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Mathematica Slovaca, Vol. 51 (2001), No. 3, 313--320

Math. Slovaca, **51** (2001), No. 3, 313–320

Mathematica Slovaca

TRIPLE POSITIVE SOLUTIONS FOR $(k, n-k)$ CONJUGATE BOUNDARY VALUE PROBLEMS

JOHN M. DAVIS* — JOHNNY HENDERSON**

(*Communicated by Milan Medveď*)

ABSTRACT. For the nth order differential equation,

$$(-1)^{n-k}y^{(n)} - f(y) = 0\,, \qquad t \in [0,1]\,,$$

satisfying the boundary conditions, $y^{(i)}(0) = 0$, $0 \le i \le k-1$, and $y^{(j)}(1) = 0$, $0 \le j \le n-k-1$, where $f\colon \mathbb{R} \to [0,\infty)$, growth conditions are imposed on f which yield the existence of at least three positive solutions.

1. Introduction

Let $n \ge 2$ and $1 \le k \le n-1$ be given. We are concerned with the existence of multiple solutions for the nth order boundary value problem

$$(-1)^{n-k}y^{(n)} - f(y) = 0\,, \qquad t \in [0,1]\,, \tag{1.1}$$

$$\begin{aligned} y^{(i)}(0) &= 0\,, \qquad 0 \le i \le k-1\,, \\ y^{(j)}(1) &= 0\,, \qquad 0 \le j \le n-k-1\,, \end{aligned} \tag{1.2}$$

where $f\colon \mathbb{R} \to [0,\infty)$ is continuous. It is fairly standard to refer to the boundary value problem (1.1), (1.2) as a $(k, n-k)$ *conjugate boundary value problem.* We will impose growth conditions on f which insure the existence of at least three positive solutions of (1.1), (1.2).

A good deal of recent attention has been directed toward obtaining triple solutions for boundary value problems for ordinary differential equations. This paper can be considered as a generalization of previous work on triple solutions for special cases of two-point boundary value problems by Avery [2],

2000 Mathematics Subject Classification: Primary 34B10.
Key words: ordinary differential equation, boundary value problem, Green's function, multiple solution, fixed point.

Chyan, Davis and Yin [5], Henderson and Thompson [9], and Wong and Agarwal [11]. Other papers on triple solutions for boundary value problems for ordinary differential equations have been written by Anderson [1], Chyan and Davis [4], and Guo and Lakshmikantham [7], and the recent papers [3], [8] and [12] were devoted to triple solutions for boundary value problems for finite difference equations.

For the most part, each of the above cited papers makes an application of a fixed point theorem by Leggett and Williams [10], which they developed using the fixed point index in ordered Banach spaces. Leggett and Williams [10] applied their fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations of the form

$$y(x) = \int_{\Omega} G(x,s)g\big(s,y(s)\big)\,\mathrm{d}s\,, \qquad \Omega \subset \mathbb{R}^N\,,$$

by making use of suitable inequalities they imposed on the kernel G and g.

In Section 2, we provide some definitions and background results, and we state the Leggett-Williams fixed point theorem. Then in Section 3, we impose growth conditions on f which allow us to apply the Leggett-Williams fixed point theorem in obtaining three positive solutions of (1.1), (1.2).

2. Background definitions and results

In this section, we provide some background material from the theory of cones in Banach spaces. We also state a fixed point theorem due to Leggett and Williams [10] for multiple fixed points of a cone preserving operator.

Let $\mathcal{B}$ be a real Banach space equipped with a norm, $\|\cdot\|$. If $\mathcal{P} \subset \mathcal{B}$ is a cone, we denote the order induced by $\mathcal{P}$ on $\mathcal{B}$ by $\leq_{\mathcal{P}}$.

DEFINITION 2.1. A map α is said to be a *nonnegative continuous concave functional on* $\mathcal{P}$ if $\alpha\colon \mathcal{P} \to [0,\infty)$ is continuous and

$$\alpha\big(tx + (1-t)y\big) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$.

DEFINITION 2.2. For numbers $0 < a < b$ and α, a nonnegative continuous concave functional on $\mathcal{P}$, define convex sets $\mathcal{P}_r$ and $\mathcal{P}(\alpha,a,b)$ by

$$\mathcal{P}_r = \big\{y \in \mathcal{P} :\ \|y\| < r\big\} \qquad \text{and} \qquad \mathcal{P}(\alpha,a,b) = \big\{y \in \mathcal{P} :\ a \leq \alpha(y)\,,\ \|y\| \leq b\big\}\,.$$

In obtaining multiple positive solutions of (1.1), (1.2), the following fixed point theorem due to Leggett and Williams [10] will be fundamental.

THEOREM 2.1 (LEGGETT-WILLIAMS FIXED POINT THEOREM). *Let $\mathcal{A}\colon \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$ be a completely continuous operator and let α be a nonnegative continuous concave functional on $\mathcal{P}$ such that $\alpha(y) \le \|y\|$ for all $y \in \overline{\mathcal{P}}_c$. Suppose there exist $0 < a < b < d \le c$ such that*

(C1) $\{y \in \mathcal{P}(\alpha, b, d) :\ \alpha(y) > b\} \ne \emptyset$ *and* $\alpha(\mathcal{A}y) > b$ *for* $y \in \mathcal{P}(\alpha, b, d)$,
(C2) $\|\mathcal{A}y\| < a$ *for* $y \in \overline{\mathcal{P}}_a$,
(C3) $\alpha(\mathcal{A}y) > b$ *for* $y \in \mathcal{P}(\alpha, b, c)$ *with* $\|\mathcal{A}y\| > d$.

Then $\mathcal{A}$ has at least three fixed points y_1, y_2, and y_3 such that $\|y_1\| < a$, $b < \alpha(y_2)$, and $\|y_3\| > a$ with $\alpha(y_3) < b$.

3. Multiple positive solutions

In this section, we will impose growth conditions on f which allow us to apply Theorem 2.1 in regard to obtaining three positive solutions of (1.1), (1.2). We will apply Theorem 2.1 in conjunction with a completely continuous operator whose kernel is the Green's function $G(t,s)$ for

$$(-1)^{n-k} y^{(n)} = 0$$

satisfying the boundary conditions (1.2). It is fairly well-known that

$$G(t,s) > 0\,, \qquad (t,s) \in (0,1) \times (0,1)\,. \tag{3.1}$$

Also, for $s \in (0,1)$, there exists $\tau(s) \in (0,1)$ such that

$$G(t,s) \le G\big(\tau(s), s\big)\,, \qquad t \in (0,1)\,, \tag{3.2}$$

and it is shown in [6] that

$$G(t,s) \ge \frac{1}{4^m} G\big(\tau(s), s\big)\,, \qquad t \in \big[1/4, 3/4\big]\,,\ \ s \in [0,1]\,, \tag{3.3}$$

where $m = \max\{k, n-k\}$.

Next, we note

$$\int_0^1 G(t,s)\,\mathrm{d}s = \frac{t^k (1-t)^{n-k}}{n!}\,, \qquad t \in [0,1]\,,$$

and as a result, we see

$$\max_{t\in[0,1]} \int_0^1 G(t,s)\,\mathrm{d}s = \int_0^1 G(k/n, s)\,\mathrm{d}s = \frac{\left(\frac{k}{n}\right)^k \left(1 - \frac{k}{n}\right)^{n-k}}{n!}\,, \tag{3.4}$$

$$\min_{t\in\left[\frac14, \frac34\right]} \int_0^1 G(t,s)\,\mathrm{d}s = \min\left\{\tfrac{3^{n-k}}{4^n n!}, \tfrac{3^k}{4^n n!}\right\} = \frac{3^{n-m}}{4^n n!}\,. \tag{3.5}$$

For notational convenience we define the following constants involving the quantities above

$$K = \left(\max_{t\in[0,1]} \int_0^1 G(t,s)\,\mathrm{d}s \right)^{-1} = \frac{n!}{\left(\frac{k}{n}\right)^k \left(1-\frac{k}{n}\right)^{n-k}}, \tag{3.6}$$

$$L = \left(\min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_0^1 G(t,s)\,\mathrm{d}s \right)^{-1} = \frac{4^n n!}{3^{n-m}}. \tag{3.7}$$

By (3.3) and (3.2) we obtain

$$\begin{aligned} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_{1/4}^{3/4} G(t,s)\,\mathrm{d}s &\geq \frac{1}{4^m} \int_{1/4}^{3/4} G(\tau(s),s)\,\mathrm{d}s \\ &\geq \frac{1}{4^m} \int_{1/4}^{3/4} G(t,s)\,\mathrm{d}s \qquad \text{for all} \quad t\in(0,1). \end{aligned}$$

Hence,

$$\begin{aligned} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_{1/4}^{3/4} G(t,s)\,\mathrm{d}s &\geq \frac{1}{4^m} \int_0^1 \mathrm{d}t \int_{1/4}^{3/4} G(t,s)\,\mathrm{d}s \\ &= \frac{1}{4^m} \int_{1/4}^{3/4} \mathrm{d}s \int_0^1 G(t,s)\,\mathrm{d}t = \frac{1}{4^m} \int_{1/4}^{3/4} \mathrm{d}s \int_0^1 G^*(s,t)\,\mathrm{d}t, \end{aligned}$$

where G^* is the Green's function for the boundary value problem

$$(-1)^k z^{(n)}(t) = 0, \qquad t\in[0,1], \tag{3.8}$$

$$\begin{aligned} z^{(i)}(0) &= 0, \qquad 0\leq i\leq n-k, \\ z^{(j)}(1) &= 0, \qquad 0\leq j\leq k-1. \end{aligned} \tag{3.9}$$

This is due to the fact that

$$\int_0^1 G^*(s,t)\,\mathrm{d}t = \frac{s^{n-k}(1-s)^k}{n!}, \qquad s\in[0,1],$$

and

$$\begin{aligned} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_{1/4}^{3/4} G(t,s)\,\mathrm{d}s &\geq \frac{1}{4^m} \int_{1/4}^{3/4} \frac{s^{n-k}(1-s)^k}{n!}\,\mathrm{d}s \\ &> \frac{1}{4^m\cdot 2L}. \end{aligned} \tag{3.10}$$

Let $\mathcal{B}$ denote the Banach space $C[0,1]$ endowed with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$, and let the cone $\mathcal{P} \subset \mathcal{B}$ be defined by

$$\mathcal{P} = \{x \in \mathcal{B} : \ x(t) \ge 0, \ \ t \in [0,1]\},$$

and finally let the nonnegative continuous concave functional $\alpha \colon \mathcal{P} \to [0, \infty)$ be defined by

$$\alpha(x) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} x(t), \qquad x \in \mathcal{P}.$$

We note that, for each $x \in \mathcal{P}$, $\alpha(x) \le \|x\|$, and also that $x \in \mathcal{B}$ is a solution of (1.1), (1.2) if and only if

$$x(t) = \int_0^1 G(t,s) f\big(x(s)\big)\, \mathrm{d}s, \qquad t \in [0,1].$$

We now present the main result of the paper.

THEOREM 3.1. *Let $0 < a < b < 4^m b \le \frac{c}{4^m}$ be such that f satisfies*

(i) $f(w) < Ka$ *for* $0 \le w \le a$,
(ii) $f(w) \ge 4^m \cdot 2Lb$ *for* $b \le w \le 4^m b$,
(iii) $f(w) \le Kc$ *for* $0 \le w \le c$.

Then the boundary value problem (1.1), (1.2) *has at least three positive solutions y_1, y_2, and y_3 satisfying $\|y_1\| < a$, $b < \alpha(y_2)$, and $\|y_3\| > a$ with $\alpha(y_3) < b$.*

Proof. We first define the completely continuous operator $\mathcal{A} \colon \mathcal{B} \to \mathcal{B}$ by

$$\mathcal{A}y(t) = \int_0^1 G(t,s) f\big(y(s)\big)\, \mathrm{d}s,$$

and we seek fixed points of $\mathcal{A}$ which satisfy the conclusion of the theorem. We observe from the positivity of f and (3.1) that $\mathcal{A}y(t) \ge 0$ on $[0,1]$ for each $y \in \mathcal{P}$. Thus, $\mathcal{A} \colon \mathcal{P} \to \mathcal{P}$.

We now show that the conditions of Theorem 2.1 are satisfied. Choose $y \in \overline{\mathcal{P}}_c$. Then $\|y\| \le c$, and by assumption (iii), $f\big(y(s)\big) \le Kc$, $s \in [0,1]$. Thus, from (3.6) we have

$$\begin{aligned} \|\mathcal{A}y\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) f\big(y(s)\big)\, \mathrm{d}s \\ &\le \max_{t\in[0,1]} \int_0^1 G(t,s) Kc \, \mathrm{d}s = c. \end{aligned}$$

Hence, $\mathcal{A}\colon \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$. In a similar way, if $y \in \overline{\mathcal{P}}_a$, then assumption (i) yields $f(y(s)) < Ka$, $s \in [0,1]$, and it follows as above that $\mathcal{A}\colon \overline{\mathcal{P}}_a \to \mathcal{P}_a$. Consequently, condition (C2) of Theorem 2.1 is fulfilled.

To verify property (C1) of Theorem 2.1, we note that $x(t) = 4^m b$, $t \in [0,1]$, belongs to $\mathcal{P}(\alpha, b, 4^m b)$, and $\alpha(x) = 4^m b > b$. So

$$\{y \in \mathcal{P}(\alpha, b, 4^m b) : \alpha(y) > b\} \neq \emptyset .$$

Furthermore, if we choose $y \in \mathcal{P}(\alpha, b, 4^m b)$, then

$$\alpha(y) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} y(t) \geq b ,$$

and so $b \leq y(s) \leq 4^m b$, $s \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Thus, for any $y \in \mathcal{P}(\alpha, b, 4^m b)$, assumption (ii) yields $f(y(s)) \geq 4^m \cdot 2Lb$, $s \in \left[\frac{1}{4}, \frac{3}{4}\right]$, and from (3.10) we obtain

$$\begin{aligned} \alpha(\mathcal{A}y) &= \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_0^1 G(t,s) f(y(s)) \, \mathrm{d}s \\ &\geq \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_{1/4}^{3/4} G(t,s) 4^m \cdot 2Lb \, \mathrm{d}s > b . \end{aligned}$$

Hence, condition (C1) of Theorem 2.1 is satisfied.

We finally exhibit that (C3) of Theorem 2.1 is satisfied. To this end, choose $y \in \mathcal{P}(\alpha, b, c)$ such that $\|\mathcal{A}y\| > 4^m b$. From (3.2) and (3.3),

$$\begin{aligned} \alpha(\mathcal{A}y) &= \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_0^1 G(t,s) f(y(s)) \, \mathrm{d}s \\ &\geq \frac{1}{4^m} \int_0^1 G(\tau(s), s) f(y(s)) \, \mathrm{d}s \\ &\geq \frac{1}{4^m} \max_{t \in [0,1]} \int_0^1 G(t,s) f(y(s)) \, \mathrm{d}s = \frac{1}{4^m} \|\mathcal{A}y\| > b . \end{aligned}$$

Therefore (C3) of Theorem 2.1 is satisfied. An application of Theorem 2.1 completes the proof. □

Remark. Hypothesis (iii) can be replaced by

$$\overline{\lim_{x \to \infty}} \frac{f(x)}{x} < K .$$

For if this is the case, then there exist $\tau > 0$ and $\sigma < K$ such that $x \geq \tau$ implies $\frac{f(x)}{x} < \sigma$. Let $\beta = \max_{x \in [0,\tau]} f(x)$. Then

$$f(x) \leq \sigma x + \beta, \qquad x \geq 0.$$

Define

$$c \geq \max\left\{\frac{\beta}{K-\sigma}, 4^{2m}b\right\}.$$

Now we observe that if $y \in \overline{\mathcal{P}}_c$, then

$$\begin{aligned}
\|\mathcal{A}y\| &\leq \max_{t \in [0,1]} \int_0^1 G(t,s)\big(\sigma y(s) + \beta\big)\, \mathrm{d}s \\
&\leq \max_{t \in [0,1]} \int_0^1 G(t,s)\big(\sigma \|y\| + \beta\big)\, \mathrm{d}s \\
&\leq \max_{t \in [0,1]} \int_0^1 G(t,s)(\sigma c + \beta)\, \mathrm{d}s < c.
\end{aligned}$$

Hence $\mathcal{A}\colon \overline{\mathcal{P}}_c \to \mathcal{P}_c$.

Acknowledgments

The authors would like to thank the referee for the remarks which greatly enhanced the final version of this paper. In particular, the referee made the observations in (3.4), (3.5) and gave the explicit representations of K and L (in terms of n and k) in (3.6), (3.7) and estimate (3.10) as well.

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Received September 29, 1998
Revised June 6, 2000

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