Vincenzo Aversa; K. P. S. Bhaskara Rao

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KELLEY'S THEOREM FROM THE DUALITY OF LP AND TYCHONOFF'S THEOREM

V. AVERSA* — K. P. S. BHASKARA RAO**

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ABSTRACT. Many mathematical problems can be reformulated as optimization problems. In the present paper we shall see how a problem on the existence of strictly positive finitely additive probabilities can be considered as an optimization problem. We use the duality of Linear Programming and Tychonoff's theorem in general topology and derive a 1959 result of J. L. Kelley on the existence of strictly positive probability charges on Boolean algebras. The original proof of Kelley used functional analytic techniques. We replace these techniques with Linear Programming and Tychonoff's theorem.

In 1959 J. L. Kelley found necessary and sufficient conditions for a Boolean algebra $\mathcal{A}$ to admit a strictly positive probability charge. In the present paper we shall give a simple proof of this theorem of Kelley as mentioned in the title. For a given finite collection $B_1, B_2, \dots, B_m$ of non zero elements of the Boolean algebra and real numbers $b_1, b_2, \dots, b_m$ such that $0 \le b_i \le 1$ for all i, we incidentally obtain necessary and sufficient conditions for the existence of a probability charge μ on $\mathcal{A}$ such that $\mu(B_i) \ge b_i$ for all i. Several proofs of Kelley's theorem are available in the literature (see [1], [4], [7]). The original proof of Kelley used functional analysis. A 1991 proof by Siu-Ah Ng used LP and non standard analysis. Our present proof is simple and elementary.

In Section 1 we shall explain the idea of the proof. In Section 2 we shall explain the use of duality of LP and in Section 3 we shall conclude the proof with the use of Tychonoff's theorem.

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Supported by research project "Analisi Reale", italian PRIN funds.
Visiting Professor at Facoltà di Economia, Università Federico II, Naples, Italy.

1. Introduction and the idea

Let $\mathcal{A}$ be a Boolean algebra with the unit element $1_{\mathcal{A}}$ and zero element $0_{\mathcal{A}}$. A *charge* on $\mathcal{A}$ is a real valued finitely additive measure. A *probability charge* is a non negative charge μ such that $\mu(1_{\mathcal{A}}) = 1$. A probability charge μ will be called a *strictly positive probability charge* if $\mu(A) > 0$ whenever $A \in \mathcal{A}$ and $A \neq 0_{\mathcal{A}}$. For a Boolean algebra $\mathcal{A}$ we shall write $\mathcal{A}^+$ for the set of all $A \in \mathcal{A}$ such that $A \neq 0_{\mathcal{A}}$.

Kelley has shown, in a simple way, that the problem of finding necessary and sufficient conditions for the existence of a strictly positive probability charge on a Boolean algebra $\mathcal{A}$ is equivalent to solving the problem of finding necessary and sufficient conditions (for a given $\mathcal{B} \subset \mathcal{A}^+$ and a real number b such that $0 \leq b \leq 1$) for the existence of a probability charge μ on $\mathcal{A}$ such that $\mu(B) \geq b$ for all $B \in \mathcal{B}$. We shall look at this problem.

First of all, if $\mathcal{B} \subset \mathcal{A}^+$ and $\mathcal{B}$ has finitely many elements, we shall find necessary and sufficient conditions for the existence of a probability charge μ on $\mathcal{A}$ such that $\mu(B) \geq b$ for all $B \in \mathcal{B}$. For this we shall formulate the problem as an LP problem, formulate its dual and use the fact that the primal has an optimal solution if and only if the dual has an optimal solution and the two optimal values are equal.

For a general $\mathcal{B} \subset \mathcal{A}^+$ and $0 \leq b \leq 1$ we shall show, using Tychonoff' theorem, that the existence of a probability charge μ on $\mathcal{A}$ such that $\mu(B) \geq b$ for all $B \in \mathcal{B}$ is equivalent to the existence of, for every finite $\mathcal{B}_0 \subset \mathcal{B}$, a probability charge $\mu_{\mathcal{B}_0}$ on $\mathcal{A}$ such that $\mu_{\mathcal{B}_0}(B) \geq b$ for all $B \in \mathcal{B}_0$. This give us the general required necessary and sufficient conditions.

2. An application of duality of LP

To start with, let us assume that $\mathcal{A}$ is a finite field of sets, $\mathcal{B} = \{B_1, B_2, \ldots \ldots, B_m\} \subset \mathcal{A}^+$ and $b_1, b_2, \ldots, b_m$ are real numbers such that $0 \leq b_i \leq 1$ for $1 \leq i \leq m$. We shall find necessary and sufficient conditions for the existence of a probability charge μ on $\mathcal{A}$ such that $\mu(B_i) \geq b_i$ for $i = 1, 2, \ldots, m$. We might as well assume that $\mathcal{A} = \mathcal{P}(\{1, 2, \ldots, k\})$, the power set of $\{1, 2, \ldots, k\}$. We wish to find conditions under which there exist real numbers $x_1, x_2, \ldots, x_k$ such that $x_j \geq 0$ for $j = 1, 2, \ldots, k$, $\sum_j x_j = 1$ and $\sum_{j \in B_i} x_j \geq b_i$ for all $i = 1, 2, \ldots, m$. We shall write this as an LP problem.

Let $\boldsymbol{e}^T = (1, 1, \ldots, 1)$, $\boldsymbol{b}^T = (b_1, b_2, \ldots, b_m)$ and $\mathbf{M}$ be the incidence matrix of $B_1, B_2, \ldots, B_m$. Thus

$$\mathbf{M} = (m_{ij}) \qquad \text{where} \quad m_{ij} = \begin{cases} 1 & \text{if } j \in B_i\,, \\ 0 & \text{if } j \notin B_i\,. \end{cases}$$

$\mathbf{M}$ is an $m \times k$ $\{0,1\}$-matrix and $m_{ij} = I_{B_i}(j)$ where I_C stands for the characteristic function of C.

Consider the LP problem

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{e}^T \boldsymbol{x} \\ &\text{such that} \\ &\mathbf{M}\boldsymbol{x} \geq \boldsymbol{b} \quad \text{and} \quad \boldsymbol{x} \geq 0 \end{aligned} \tag{$*$}$$

where $\boldsymbol{x}^T = (x_1, x_2, \ldots, x_k)$ is a vector of real variables. Since there is always an $\boldsymbol{x} \geq 0$ such that $\mathbf{M}\boldsymbol{x} \geq \boldsymbol{b}$, the optimum value exists and let it be a_0.

If there is a probability charge μ on $\mathcal{A}$ such that $\mu(B_i) \geq b_i$ for $i = 1, 2, \ldots, m$, then the optimum value $a_0 \leq 1$. Conversely, let $a_0 \leq 1$; in case $a_0 = 0$, $b_i = 0$ for all i. In this case any $\boldsymbol{x} \geq 0$ such that $\sum x_i = 1$ will be the required probability charge. Let $a_0 \neq 0$ and $\boldsymbol{x}_0$ be an optimum solution, i.e., $\boldsymbol{e}^T \boldsymbol{x}_0 = a_0 \leq 1$, $\boldsymbol{x}_0 \geq 0$ and $\mathbf{M}\boldsymbol{x}_0 \geq \boldsymbol{b}$; then the vector $\boldsymbol{x}_1$ defined by $\boldsymbol{x}_1 = \frac{1}{a_0}\boldsymbol{x}_0$ will satisfy the conditions $\boldsymbol{e}^T \boldsymbol{x}_1 = 1$, $\mathbf{M}\boldsymbol{x}_1 = \frac{1}{a_0}\mathbf{M}\boldsymbol{x}_0 \geq \frac{1}{a_0}\boldsymbol{b} \geq \boldsymbol{b}$, and $\boldsymbol{x}_1 \geq 0$. This $\boldsymbol{x}_1$ gives us a probability charge on $\mathcal{A}$ satisfying the required conditions.

Let us look at the dual of $(*)$. The dual of $(*)$ is

$$\begin{aligned} &\text{maximize} \quad \boldsymbol{b}^T \boldsymbol{y} \\ &\text{such that} \\ &\mathbf{M}^T \boldsymbol{y} \leq \boldsymbol{e} \quad \text{and} \quad \boldsymbol{y} \geq 0\,. \end{aligned} \tag{$**$}$$

Of course, the optimal value of the primal $(*)$ is ≤ 1 if and only if the optimal value of the dual $(**)$ is ≤ 1. This gives us the following theorem:

THEOREM 1. *Let $\mathcal{A}$ be a field of subsets of a set X. Let $B_1, B_2, \ldots, B_m$ be nonempty sets from $\mathcal{A}$. Let $b_1, b_2, \ldots, b_m$ be real numbers such that $0 \leq b_i \leq 1$ for all i. The following are equivalent:*

i) *There is a probability charge μ on $\mathcal{A}$ such that $\mu(B_i) \geq b_i$ for all $1 \leq i \leq m$;*

ii) $\sum_{i=1}^{m} b_i y_i \leq 1$ *whenever $y_1, y_2, \ldots, y_m \geq 0$ are such that $\sum_{i:\, x \in B_i} y_i \leq 1$ for every x;*

iii) $\sum_{i=1}^{m} b_i y_i \leq \max_x \sum_{i:\, x \in B_i} y_i$ *whenever $y_1, y_2, \ldots, y_m \geq 0$;*

iv) $\sum_{i=1}^{m} k_i b_i \leq \max_x \sum_{i:\, x \in B_i} k_i$ *whenever $k_1, k_2, \ldots, k_m$ are non negative integers.*

P r o o f.

The argument before the statement of Theorem 1 proves (i) $\Longleftrightarrow$ (ii).

To prove (ii) $\Longrightarrow$ (iii), let $y_1, y_2, \ldots, y_m$ be ≥ 0 and let $\max_x \sum_{i:x\in B_i} y_i = z$. Let $y_i' = y_i/z$ for $1 \leq i \leq m$. Then $\sum_{i:x\in B_i} y_i' \leq 1$ for all x. Hence $\sum b_i y_i' \leq 1$ which is equivalent to say that $\sum b_i y_i \leq z$. Thus (iii).

(iii) $\Longrightarrow$ (ii) is clear.

(iii) says that $\sum b_i y_i \leq \max_x \sum_{i:x\in B_i} y_i$ for every vector $(y_1, y_2, \ldots, y_m)$ such that $y_i \geq 0$ for all i. But $\{(y_1, y_2, \ldots, y_m) : \ y_i \geq 0$ and y_i all rationals$\}$ is a dense subset of $\{(y_1, y_2, \ldots, y_m) : \ y_i \geq 0\}$. Also any $(y_i, y_2, \ldots, y_m)$ such that $y_i \geq 0$ and y_i are all rationals can be written as $(k_1/k, k_2/k, \ldots, k_m/k)$ for some non negative integers $k_1, k_2, \ldots, k_m$. Hence (iii) $\Longleftrightarrow$ (iv). $\square$

We note that the condition (iv) can also be written as

$$\sum_{i=1}^{m} k_i b_i \leq \max_x \sum_{i=1}^{m} k_i I_{B_i}(x)\,.$$

The above theorem takes the following form if all b_i are equal.

THEOREM 2. *Let $\mathcal{A}$ be a field of subsets of a set X. Let $B_1, B_2, \ldots, B_m$ be non empty sets from $\mathcal{A}$. Let b be such that $0 \leq b \leq 1$. Then there exists a probability charge μ on $\mathcal{A}$ such that $\mu(B_i) \geq b$ for all i if and only if*

$$b \leq \max_x \frac{1}{\sum k_i} \sum_{i=1}^{m} k_i I_{B_i}(x)$$

for every $k_1, k_2, \ldots, k_m \geq 0$, integers.

K e l l e y called $\inf_{k_1,k_2,\ldots,k_m \geq 0} \max_x \sum_{i=1}^{m} k_i I_{B_i}(x)$ the *intersection number* of the sets $B_1, B_2, \ldots, B_m$.

The above theorem says that there is a probability charge μ on $\mathcal{A}$ such that $\mu(B_i) \geq b$ for all i if and only if b is $\leq$ the intersection number of $B_1, B_2, \ldots, B_m$. Since $\mu(B_i) = b_i$ can be written as $\mu(B_i) \geq b_i$ and $\mu(X - B_i) \geq 1 - b_i$, we remark that the equivalence (ii) $\Longleftrightarrow$ (iv) can be used to give neces sary and sufficient conditions for the existence of a probability charge μ on $\mathcal{A}$ such that $\mu(B_i) = b_i$ for all i. Namely, the following can be proved:

THEOREM 3. *Let $\mathcal{A}$ be a field of subsets of a set X. Let $B_1, B_2, \ldots, B_m$ be nonempty sets from $\mathcal{A}$ and $b_1, b_2, \ldots, b_m$ be real numbers such that $0 \leq b_i \leq 1$ for all i.*

Then a probability charge μ on $\mathcal{A}$ such that $\mu(B_i) = b_i$ for all i exits if and only if:

$$\min_x \sum_{i=1}^{m} k_i I_{B_i}(x) \leq \sum_{i=1}^{m} k_i b_i \leq \max_x \sum_{i=1}^{m} k_i I_{B_i}(x)$$

whenever $k_1, k_2, \ldots, k_m$ are non negative integers.

3. Use of Tychonoff's theorem

Let $\mathcal{B} \subset \mathcal{A}^+$ be infinite and let $0 \leq b \leq 1$. We wish to find necessary and sufficient conditions for the existence of a probability charge μ on $\mathcal{A}$ such that $\mu(B) \geq b$ for all $B \in \mathcal{B}$. Tychonoff's theorem comes in naturally in the following theorem:

THEOREM 4. *Let $\mathcal{A}$ be a field of subsets of a set X, $\mathcal{B} \subset \mathcal{A}^+$ be a possibly infinite subcollection and let $0 \leq b \leq 1$. If for every finite $\mathcal{B}_0 \subset \mathcal{B}$ there is a probability charge μ on $\mathcal{A}$ such that $\mu(B) \geq b$ for all $B \in \mathcal{B}_0$, then there is a probability charge μ on $\mathcal{A}$ such that $\mu(B) \geq b$ for all $B \in \mathcal{B}$.*

Proof. Our proof goes along the expected line. Consider the $\mathcal{A}$-fold product $[0,1]^{\mathcal{A}}$ with the product topology. For $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{B}_0 = \{B_1, B_2, \ldots, B_m\}$, let

$$Z_{\mathcal{B}_0} = \{\mu \in [0,1]^{\mathcal{A}} : \mu \text{ is a probability charge on } \mathcal{A} \text{ and } \mu(B_i) \geq b \text{ for all } B_i \in \mathcal{B}_0\}.$$

For every finite $\mathcal{B}_0 \subset \mathcal{B}$, $Z_{\mathcal{B}_0}$ is a nonempty closed set and $\{Z_{\mathcal{B}_0} : \mathcal{B}_0 \subset \mathcal{B}, \mathcal{B}_0 \text{ finite}\}$ has the finite intersection property. Hence $\bigcap Z_{\mathcal{B}_0} \neq \emptyset$. This exactly means that there is a probability charge μ on $\mathcal{A}$ such that $\mu(B) \geq b$ for all $B \in \mathcal{B}$. □

Thus, there is a probability charge μ on $\mathcal{A}$ such that $\mu(B) \geq b$ for all $B \in \mathcal{B}$ if and only if the intersection number of $B_1, B_2, \ldots, B_m$ is $\geq b$ for every $\{B_1, B_2, \ldots, B_m\} \subset \mathcal{B}$ and $m \geq 1$.

The rest of the argument for the proof of Kelley's theorem is clear and we get that for a Boolean algebra $\mathcal{A}$ to admit a strictly non zero probability charge μ it is necessary and sufficient that $\mathcal{A}^+$ can be written as a countable union of subfamilies $\mathcal{B}_1, \mathcal{B}_2 \ldots$ of $\mathcal{A}^+$ such that for each i there is a c_i such that $0 < c_i \leq 1$ and the intersection number of $B_1, B_2, \ldots, B_m$ is $\geq c_i$ for every finite subcollection $B_1, B_2, \ldots, B_m \subset \mathcal{B}_i$, $m \geq 1$.

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* *Dipartimento di Matematica e Statistıca*
Vıa Cinzia
I-80126 Napoli
ITALY
E-maıl: aversa@unina.it

** *Indian Statistical Institute*
Bangalore Centre
8th Mile, Mysore Road
R. V. College Post
Bangalore 560 059
INDIA

Current address:
Southwestern college
Winfield, KS
U.S.A
E-mail: brao@sckans.edu