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ON k-PLY DOMATIC NUMBERS OF GRAPHS

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In the paper we shall generalize the domatic number of a graph introduced by E. J. Cockayne and S. T. Hedetniemi [1].

Let G be an undirected graph without loops and multiple edges, let k be a positive integer. A k-ply dominating set in G is a subset D of the vertex set $V(G)$ of G with the property that to each vertex $x \in V(G) - D$ there exist pairwise distinct vertices $y_1, \ldots, y_k$ of D which are all adjacent to x. A k-ply domatic partition of G is a partition of $V(G)$, all of whose classes are k-ply dominating sets in G. The maximum number of classes of a k-ply domatic partition of G is called the k-ply domatic number of G and denoted by $d^k(G)$.

The k-ply domatic number is defined for every graph G and every positive integer k, because in every graph there exists at least one k-ply domatic partition for every k, namely the partition consisting of one class.

The k-ply domatic number $d^k(G)$ of G is to be distinguished from the k-domatic number $d_k(G)$ of G introduced in [3].

For $k = 1$ the concepts of a k-ply dominating set, a k-ply domatic partition and a k-ply domatic number are the usual concepts of a dominating set, a domatic partition and a domatic number, as they are used in [1].

We shall describe some properties of k-ply domatic numbers of graphs.

Proposition 1. *Let G be an undirected graph, let k, l be two positive integers. Let $D_1, \ldots, D_l$ be pairwise disjoint k-ply dominating sets in G. Then $\bigcup_{i=1}^{l} D_i$ is a kl-ply dominating set in G.*

Proof. Denote $D = \bigcup_{i=1}^{l} D_i$. Let $x \in V(G) - D$. As the sets $D_1, \ldots, D_l$ are k-ply dominating, for each $i = 1, \ldots, l$ there exist pairwise distinct vertices $y_{i1}, \ldots, y_{ik}$ of D_i which are adjacent to x. As the sets $D_1, \ldots, D_l$ are pairwise disjoint, the vertices y_{ij} for $i = 1, \ldots, l$ and $j = 1, \ldots, k$ are pairwise distinct and D is a kl-ply dominating set in G.

The converse assertion is not true. In the circuit C_4 of the length 4 each pair of non-adjacent vertices is a k-ply dominating set for $k = 2$, but no proper subset of it is a k-ply dominating set for $k = 1$.

Proposition 2. *Let G be an undirected graph, let k, m be positive integers, $k \leqq m$. Then each m-ply dominating set in G is also a k-ply dominating set in G.*

Proof is straightforward.

For every real number a the symbol $[a]$ will denote the greatest integer which is less than or equal to a and the symbol $]a[$ will denote the least integer which is greater than or equal to a.

Proposition 3. *Let G be an undirected graph, let k, m be two positive integers. Then*

$$d^m(G) \geqq \left[\frac{d^k(G)}{\left]\frac{m}{k}\right[} \right].$$

Proof. Denote $d^k(G) = a$, $]m/k[= b$. Let $\mathcal{P} = \{D_1, \ldots, D_a\}$ be a k-ply domatic partition of G with a classes. The number set $\{1, \ldots, a\}$ can be partitioned into $[a/b]$ classes $C_1, \ldots, C_{[a/b]}$ such that one of them has $a + b - [a/b]b$ elements and all other classes have b elements each. Now let $D_i^* = \bigcup_{j \in C_i} D_j$. Each D_i^* is the union of at least b pairwise disjoint k-ply dominating sets in G, therefore it is bk-ply dominating and $\mathcal{P}^* = \{D_1^*, \ldots, D_{[a/b]}^*\}$ is a bk-ply domatic partition of G. We have $bk = k]m/k[\geqq m$, thus $\mathcal{P}^*$ is an m-ply domatic partition of G (see the proof of Proposition 2). The m-ply domatic number of G is at least $[a/b] = \left[\frac{d^k(G)}{\left]\frac{m}{k}\right[} \right]$.

Proposition 4. *Let G be an undirected graph, let k be a positive integer. Let $\delta(G)$ be the minimum degree of a vertex of G. Then*

$$d^k(G) \leqq [\delta(G)/k] + 1.$$

Proof. Let u be a vertex of G of degree $\delta(G)$. Let $d^k(G) = d$ and let $\{D_1, \ldots, D_d\}$ be a k-ply domatic partition of G with d classes. Without loss of generality let $u \in D_d$. For each $i = 1, \ldots, d-1$ there exist vertices $x_{i1}, \ldots, y_k$ adjacent to u and contained in D_i. The vertices x_{ij} for $i = 1, \ldots, d-1$ and $j = 1, \ldots, k$ are pairwise distinct, therefore there are at least $k(d-1)$ vertices adjacent to u and $k(d-1) \leqq \delta(G)$. This implies $d \leqq \delta(G)/k + 1$; as d is an integer, we have $d = d^k(G) \leqq [\delta(G)/k] + 1$.

Similarly as in [1] a graph G for which $\delta(G) = k(d^k(G) - 1)$ holds will be called k-ply domatically full.

Proposition 5. *Let K_n be a complete graph with n vertices, let k be a positive integer. Then*

$$d^k(K_n) = [n/k].$$

Proof. Let $\mathscr{P}$ be a partition of the vertex set of K_m into $[n/k]$ classes such that one of them has $k+n-k[n/k]$ vertices and all the others have k vertices each. Then $\mathscr{P}$ is evidently a k-ply domatic partition of K_n and $d^k(K_n) \geqq [n/k]$. On the other hand, no k-ply dominating set can have less than k vertices, therefore $d^k(K_n)$ cannot be greater than $[n/k]$ and we have $d^k(K_n)=[n/k]$.

Now we shall prove a lemma.

Lemma. *Let G be a bipartite graph on the vertex sets A, B, let k be a positive integer. Let D be a k-ply dominating set in G. Then either $A \subseteq D$, or $B \subseteq D$, or $|A \cap D| \geqq k$ and $|B \cap D| \geqq k$.*

Proof. Suppose that $|A \cap D| < k$ and $B - D \neq \emptyset$. Let $x \in B - D$. The vertex x is adjacent only to the vertices of A. The vertices of D adjacent to x are only those of $A \cap D$; but there are less than k such vertices, therefore D is not a k-ply dominating set in G, which is a contradiction. Therefore $|A \cap D| < k$ implies $B - D = \emptyset$, i.e. $B \subseteq D$. Analogously $|B \cap D| < k$ implies $A \subseteq D$ and this proves the assertion.

With the help of this lemma we shall prove a theorem.

Theorem 1. *Let $K_{m,n}$ be a complete bipartite graph on the vertex sets A, B such that $|A| = m$, $|B| = n$, let k be a positive integer. Then*

$$\begin{aligned} d^k(K_{m,n}) &= 1 \quad \text{for} \quad \min(m, n) < k, \\ d^k(K_{m,n}) &= 2 \quad \text{for} \quad k \leqq \min(m, n) < 2k, \\ d^k(K_{m,n}) &= [\min(m, n)/k] \quad \text{for} \quad \min(m, n) \geqq 2k. \end{aligned}$$

Proof. Without loss of generality let $m \geqq n$, i.e. $\min(m, n) = n$. Let $n < k$ and suppose that $d^k(K_{m,n}) \geqq 2$. Then there exists a k-ply domatic partition $\{D_1, D_2\}$ of $K_{m,n}$. We have $|B \cap D_1| \leqq |B| = n < k$; according to Lemma either $A \subseteq D_1$, or $B \subseteq D_1$. Analogously also either $A \subseteq D_2$, or $B \subseteq D_2$. Without loss of generality let $A \subseteq D_1$. As $D_1 \cap D_2 = \emptyset$, we have $B \subseteq D_2$ and this implies $A = D_1$, $B = D_2$. But then $|D_2| = n < k$ and D_2 is not a k-ply dominating set in $K_{m,n}$, which is a contradiction. We have $d^k(K_{m,n}) = 1$. Now let $k \leqq n < 2k$. Then $|A| \geqq k$, $|B| \geqq k$ and $\{A, B\}$ is a k-ply domatic partition of $K_{m,n}$, which implies $d^k(K_{m,n}) \geqq 2$. Suppose that it is greater. Then there exists a k-ply domatic partition $\{D_1, D_2, D_3\}$ of $K_{m,n}$. As $n < 2k$, at most one of the sets D_1, D_2, D_3 may have its intersection with B of the cardinality at least k. Thus without loss of generality we may suppose that $|B \cap D_1| < k$, $|B \cap D_2| < k$. This implies that either $A \subseteq D_1$, or $B \subseteq D_1$ and similarly for D_2. Without loss of generality let $A \subseteq D_1$, $B \subseteq D_2$. But then D_3 is disjoint with $A \cup B = V(K_{m,n})$, which is a contradiction. We have proved that $d^k(K_{m,n}) = 2$. Now let $n \geqq 2k$. Let $l = [n/k]$. Then there exists a partition $\{D'_1, \ldots, D'_l\}$ of A and a partition $\{D''_1, \ldots, D''_l\}$ of B such that $|D'_i| = |D''_i| = k$ for $i = 1, \ldots, l-1$ and $|D'_l| = m + k - kl$, $|D''_l| = n + k - kl$. Put $D_i = D'_i \cup D''_i$ for $i = 1, \ldots, l$. Then $\{D_1, \ldots, D_l\}$ is a k-ply domatic partition of $K_{m,n}$ and $d^k(K_{m,n}) \geqq l = [n/k]$. Any partition of $A \cup B$ with more than $[n/k]$ classes has the property that at least one of its classes has its intersection with B of the cardinality less than k and analogously to the preceding cases we can prove that it cannot be a k-ply domatic partition of $K_{m,n}$. Thus $d^k(K_{m,n}) = [n/k]$.

Theorem 2. *Let C_n be a circuit of the length n. Then $d^2(C_n) = 2$ for n even and $d^2(C_n) = 1$ for n odd.*

Proof. Let the vertices of C_n be $u_1, \ldots, u_n$ and the edges $u_i u_{i+1}$ for $i = 1, \ldots, n-1$ and $u_n u_1$. Suppose that n is even. Put $D_1 = \{u_i \mid i \text{ odd}\}$, $D_2 = \{u_i \mid i \text{ even}\}$. Evidently $\{D_1, D_2\}$ is a doubly domatic (i.e. k-ply domatic for $k = 2$) partition of C_n and $d^2(C_n) \geqq 2$. According to Proposition 4 it cannot be greater, therefore $d^2(C_n) = 2$. Now suppose that n is odd. Then C_n is not bipartite and in each partition $\mathcal{P}$ of $V(C_n)$ with two classes at least one class D contains a pair of adjacent vertices u, v. Any of the vertices u, v is adjacent to at most one vertex not belonging to D, thus no class of $\mathcal{P}$ distinct from D is a doubly dominating set in C_n and $\mathcal{P}$ is not a doubly domatic partition of C_n. We have $d^2(C_n) = 1$.

Analogously to [2] we shall study k-ply domatically critical graphs.

A graph G is called k-ply domatically critical (for a given positive integer k) if $d^k(G') < d^k(G)$ for each proper spanning subgraph G' of G.

Theorem 3. *Let G be a k-ply domatically critical graph for a positive integer k, let $d^k(G) = d$. Then the vertex set $V(G)$ of G is the union of pairwise disjoint sets $V_1, \ldots, V_d$ such that for any i, j from the numbers $1, \ldots, d$ such that $i \neq j$ the subgraph G_{ij} of G induced by the set $V_i \cup V_j$ is a bipartite graph on the sets V_i, V_j with the property that each vertex of G_{ij} has degree at least k in it and each edge of G_{ij} is incident with at least one vertex of degree k in G_{ij}.*

Proof. Let $\{V_1, \ldots, V_d\}$ be a k-ply domatic partition of G. Any V_i (for $i = 1, \ldots, d$) is an independent set in G; otherwise by deleting an edge joining two vertices of V_i the k-ply domatic number of G would not be diminished and G would not be a k-ply domatically critical graph. Let G_{ij} be the subgraph of G induced by $V_i \cup V_j$ for some i and j, $i \neq j$. As V_i, V_j are independent sets, the graph G_{ij} is a bipartite graph on the sets V_i, V_j. Any vertex of V_i must be adjacent to at least k vertices of V_j, therefore its degree in G_{ij} is at least k; analogously for each vertex of V_j. Let e be an edge of G_{ij}, let v_i (or v_j) be its end vertex in V_i (or V_j respectively). If the degrees of v_i and v_j were both greater than k, then the graph G' obtained from G by deleting e would have also the k-ply domatic number equal to d and G would not be k-ply domatically critical; this proves the assertion.

Theorem 4. *A regular graph with n vertices which is k-ply domatically full (for a positive integer k) with the k-ply domatic number d exists if and only if d divides n and $kd \leqq n$. The vertex set of such a graph G is the union of pairwise disjoint sets*

$V_1, \ldots, V_d$ such that $|V_i| = n/d$ for $i = 1, \ldots, d$ and the subgraph G_{ij} of G induced by the set $V_i \cup V_j$ for any i, j such that $i \neq j$ is a regular bipartite graph of degree k on the sets V_i, V_j.

Proof. Let G be a regular graph of degree r with n vertices which is k-ply domatically full with the k-ply domatic number d. Then $r = \delta(G) = k(d-1)$. Let $\{V_1, \ldots, V_d\}$ be a k-ply domatic partition of G with d classes. Let $i \in \{1, \ldots, d\}$, $u \in V_i$. Then the vertex u is adjacent exactly to k vertices of any V_j for $j \neq i$ and to no vertex of V_i. As i and u were chosen arbitrarily, the subgraph G_{ij} of G induced by $V_i \cup V_j$ for any i and j, $i \neq j$ is a regular bipartite graph of degree k. The number of its edges is $k|V_i| = k|V_j|$, which implies $|V_i| = |V_j|$. As i and j were chosen arbitrarily, all the sets $V_1, \ldots, V_d$ have equal cardinalities, thus $|V_i| = n/d$ for each i and d must divide n. The condition $kd \leqq n$ is evident. On the other hand, suppose that d divides n and $kd \leqq n$; we shall construct the graph G. We take the vertex set $V(G)$ with n vertices and a partition $\{V_1, \ldots, V_d\}$ of $V(G)$ with d classes, each of which has the cardinality n/d. For any i and j, $i \neq j$, we construct a regular bipartite graph G_{ij} of degree k on the sets V_i, V_j; this is always possible. The graph with the vertex set $V(G)$ and with the edge set equal to the union of edge sets of all G_{ij} is the required graph G. Evidently $\{V_1, \ldots, V_d\}$ is a k-ply domatic partition of this graph. According to Proposition 4 the k-ply domatic number of G cannot exceed d, hence it is equal to d. The graph G is a regular graph of degree $k(d-1)$.

Note that this graph G is also k-ply domatically critical. For any proper spanning subgraph G' of G we have $\delta(G') \leqq k(d-1)-1$, thus $d^k(G') \leqq [(k(d-1)-1)/k] + 1 = d - 1 < d^k(G)$.

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О k-КРАТНО ДОМАТИЧЕСКИХ ЧИСЛАХ ГРАФОВ

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Резюме

Подмножество D множества вершин $V(G)$ графа G называется k-кратно доминантным, если для каждой вершины $x \in V(G) - D$ существует k попарно различных вершин множества D, смежных с x. Разбиение множества $V(G)$, все классы которого являются k-кратно доминантными множествами в G, называется k-кратно доматическим разбиением графа G. Максимальное число классов k-кратно доматического разбиения графа G называется k-кратно доматическим числом графа G и обозначается через $d^k(G)$. Описаны некоторые свойства числа $d^k(G)$.