int}\, L^+(f, x_0), \qquad f \cap M = \emptyset.$$

We may assume that

$$M \subset [x_0, x_0 + \delta] \times [m_1, m_2],$$

where $m_1, m_2 \in L^+(f, x_0)$, $m_1 < m_2$. Then there exist points $c, d \in [x_0, x_0 + \delta]$ such that $c < d, f(c) < m_1, f(d) > m_2$. The function $f | [c, d]$ is connected ([2]). The set

$$M_1 = \{([c, d] \times [m_1, m_2]) \cap M\} \cup \{\{c, d\} \times [m_1, m_2]\}$$

is a continuum with the projection onto the x-axis equalled to $[c, d]$. This continuum fulfils all requirements of the Definition 1 for x_0 but $M \cap f = \emptyset$. The contradiction ends the proof.

4. Sufficient condition.

Before we prove the sufficient condition we shall give some useful lemmas.

Lemma 1. *Every $\mathfrak{c}$-dense in itself set of type F_σ is a countable union of $\mathfrak{c}$-dense in itself closed sets.*

Proof. Let D be an F_σ set, $\mathfrak{c}$-dense in itself. Then

$$D = \bigcup_{n=1}^{\infty} B_n,$$

where

$$B_n \ (n = 1, 2, \ldots) \text{ are closed sets.}$$

The set $\bigcup_{n=1}^{\infty} (B_n \setminus B_n^c)$ is countable; let then

$$\bigcup_{n=1}^{\infty} (B_n \setminus B_n^c) = \bigcup_{n=1}^{\infty} \{x_n\}.$$

Let $U_{1,n}$ be an open neighbourhood of the point x_n for $n = 1, 2, \ldots$. Since $x_n \in D^c$, then there exists $k_{1,n}$ such that

$$\mathrm{card}\,(B_{k_{1,n}}^c \cap U_{1,n}) = \mathfrak{c}.$$

Now let $I_{1,n}$ be any closed interval contained in $U_{1,n}$ which fulfils the following condition

$$\operatorname{card}(B^c_{k_{1,n}} \cap I_{1,n}) = \mathfrak{c},$$
$$\varrho(x_n, I_{1,n}) > 0.$$

Suppose that we have chosen the sets $U_{m,n}$, $B_{k_{m,n}}$ and intervals $I_{m,n} \subset U_{m,n}$ such that

$$\operatorname{card}(B_{k_{m,n}} \cap I_{m,n}) = \mathfrak{c}$$
$$\varrho(x_n, I_{m,n}) > 0.$$

Now let $U_{m+1,n}$ be any open neigbourhood of the point x_n and disjoint with intervals $I_{1,n}, \ldots, I_{m,n}$. Then there exists $k_{m+1,n}$ with

$$\operatorname{card}(B_{k_{m+1,n}} \cap U_{m+1,n}) = \mathfrak{c}.$$

Let $I_{m+1,n}$ be any nondegenerated closed interval contained in $U_{m+1,n}$ such that

$$\operatorname{card}(B_{k_{m+1,n}} \cap I_{m+1,n}) = \mathfrak{c},$$
$$\varrho(x_n, I_{m+1,n}) > 0.$$

In this way we have defined for every point x_n a sequence of sets $(B_{k_{m,n}})_{m=1}^{\infty}$ and a sequence of intervals $(I_{m,n})_{m=1}^{\infty}$ so that

$$\lim_{m \to \infty} \varrho(x_n, I_{m,n}) = 0,$$
$$\operatorname{card}(B_{k_{m,n}} \cap I_{m,n}) = \mathfrak{c}.$$

Now let

$$C_n = \bigcup_{m=1}^{\infty} (B^c_{k_{m,n}} \cap I_{m,n}) \cup \{x_n\}.$$

It is easy to see that C_n is a closed set for which every of its points is a point of condensation of C_n. Let now

$$D_n = B^c_n \cup C_n.$$

One can prove that $D = \bigcup_{n=1}^{\infty} D_n$ and the sets D_n have all the required properties.

Lemma 2. *If $D \subset I = [0, 1]$ is an F_σ-set, $\mathfrak{c}$-dense in itself, then there exists a Darboux function $f: I \to R$ such that $\mathscr{C}\operatorname{ted}(f) = I \setminus D$.*

Proof. In view of Lemma 1, there exists a sequence D_n of closed sets $\mathfrak{c}$-dense in themselves and such that

$$D = \bigcup_{n=1}^{\infty} D_n .$$

We can assume that $D_n \subset D_{n+1}$, $n = 1, 2, \ldots$.
Let us denote by E_n the set of bilateral accumulation points of the set D_n. Thus each of the sets E_n is $\mathfrak{c}$-dense in itself and $\bar{E}_n = D_n$.

Now we shall prove that there exists a class $\{A_{n,\alpha}\}_{\substack{n \in N \\ \alpha < \Omega}}^{\infty}$ (where Ω denotes, as usually, the least uncountable ordinal) of sets fulfilling the following conditions:

$$A_{n,\alpha} \cap A_{m,\beta} = \emptyset , \qquad \text{for } (n, \alpha) \neq (m, \beta),\ n, m \in N,\ \alpha, \beta < \Omega$$
$$A_{n,\alpha} \subset E_n ,\ \bar{A}_{n,\alpha} = D_n ,\ n \in N ,\ \alpha < \Omega .$$

Let $(I_k)_{k=1}^{\infty}$ denote a sequence of all intervals whose ends are rational numbers and $I_k \subset I$. Let $A_{11} \subset E_1$ be any countable set which has common points with every nonvoid set $I_k \cap E_1$, $k = 1, 2, \ldots$. Assume now that we have chosen the sets $A_{11}, A_{21}, \ldots, A_{n1}$ such that $A_{i1} \subset E_i$, $\bar{A}_{i1} = D_i$, $A_{i,1} \cap A_{j1} = \emptyset$, for $i, j = 1, \ldots, n$, $i \neq j$ and such that A_{i1} is a countable set.

Now let $A_{n+1,1} \subset E_{n+1} \setminus \bigcup_{i=1}^{n} A_{i,1}$ be a countable set which has common points with every nonvoid set

$$I_k \cap E_{n+1} , \qquad k = 1, 2, \ldots .$$

Since E_{n+1} is a $\mathfrak{c}$-dense in itself set, then $\bar{A}_{n+1,1} = D_{n+1}$.

Assume now that for an ordinal $\alpha < \Omega$ we have chosen countable sets $A_{n,\beta}$ for $n \in N$, $\beta < \alpha$ such that

$$A_{n,\beta} \cap A_{n',\beta'} = \emptyset \quad \text{for } (n, \beta) \neq (n', \beta'),\ n, n' \in N,\ \beta, \beta' < \alpha ,$$
$$A_{n,\beta} \subset E_n ,\ \bar{A}_{n,\beta} = D_n ,\ n \in N ,\ \beta < \alpha .$$

Let $A_{1,\alpha} \subset E_1 \setminus \bigcup_{n \in N} \bigcup_{\beta < \alpha} A_{n,\beta}$ be a countable set which has common points with every nonvoid set $(I_k \cap E_1)$ for $k = 1, 2, ..$ The set E_1 is $\mathfrak{c}$-dense in itself and $\bigcup_{\beta < \alpha} \bigcup_{n=1}^{\infty} A_{n,\beta}$ is countable; then $\bar{A}_{1,\alpha} = D_1$.

Assume, finally, that we have chosen countable sets $A_{m,\beta}$, $m \in N$, $\beta < \alpha$ and $A_{1,\alpha}, \ldots, A_{n,\alpha}$ which have the adequate properties. Let $A_{n+1,\alpha} \subset$ $\subset E_{n+1} \setminus \left(\bigcup_{m \in N} \bigcup_{\beta < \alpha} A_{m,\beta} \cup \bigcup_{i=1}^{n} A_{i,\alpha} \right)$ be a coutable set with common points with every nonvoid set $(I_k \cap E_{n+1})$.

We have defined in this way a class $\{A_{n,\alpha}\}$ fulfilling all required condition.

Let $(M_\alpha)_{\alpha<\Omega}$ be a sequence of all continua contained in I^2 with nondegenerate projections on the x-axis. Let $g: I \to R$ be defined as follows

$$g(x) = \begin{cases} \min\{y \in R\colon (x, y) \in M_\alpha\} & \text{if } x \in A_{n,\alpha} \text{ and } \\ & \left\{y \in \left[\frac{1}{n+1}, \frac{1}{n}\right]: (x, y) \in M_\alpha\right\} \neq \emptyset, \\ 0 & \text{for the remaining } x \text{ from } I. \end{cases}$$

If $x_0 \notin D$, then x_0 is a point of continuity of g. If $x_0 \in D$, and $n_0 = \min\{n \in N | x_0 \in D_n\}$, then $L(g, x_0) = \left[0, \frac{1}{n_0}\right]$. From the definition of the function g it follows that the graph of g has common points with every continuum fulfilling the condition from Definition 1. Thus g is a connected function.

Now let us define a function $f: I \to R$ in the following way.

$$f(x) = \begin{cases} g(x) & \text{if for every } n \in N\ g(x) \neq \frac{1}{n(n+1)} \cdot \frac{x^2}{1+x^2} + \frac{1}{n+1}, \\ 0 & \text{for the remaining } x \text{ from } I. \end{cases}$$

Since for every $y > 0$

$$\{x \in I | f(x) = y\} = \{x \in I | g(x) = y\}$$

or there is one point $x_y \in I$ such that

$$\{x \in I | g(x) = y\} = \{x \in I | f(x) = y\} \cup \{x_y\},$$

then f has the Darboux property. Analogously, as previously, if $x_0 \notin D$, then x_0 is a point of continuity of f. If $x_0 \in D$, and $n_0 = \min\{n \in N | x_0 \in D_n\}$, then $L(f, x_0) = \left[0, \frac{1}{n_0}\right]$ and for example $L^+(f, x_0)$ (simultaneoulsy $L^-(f, x_0) = \left[0, \frac{1}{m}\right]$ for some $m > n_0$). Let n be an integer such that $n > n_0$ and

$$M = \{(x, y) \in R^2 | x \in [x_0, 1], \quad y = \frac{1}{n(n+1)} \cdot \frac{x^2}{1+x^2} + \frac{1}{n+1}.$$

M is a continuum fulfilling conditions in Definition 1 but M has no common point with the graph of f. Thus x_0 is not a connectivity point of f.

In this way we have proved that $\mathscr{C}\text{ted}(f) = I \setminus D$.

Lemma 3. *If $A \subset I$ is a countable set dense in I, then there exists an ascending sequence $(K_m)_{m=1}^\infty$ of perfect and nowhere dense sets such that*

$$K_m = I \setminus \left(\bigcup_{n=1}^{\infty} (a_n^{(m)}, c_n^{(m)}) \cup \bigcup_{n=1}^{\infty} (b_n^{(m)}, d_n^{(m)}) \right), \tag{4}$$

where $$A = \bigcup_{n,m=1}^{\infty} \{a_n^{(m)}, c_n^{(m)}\},\ b_n^{(m)}, d_n^{(m)} \notin A,$$

and between any two intervals $(a_n^{(m)}, c_n^{(m)})$, $(a_k^{(m)}, c_k^{(m)})$ *there is at least one interval* $(b_p^{(m)}, d_p^{(m)})$.

Proof. Let us denote by (x_n) a sequence of the points of A, where $x_n \neq x_m$ for $n \neq m$. Let

$$a_1^{(1)} = x_1, \qquad c_1^{(1)} = x_{n_1},$$

where $n_1 = \min\{n \in N | x_n \in (a_1^{(1)}, 1)\}$.

Let $(b_1^{(1)}, d_1^{(1)})$ and $b_2^{(1)}, d_2^{(1)})$ be any two intervals such that $(b_1^{(1)}, d_1^{(1)}) \subset (0, a_1^{(1)})$, $(b_2^{(1)}, d_2^{(1)}) \subset (c_1^{(1)}, 1)$, and $b_1^{(1)}, b_2^{(1)}, d_1^{(1)}, d_2^{(1)} \notin A$. Now let

$$a_2^{(1)} = x_{n_2},\ a_3^{(1)} = x_{n_3},\ c_2^{(1)} = x_{n_2'},\ c_3^{(1)} = x_{n_3'},$$

$$a_4^{(1)} = x_{n_4},\ a_5^{(1)} = x_{n_5},\ c_4^{(1)} = x_{n_4'},\ c_5^{(1)} = x_{n_5'},$$

where
$$\begin{aligned}
n_2 &= \min\{n \in N | x_n \in (0, b_1^{(1)})\},\\
n_2' &= \min\{n \in N | x_n \in (a_2^{(1)}, b_1^{(1)})\},\\
n_3 &= \min\{n \in N | x_n \in (d_1^{(1)}, a_1^{(1)})\},\\
n_3' &= \min\{n \in N | x_n \in (a_3^{(1)}, a_1^{(1)})\},\\
n_4 &= \min\{n \in N | x_n \in (c_1^{(1)}, b_2^{(1)})\},\\
n_4' &= \min\{n \in N | x_n \in (a_4^{(1)}, b_2^{(1)})\},\\
n_5 &= \min\{n \in N | x_n \in (d_2^{(1)}, 1)\},\\
n_5' &= \min\{n \in N | x_n \in (a_5^{(1)}, 1)\}.
\end{aligned}$$

Now between any two of the chosen intervals $(a_i^{(1)}, c_i^{(1)})$ and $b_i^{(1)}, d_i^{(1)})$ we select intervals $(b_3^{(1)}, d_3^{(1)}), \ldots, (b_{10}^{(1)}, d_{10}^{(1)})$ such that $b_j^{(1)}, d_j^{(1)} \notin A$ for $j = 3, \ldots, 10$.

Continuing this process we infer sequences $(a_n^{(1)}, c_n^{(1)})$ and $(b_n^{(1)}, d_n^{(1)})$ of intervals such that $a_n^{(1)}, c_n^{(1)} \in A$, $b_n^{(1)}, d_n^{(1)} \notin A$. Let

$$K_1 = I \setminus \left(\bigcup_{n=1}^{\infty} (a_n^{(1)}, c_n^{(1)}) \cup \bigcup_{n=1}^{\infty} (b_n^{(1)}, d_n^{(1)}) \right).$$

In every interval of the form $(a_n^{(1)}, c_n^{(1)})$ or $(b_n^{(1)}, d_n^{(1)})$ we select now, in anlogous way, adequate sequences of intervals $(a_n^{(2)}, c_n^{(2)})$, $(b_n^{(2)}, d_n^{(2)})$ fulfilling the conditions:

— every interval $(a_n^{(2)}, c_n^{(2)})$ and $(b_n^{(2)}, d_n^{(2)})$ is contained in some interval $(a_m^{(1)}, c_m^{(1)})$ or $(b_m^{(1)}, d_m^{(1)})$,

— between any two intervals $(a_n^{(2)}, c_n^{(2)})$ and $(a_m^{(2)}, c_m^{(2)})$ there is some interval $(b_p^{(2)}, d_p^{(2)})$,

— $a_n^{(2)}, c_n^{(2)} \in A$, $b_n^{(2)}, d_n^{(2)} \notin A$.

Let

$$K_2 = I \setminus \left(\bigcup_{n=1}^{\infty} (a_n^{(2)}, c_n^{(2)}) \cup \bigcup_{n=1}^{\infty} (b_n^{(2)}, d_n^{(2)}) \right).$$

Continuing this proces we obtain a sequence of sets K_m fulfilling all our requirements.

Lemma 4. *If A is a countable set which is dense in itself and nowhere dense in I, then there exists a sequence $(K_m)_{m=1}^{\infty}$ of nowhere dense perfect sets fulfilling the condition* (4) *in Lemma* 3.

The proof of this lemma is analogous to the proof of Lemma 3.

The only difference is that we choose all intervals of the form $(b_n^{(m)}, d_n^{(m)})$ in such a way that $(b_n^{(m)}, d_n^{(m)}) \cap A = \emptyset$.

Lemma 5. *Let a nowhere dense perfect set K be of the form*

$$K = I \setminus \left(\bigcup_{n=1}^{\infty} (a_n, c_n) \cup \bigcup_{n=1}^{\infty} (b_n, d_n) \right),$$

where between any two different intervals (a_n, c_n), (a_m, c_m) there is some interval (b_k, d_k) and, conversely, between any different intervals (b_n, d_n) and (b_m, d_m) there is some interval (a_k, c_k).

Then there exists a Darboux function $f: I \to R$ such that

$$\mathscr{C}\,\mathrm{ted}(f) = I \setminus \left(\bigcup_{n=1}^{\infty} \{a_n, c_n\} \cup \{0, 1\} \right).$$

Proof. Define four functions

$$\varphi_{a,c}: (a, c) \to R, \ \Phi_{a,c}: (a, c) \to R, \ \psi_{b,d}: (b, d) \to R, \ \Psi_{b,d}: (b, d) \to R$$

in the following way:

$$\varphi_{a,c}(x) = \begin{cases} \frac{1}{2} \cdot \frac{x-a}{c-a} + \frac{1}{2} \left(\frac{x-a}{c-a} + 1 \right) . \sin \frac{\pi (c-a)}{4(x-a)} & \text{for } x \in \left(a, \frac{a+c}{2} \right], \\ \frac{1}{2} \cdot \frac{c-x}{c-a} + \frac{1}{2} \left(\frac{c-x}{c-a} + 1 \right) . \sin \frac{\pi (c-a)}{4(c-x)} & \text{for } x \in \left(\frac{a+c}{2}, c \right), \end{cases}$$

$$\psi_{b,d}(x) = \begin{cases} \sin \frac{\pi (d-b)}{4(x-b)} & \text{for } x \in \left(b, \frac{b+d}{2} \right], \\ \sin \frac{\pi (d-b)}{4(d-x)} & \text{for } x \in \left(\frac{b+d}{2}, d \right). \end{cases}$$

$$\Phi_{a,c}(x) = \max \left(\varphi_{a,c}(x), \frac{1}{c-a} . \varrho(x, \{a, c\}) \right) \qquad \text{for } x \in (a, c),$$

$$\Psi_{b,d}(x) = \begin{cases} \psi_{b,d}(x) & \text{for } x \in \{x \in (b, d) | \psi_{b,d}(x) < 0\}, \\ \min(x \,.\, \psi_{b,d}(x),\ x - \varrho(x, \{b, d\})) & \text{for the remaining } x \text{ from } (b, d). \end{cases}$$

Since K is a nowhere dense set, then there exists a subsequence $(a_{k_n^{(1)}}, c_{k_n^{(1)}})_{n=1}^{\infty}$ of intervals of the sequence $((a_n, c_n))_{n=1}^{\infty}$ such that

— $a_{k_n^{(1)}} < a_{k_{n+1}^{(1)}}$ for $n = 1, 2, \ldots$

— $a_{k_n^{(1)}} \xrightarrow[n \to \infty]{} 1$,

— (a_1, c_1) is one of the intervals of this sequence.

Let now $((a_{k_n^{(2)}}, c_{k_n^{(2)}}))_{n=1}^{\infty}$ be a sequence of intervals fulfilling the following conditions:

— $a_{k_n^{(2)}} > a_{k_{n+1}^{(2)}}$ for $n = 1, 2, \ldots$

— $a_{k_n^{(2)}} \xrightarrow[n \to \infty]{} 0$

— no interval $(a_{k_n^{(2)}}, c_{k_n^{(2)}})$ is contained in the sequence $((a_{k_n^{(1)}}, c_{k_n^{(1)}}))_{n=1}^{\infty}$.

Suppose now that we have chosen $2m$ of such sequences. Now let $(a_{k_n^{(2m+1)}}, c_{k_n^{(2m+1)}})_{n=1}^{\infty}$ be a subsequence of $((a_n, c_n)_{n=1}^{\infty}$

such that

— $a_{k_n^{(2m+1)}} < a_{k_{n+1}^{(2m+1)}}$ for $n = 1, 2, \ldots$

— $a_{k_n^{(2m+1)}} \to a_m$

— (a_{m+1}, c_{m+1}) is one of the terms of that sequence if it is not contained in previously chosen sequences

— no interval $(a_{k_n^{(2m+1)}}, c_{k_n^{(2m+1)}})$ is contained in the sequences

$$((a_{k_n^{(1)}}, c_{k_n^{(1)}}))_{n=1}^{\infty}, \ldots, ((a_{k_n^{(2m)}}, c_{k_n^{(2m)}}))_{n=1}^{\infty}.$$

In this way we have chosen an infinite seuence of sequences of intervals from the sequence $((a_n, c_n))_{n=1}^{\infty}$ such that every interval (a_n, c_n) is exactly one term in exactly one of those sequences.

Now let $\varkappa: K \setminus \bigcup_{n=1}^{\infty} \{a_n, c_n\} \to R$ be a function meeting every continuum which is contained in the set $\{(x, y) \in I \times R,\ -1 \leqq y < x\}$. Let

$$f(x) = \begin{cases} x + \dfrac{1}{m} \cdot \Phi_{a_{k_n^{(m)}}, c_{k_n^{(m)}}}(x) & \text{for } x \in (a_{k_n^{(m)}}, c_{k_n^{(m)}}), \quad n, m = 1, 2, \ldots \\ \Psi_{b_n, d_n}(x) & \text{for } x \in (b_n, d_n), \quad n = 1, 2, \ldots \\ a_m + \dfrac{1}{2m+2} & \text{for } x = a_m, \quad m = 1, 2, \ldots \\ c_m + \dfrac{1}{2m+3} & \text{for } x = c_m, \quad m = 1, 2, \ldots \\ \varkappa(x) & \text{for the remaining } x \text{ from } I. \end{cases}$$

The function defined in this way has to the following properties:
— f is continuous at every point of the set

$$\bigcup_{n=1}^{\infty} (a_n, c_n) \cup \bigcup_{n=1}^{\infty} (b_n, d_n),$$

— $L(f, x) = [-1, x]$ for $x \notin \bigcup_{n=1}^{\infty} [a_n, c_n] \cup \bigcup_{n=1}^{\infty} (b_n, d_n)$,

— $\left[-1, \dfrac{1}{2m+2}\right] \subset L^{-}(f, a_m)$, $\left[-1, \dfrac{1}{2m+3}\right] \subset L^{+}(f, c_m)$,

— $f(x) \neq x$ for $x \in I$.

It follows from those properties that f has the Darboux property in I, but the set $\{0, 1\} \cup \bigcup_{n=1}^{\infty} \{a_n, c_n\}$ is the set of nonconnectivity points of f, because an adequate part of the segment $\{(x, y) \in R^2 | x \in [0, 1], y = x\}$ is a continuum disjoint with the graph of f but fulfilling all remaining requirements of Definition 1.

Analogously, omitting only the first two steps of the previous construction of sequences $((a_{k_n^{(m)}}, c_{k_n^{(m)}}))_{n=1}^{\infty}$, one can prove the following lemma

Lemma 6. *If a set K fulfils all suppositions of the Lemma 5, then there exists a function $f: I \to R$ with the Darboux property and such that $\mathscr{C}\,\mathrm{ted}(f) = I \setminus \bigcup_{n=1}^{\infty} (a_n, c_n\}$.*

Corollary 1. If K fulfils all supositions of the Lemma 5, $g: I \to R$ is a continuous function which is constant on no subinterval of I, ε-arbitrary positive number, then there exist functions $f_i: I \to R$, $i = 1, 2, 3, 4$ such that
— f_i is a Darboux function; $i = 1, 2, 3, 4$,
— $f_i(x) \neq g(x)$ for $x \in I$, $i = 1, 2, 3, 4$,
— $|f_i(x) - g(x)| < \varepsilon$ for $x \in I$, $I = 1, 2, 3, 4$,

— $\mathscr{C}\operatorname{ted}(f_1) = I \setminus \left(\bigcup_{n=1}^{\infty} \{a_n, c_n\} \right)$,

$\mathscr{C}\operatorname{ted}(f_2) = I \setminus \left(\bigcup_{n=1}^{\infty} \{a_n, c_n\} \cup \{0\} \right)$,

$\mathscr{C}\operatorname{ted}(f_3) = I \setminus \left(\bigcup_{n=1}^{\infty} \{a_n, c_n\} \cup \{1\} \right)$,

$\mathscr{C}\operatorname{ted}(f_4) = I \setminus \left(\bigcup_{n=1}^{\infty} \{a_n, c_n\} \cup \{0, 1\} \right)$,

more exactly:

$g(a_n) \in \operatorname{int} L^+(f_i, a_n)$, $g(c_n) \in \operatorname{int} L^-(f_i, c_n)$, $i = 1, 2, 3, 4$,

$g(0) \in \operatorname{int} L^+(f_j, 0)$, $j = 2, 4$,

$g(1) \in \operatorname{int} L^-(f_k, 1)$ $k = 3, 4$,

and

$f_i(x) \neq g(x)$ for $x \in I$.

Lemma 7. *For every countable set A dense i the interval I there exists a Darboux function $f\colon I \to R$ such that $\mathscr{C}\operatorname{ted}(f) = I \setminus A$.*

Proof. According to Lemma 3, there exists an ascending sequence of nowhere dense perfect sets K_m such that

$$K_m = I \setminus \left(\bigcup_{n=1}^{\infty} (a_n^{(m)}, c_n^{(m)}) \cup \left(\bigcup_{n=1}^{\infty} (b_n^{(m)}, d_n^{(m)}) \right) \right),$$

where $A \setminus \{0, 1\} = \bigcup_{n,m=1}^{\infty} \{a_n^{(m)}, c_n^{(m)}\}$, $b_n^{(m)}, d_n^{(m)} \notin A$ and between any two intervals $(a_n^{(m)}, c_n^{(m)})$, $(a_k^{(m)}, c_k^{(m)})$ there is some interval $(b_p^{(m)}, d_p^{(m)})$.

Consider $m = 1$. Depending on the fact if 0 or 1 belongs to A we define a function $h_1\colon I \to R$ as in the Corollary 1 such that the following conditions are fulfilled:

— h_1 has the Darboux property on I,

— $h_1(x) \neq x + \frac{1}{2}$ for $x \in I$,

— $L(h_1, x) \subset \left[0, x + \frac{1}{2}\right]$ for $x \notin \bigcup_{n=1}^{\infty} [a_n^{(1)}, c_n^{(1)}]$,

— h_1 is constant on no subinterval of I,

(5) — h_1 is continuous for $x \in \bigcup_{n=1}^{\infty} [(a_n^{(1)}, c_n^{(1)}) \cup (b_n^{(1)}, d_n^{(1)})]$,

— $\mathscr{C}\,\mathrm{ted}(h_1) = I \setminus \bigcup_{n=1}^{\infty} \{a_n^{(1)}, c_n^{(1)}\}$ if $0, 1 \notin A$,

— $\mathscr{C}\,\mathrm{ted}(h_1) = I \setminus \left(\bigcup_{n=1}^{\infty} \{a_n^{(1)}, c_n^{(1)}\} \cup \{0\} \right)$ if $0 \in A$, $1 \notin A$,

— $\mathscr{C}\,\mathrm{ted}(h_1) = I \setminus \left(\bigcup_{n=1}^{\infty} \{a_n^{(1)}, c_n^{(1)}\} \cup \{1\} \right)$ if $0 \notin A$, $1 \in A$,

— $\mathscr{C}\,\mathrm{ted}(h_1) = I \setminus \left(\bigcup_{n=1}^{\infty} \{a_n^{(1)}, c_n^{(1)}\} \cup \{0, 1\} \right)$ if 0, $1 \in A$.

Now let $m = 2$. Define a function $h_2 : I \to R$ in the following way. If $x \notin \bigcup_{n=1}^{\infty} [(a_n^{(1)}, c_n^{(1)}) \cup (b_n^{(1)}, d_n^{(1)})]$, then let $h_2(x) = h_1(x)$. On each interval $(a_n^{(1)}, c_n^{(1)})$ or $(b_n^{(1)}, d_n^{(1)})$ we define h_2 (like f_4 or f_1 in the Corollary 1) in such a way that there are fulfilled the following conditions:

— $h_2 | [a_n^{(1)}, c_n^{(1)}]$, $h_2 | [b_n^{(1)}, d_n^{(1)}]$ have the Darboux property,

— $h_2(x) \neq h_1(x)$ for $x \in \bigcup_{n=1}^{\infty} [(a_n^{(1)}, c_n^{(1)}) \cup (b_n^{(1)}, d_n^{(1)})]$

— h_2 is constant on no subinterval of I,

— h_2 is continuous for $x \in \bigcup_{n=1}^{\infty} [(a_n^{(2)}, c_n^{(1)}) \cup (b_n^{(2)}, d_n^{(2)})]$,

— $|h_2(x) - h_1(x)| \leqq \frac{1}{2}$,

— $h_2(x) \geqq 0$,

— $\mathscr{C}\,\mathrm{ted}(h_2 | (a_n^{(1)}, c_n^{(1)})) = (a_n^{(1)}, c_n^{(1)}) \setminus \bigcup_{n=1}^{\infty} \{a_n^{(2)}, c_n^{(2)}\}$,

— $\mathscr{C}\,\mathrm{ted}(h_2 | (b_n^{(1)}, d_n^{(1)})) = (b_n^{(1)}, d_n^{(1)}) \setminus \bigcup_{n=1}^{\infty} \{a_n^{(2)}, c_n^{(2)}\}$.

Continuing this process we can define a sequence (h_m) of functions such that the following conditions are fulfilled for $m \in N$:

— h_m has the Darboux property on the interval I,

— $h_{m+1}(x) \neq h_m(x)$ for $x \in \bigcup_{n=1}^{\infty} [(a_n^{(m)}, c_n^{(m)}) \cup (b_n^{(m)}, d_n^{(m)})]$,

— $|h_{m+1}(x) - h_1(x)| \leqq \frac{1}{2^m}$ for $x \in I$,

— h_m is constant on no subinterval of I,

— h_m is continuous for $x \in \bigcup_{n=1}^{\infty} [(a_n^{(m)}, c_n^{(m)}) \cup (b_n^{(m)}, d_n^{(m)})]$,

— $h_{m+1}(x) = h_m(x)$ for $x \in I \setminus \bigcup_{n=1}^{\infty} ([a_n^{(m)}, c_n^{(m)}] \cup [b_n^{(m)}, d_n^{(m)}])$,

— $h_m(x) \geqq 0$,

— $\mathscr{C}\,\mathrm{ted}\,(h_m|(a_n^{(m-1)}, c_n^{(m-1)})) = (a_n^{(m-1)}, c_n^{(m-1)}) \setminus \bigcup_{n=1}^{\infty} \{a_n^{(m)}, c_n^{(m)}\}$,

— $\mathscr{C}\,\mathrm{ted}\,(h_m|(b_n^{(m-1)}, d_n^{(m-1)})) = (b_n^{(m-1)}, d_n^{(m-1)}) \setminus \bigcup_{n=1}^{\infty} \{a^{(m)}, c^{(m)}\}$

for $m = 2, \ldots$ and (5) for $m = 1$.

The sequence (h_m) is uniformly convergent. Let then

$$f = \lim_{n \to \infty} h_m.$$

One can prove that the function f has all the required properties.

Corollary 2. For an arbitrary interval $[\alpha, \beta]$, a coutable set $A \subset [\alpha, \beta]$, an arbitrary $M > 0$ there exists $f_{\alpha,\beta}^M: [\alpha, \beta] \to R$ such that $\mathscr{C}\,\mathrm{ted}\,(f_{\alpha,\beta}^M) = [\alpha, \beta] \setminus A$, $0 \leqq f_{\alpha,\beta}^M(x)$, $\mathrm{osc}\,(f_{\alpha,\beta}^M[\alpha, \beta]) = M$, $\lim_{x \to \alpha+} \inf f_{\alpha,\beta}^M(x) = 0 = \lim_{x \to \beta-} \inf f_{\alpha,\beta}^M(x)$.

The proof of the next lemma is similar to the proof of Lemma 7, so we omit it.

Lemma 8. *For every countable set A dense in itself and nowhere dense in I there exists a function $f: I \to R$ with the Darboux property and such* $\mathscr{C}\,\mathrm{ted}\,(f) = I \setminus A$, $\lim_{x \to 0+} \inf f(x) = 0 = \lim_{x \to 1-} \inf f(x)$.

Corollary 3. For every interval $[\alpha, \beta]$, a countable set $A \subset [\alpha, \beta]$ dense in itself and nowher dense in $[\alpha, \beta]$ and an arbitrary number $M > 0$ there exists a function

$$g_{\alpha,\beta}^M: [\alpha, \beta] \to R \text{ such that } \mathscr{C}\,\mathrm{ted}\,(g_{\alpha,\beta}^M) = [\alpha, \beta] \setminus A,$$

$$0 \leqq g_{\alpha,\beta}^M(x),\ \mathrm{osc}\,(g_{\alpha,\beta}^M, [\alpha, \beta]) = M \text{ and}$$

$$\lim_{x \to \alpha+} \inf g_{\alpha,\beta}^M(x) = 0 = \lim_{x \to \beta-} \inf g_{\alpha,\beta}^M(x).$$

Now we can prove the main theorem.

Theorem 2. *Let $E \subset I$ be any set of type G_σ such that $I \setminus E$ is dense in itself. Then there exists a function $f: I \to R$ with the Darboux property and such that* $\mathscr{C}\,\mathrm{ted}\,(f) = E$.

Proof. The set $A = I \setminus E$ is of type F_σ and it is dense in itself. Let

$$A_1 = A \cap A^c.$$

It is obvious that A_1 is a set of type F_σ, $\mathfrak{c}$-dense in itself, and $A \setminus A_1$ is a countable set. Let now

$$B = \overline{A \setminus A_1}.$$

Then $B = \bigcup_{n=1}^{\infty} I_n \cup C$, where I_n are nondegenerate components of B, and C a set containing no intervals. Now let

$$A_2 = \bigcup_{n=1}^{\infty} ((A \setminus A_1) \cap I_n,$$

$$A_4 = (\bar{A}_2 \cap A) \setminus (A_1 \cup A_2),$$

$$A_3 = A \setminus (A_1 \cup A_2 \cup A_4).$$

One can easily prove that the following conditions are fulfilled:

— $A = A_1 \cup A_2 \cup A_3 \cup A_4$,
— A_1 is a $\mathfrak{c}$-dense in itself set of type F_σ,
— A_2 is a countable set dense in some union of closed nondegenerate intervals I_n,
— A_3 is a countable set dense in itself and nowhere dense,
— A_4 is the set of accumulation points ofthe set A_2, belonging to A which do not belong to $A_1 \cup A_2$.
— $A_i \cap A_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, 3, 4$.

Now let $g_1 : I \to R$ be a function (constructed in Lemma 2) for which $\mathscr{C}\,\mathrm{ted}\,(g_1) = I \setminus A_1$.

Let $A_4 = \bigcup_{n=1}^{\infty} \{x_m\}$. For every point $x_m \in A_4$ there exists a subsequence $(I_{k_n^{(m)}})_{n=1}^{\infty}$ of $(I_n)_{n=1}^{\infty}$ such that $I_{k_n^{(m)}} \cap I_{k_l^{(j)}} = \emptyset$ for $k_n^{(m)} \neq k_l^{(j)}$ and $\varrho(x_m, I_{k_n^{(m)}}) \xrightarrow[n \to \infty]{} 0$.

Now let $(I_{k_n^{(0)}})_{n=1}^{\infty}$ be a sequence of all those intervals I_n which are not contained in sequences $(I_{k_n^{(m)}})_{n=1}^{\infty}$. Let a function $g_2 : I \to R$ (behaving denotations from the Corollary 2 for the set A_2) be definedas follows:

$$g_2(x) = \begin{cases} (-1)^{k_m^{(m)}} \cdot f_{\alpha,\beta}^{1/m}(x) & \text{for } x \in I, \text{ where } I_{k_n^{(m)}} = [\alpha, \beta], \\ f_{\alpha,\beta}^{1/k_n^{(0)}}(x) & \text{for } x \in I_{k_n^{(0)}}, \text{ where } I_{k_n^{(0)}} = [\alpha, \beta], \\ 0 & \text{for all remaining } x \in I. \end{cases}$$

There exists an open set $U = \bigcup_{n=1}^{\infty} (\alpha_n, \beta_n)$ such that $A_3 \subset U$ and $U \cap \cap (\bar{A}_1 \cup A_2) = \emptyset$. Respecting now denotations from the Corollary 3 for the set A_3, let $g_3: I \to R$ be defined as follows:

$$g_3(x) = \begin{cases} q_{\alpha_n, \beta_n}^{1/n}(x) & \text{for } x \in (\alpha_n, \beta_n) \\ 0 & \text{for all remained } x \in I. \end{cases}$$

Now let $g_4: I \to R$ be any continuous and nowhere constant function for which the sum of g_4 and each of the functions appeared in all the previous constructions of the functions $g_1, g_2\ g_3$ as well as their sums and limits is constant on no subinterval of I. In this way the function

$$f = g_1 + g_2 + g_3 + g_4$$

has all the required properties.

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О ТОЧКАХ СВЯЗНОСТИ ФУНКЦИЙ ДАРБУ

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Резюме

В работе показано, что для того чтобы E было множеством несвязности некоторой функции Дарбу необходимо и дистаточно, чтобы E было F_σ — множеством плотным в себе.