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ALL ORDERED SETS HAVING AMENABLE LATTICE ORDERS

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ABSTRACT. Kolibiar, Rosenberg and Schweigert proved that all compatible orders $\leq$ on the set P of a lattice $\mathbf{L} = (P; \leq^*)$ stem from 2-factor subdirect representations of $\mathbf{L}$. We denote this by $\mathbf{P}\#\mathbf{L}$ and call $\leq^*$ *amenable lattice order* of an ordered set $\mathbf{P} = (P; \leq)$. In this paper, we first give necessary and sufficient conditions for an order to be compatible with a lattice. We show that an ordered set has an amenable lattice order just if each its order component has. Further, we prove that there is a bijection between the connected compatible orders of a lattice and the pairs of complementary congruence relations on the lattice. Finally, we characterize all ordered sets having an amenable lattice order.

1. Introduction

Let $\mathbf{L}$ be a (semi)lattice on the underlying set P and $\leq^*$ be the corresponding (semi)lattice order of $\mathbf{L}$; that is, $\mathbf{L} = (P; \wedge, \vee, \leq^*)$ ($\mathbf{L} = (P; \wedge, \leq^*)$ resp.). Consider an ordered set $\mathbf{P} = (P; \leq)$ such that $\leq$ is a sub(semi)lattice of $\mathbf{L}^2$. Then we say that $\leq$ is a *compatible ordering* of $\mathbf{L}$. On the other hand, if $\mathbf{P} = (P; \leq)$ is a fixed ordered set and we consider some (semi)lattice order $\leq^*$ on the set P; that is, $\mathbf{L} = (P; \leq^*)$ is a (semi)lattice such that $\leq$ is a compatible ordering of $\mathbf{L}$, then we say that $\leq^*$ is an *amenable* (*semi*)*lattice order of* $\mathbf{P} = (P; \leq)$ or it is said that $\leq^*$ is a (*semi*)*lattice order amenable with* $\leq$.

Bounded compatible orderings of lattices were studied by G. Czédli, A. P. Huhn and L. Szabó in [2]. I. G. Rosenberg and D. Schweigert studied compatible orderings and tolerances of lattices in [8]. Compatible orderings in semilattices were studied by M. Kolibiar in [5]. In [2] and [8] it was shown that compatible lattice orderings in a lattice $\mathbf{L}$ are in one-to-one correspondence with the set of all direct decomposition of $\mathbf{L}$ into two factors.

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In [5] and [8] there was proved that all compatible orders $\leq$ on the set P of a lattice $\mathbf{L} = (P, \leq^*)$ stem from two factor subdirect representations of $\mathbf{L}$. We denote this by $\mathbf{P}\#\mathbf{L}$.

In Section 2, we first prove some properties of compatible orders of a (semi)lattice and give necessary and sufficient conditions for an order to be compatible with a lattice (Theorem 2). In Section 3, we show that an ordered set has an amenable lattice order just if each its order components has (Theorem 3). Further we prove that there is a bijection between the connected compatible orders of a lattice and the pairs of complementary congruence relations on the lattice (Corollary 5). It is shown that if $\mathbf{P}$ is a connected ordered set having an amenable lattice order, then $\mathbf{P}$ satisfies the upper bound property and the lower bound property (Lemma 6). Finally we characterize all ordered sets having an amenable lattice order (Theorem 4, Theorem 5). There are given two examples of ordered sets which have no amenable lattice orders.

2. Amenable lattice orders and subdirect representations

Let P be a set and let $\leq$ and $\leq^*$ be orders defined on P. If $a, b \in P$ with $a \leq b$, we define $[a, b]$ be the set of elements in P between a and b; that is,

$$[a, b] = \{x \in P : \ a \leq x \leq b\}.$$

Similarly, we define $[a, b]^* = \{x \in P : \ a \leq^* x \leq^* b\}$. We have the following.

Lemma 1. *Let $\leq$ be a compatible ordering of a lattice $\mathbf{L} = (P; \wedge, \vee, \leq^*)$. Then*

(i) *$a \leq b$ implies that $a \wedge b$ and $a \vee b$ belong to $[a, b]$,*
(ii) *$a \leq b$ and $a \leq^* b$ imply that $[a, b] = [a, b]^*$,*
(iii) *$a \leq b$ and $b \leq^* a$ imply that $[a, b] = [b, a]^*$.*

Proof.

(i) follows immediately from the definition of compatible ordering $\leq$.

To prove (ii), let $a \leq b$ and $a \leq^* x \leq^* b$. Then $a = a \wedge x \leq b \wedge x = x = a \vee x \leq b \vee x = b$ since $\leq$ is compatible with $\vee$ and $\wedge$. Conversely, let $a \leq^* b$ and $a \leq x \leq b$. Then $a \leq x \leq b$ implies $a \leq a \wedge x$ and $b \wedge x \leq b$, which together with $a \wedge b = a$ yields $a \wedge x = a \wedge b \wedge x = a \wedge b = a$. Therefore, $a \leq^* x$. A similar argument with $a \vee b = b$ gives $x \leq^* b$.

By duality, we get (iii). □

Given $a, b \in P$ we write $a \prec b$ (resp $a \prec^* b$) if $a < b$ (resp. $a <^* b$) and the interval $[a, b]$ (resp. $[a, b]^*$) is a two element set.

COROLLARY 1. *Let $\leq^*$ be an amenable lattice order of* $\mathbf{P} = (P; \leq)$. *If* $a \prec b$, *then* $a \prec^* b$ *or* $b \prec^* a$.

Proof. Assume that $a \prec b$. Then $a \leq b$ implies $a \vee b \in [a, b] = \{a, b\}$; hence, $a \vee b = a$ or $a \vee b = b$; that is, $a \leq^* b$ or $b \leq^* a$. By Lemma 1, we have $[a, b]^* = [a, b] = \{a, b\}$ or $[b, a]^* = [a, b] = \{a, b\}$ which shows that a $\prec^* b$ or $b \prec^* a$. □

For a (semi)lattice $\mathbf{L}$, we denote the lattice of congruences by $\operatorname{Con}\mathbf{L}$ with smallest element ω; the identity relation. The dual of an ordered set $\mathbf{P} = (P; \leq)$ is denoted by $\mathbf{P}^\partial = (P; \leq^\partial)$. The set of all equivalence classes of an equivalence relation θ on an order set $\mathbf{P}$ and the equivalence class containing an $a \in P$ are denoted by P/θ and $[a]\theta$; respectively.

Let $\mathbf{L} = (P; \leq^*)$ be a (semi)lattice. If θ_1 and θ_2 are congruence relations of $\mathbf{L}$ with $\theta_1 \cap \theta_2 = \omega$, M. Kolibiar [5; Lemma 2.1] showed that there exists an injective map $a \mapsto (a_1, a_2)$ from P into $P/\theta_1 \times P/\theta_2$ and the binary relation $\leq$ defined on P by

$$a \leq b \iff \begin{cases} a_1 \geq^* b_1 \ (\text{in } P/\theta_1) \text{ and } a_2 \leq^* b_2 \ (\text{in } P/\theta_2) \\ \text{or} \\ \text{the image of } a \text{ is smaller than the image of } b \\ \text{in the direct product } (P/\theta_1)^\partial \times P/\theta_2 \end{cases}$$

is a compatible ordering of $\mathbf{L}$.

DEFINITION 1. We say that $\leq$ *stems* from the 2-factor subdirect representation (θ_1, θ_2) of $\mathbf{L}$ and we will write $\mathbf{P}\#\mathbf{L}$ where $\mathbf{P} = (P; \leq)$.

Now let $\leq$ be a compatible ordering of a lattice $\mathbf{L} = (P; \wedge, \vee, \leq^*)$. Define relations θ_1 and θ_2 on P as follows:

$$\begin{aligned} a\,\theta_1\,b &\iff a \leq^* u \geq^* b \ \&\ a \leq u \geq b, \\ a\,\theta_2\,b &\iff a \leq^* v \geq^* b \ \&\ a \geq v \leq b \end{aligned} \tag{2.1}$$

for some $u, v \in P$; or

$$\begin{aligned} a\,\theta_1\,b &\iff a \geq^* u \leq^* b \ \&\ a \geq u \leq b, \\ a\,\theta_2\,b &\iff a \geq^* v \leq^* b \ \&\ a \leq v \geq b \end{aligned} \tag{2.2}$$

for some $u, v \in P$.

Kolibiar [5], Rosenberg and Schweigert [8] proved that θ_1 and θ_2, as defined either by (2.1) or (2.2), are congruence relations of the semilattices $(P; \vee)$ or $(P; \wedge)$, respectively. We prove the following.

Lemma 2. *If* $\mathbf{L} = (P; \leq^*)$ *is a semilattice, then the congruence relations* θ_1 *and* θ_2 *as defined in* (2.1) *or* (2.2) *are the transitive closure of* R_1 *and* R_2 *where* R_1 *is the set of all pairs* $(a,b) \in P^2$ *such that either*

$$(a \leq^* b \ \textit{and}\ a \leq b) \ \textit{or}\ (a \geq^* b \ \textit{and}\ a \geq b)$$

and R_2 *is the set of all pairs* $(a,b) \in P^2$ *such that either*

$$(a \leq^* b \ \textit{and}\ a \geq b) \ \textit{or}\ (a \geq^* b \ \textit{and}\ a \leq b).$$

Moreover, if $\mathbf{L}$ *is a lattice, then the congruence relations* θ_1 *and* θ_2 *defined in* (2.1) *are the same congruence relations defined in* (2.2).

Proof. It is enough to show that $R_i \subseteq \theta_i \subseteq \bar{R}_i$ $(i = 1,2)$. Let $(a,b) \in R_1$. Then either $(a \leq^* b$ and $a \leq b)$ or $(b \leq^* a$ and $b \leq a)$. Hence, either $(a \leq^* b \geq^* b$ and $a \leq b \geq b)$ or $(b \leq^* a \geq^* a$ and $b \leq a \geq a)$ proves $a\,\theta_1\,b$.

Now let $a\,\theta_1\,b$. Then there is $u \in P$ such that $a \leq^* u \geq^* b$ and $a \leq u \geq b$ which shows that (a,u) and (u,b) are elements of R_1; hence $(a,b) \in \bar{R}_1$.

We can prove $R_2 \subseteq \theta_2 \subseteq \bar{R}_2$ analogously. □

The following useful result is proved by Kolibiar [5], Rosenberg and Schweigert [8].

Theorem 1. ([5], [8]) *The following statements are equivalent for a compatible ordering* $\leq$ *of a semilattice* $\mathbf{L} = (P; \leq^*)$ *and the corresponding congruence relations* θ_1 *and* θ_2,

(i) $\theta_1 \cap \theta_2 = \omega$ *and* $\leq$ *stems from the subdirect representation given by* θ_1 *and* θ_2,

(ii) *each interval* $\{x \in P : \ a \leq x \leq b\}$ *is a convex subset of* $\mathbf{L}$,

(iii) *if* $a \leq^* b \leq^* c$, *then* $a \leq c$ *implies* $a \leq b \leq c$, *and* $c \leq a$ *implies* $c \leq b \leq a$.

In the proof of Theorem 1, we can see that $a \leq^* b$ and $a\,\theta_1\,b$ imply that $a\,\theta_1\,b\,\theta_2\,b$ and $a \leq^* b \geq^* b$; hence $a \leq b$. Analogously, if $a \leq^* b$ and $a\,\theta_2\,b$, then $a \geq b$.

Corollary 2. *For* $a,b \in P$,

(i) *if* $a \leq^* b$ *and* $a\,\theta_1\,b$, *then* $a \leq b$,

(ii) *if* $a \leq^* b$ *and* $a\,\theta_2\,b$, *then* $b \leq a$.

Now, if $\mathbf{L}$ is a lattice, Lemma 1 shows that condition (iii) of Theorem 1 always holds. In [5] and [8], they showed that the map $\leq \mapsto (\theta_1, \theta_2)$ induced a bijection between the set of compatible orderings of a lattice and the set of orders stemming from 2-factor subdirect representation of the lattice. We have the following as its corollary.

COROLLARY 3. *Let* $\mathbf{L}$ *be a lattice. Then every congruence* θ *on* $\mathbf{L}$ *gives rise to compatible orders* $\leq$ *and* $\leq^{\partial}$ *where* $\leq$ *is given by*

$$a \leq b \iff a \leq^* b \ \&\ a\,\theta\, b\,.$$

Morover, if $\mathbf{L}$ *is subdirectly irreducible, then every compatible ordering of* $\mathbf{L}$ *arises in this way.*

Proof. If $\mathbf{L}$ is subdirectly irreducible, then $\theta_1 \cap \theta_2 = \omega$ implies $\theta_1 = \omega$ or $\theta_2 = \omega$. □

If an order $\leq$ is compatible with $\vee$ and $\wedge$ of a lattice $\mathbf{L} = (P; \leq^*)$ and θ_1, θ_2 are defined as in (2.1), then it follows from Theorem 1 that θ_1 and θ_2 are congruence relations of $\mathbf{L}$ with $\theta_1 \cap \theta_2 = \omega$ and one can easily show that

(a) $(\theta_1 \cap \leq^*)$ and $(\theta_2 \cap \geq^*)$ are suborders of $\leq$,
(b) $(\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*) = \{(x,y) \in P^2 :\ x\,\theta_1\, u\,\theta_2\, y$ and $x \leq u \geq y$ for some $u \in P\}$ is compatible with $\vee$ and $\wedge$ of $\mathbf{L}$ and is equal to $\leq$.

Therefore, $a\theta_1 b$ implies $(a\wedge b)\theta_1 a\theta_2 a\theta_1 (a\vee b)\theta_2 (a\vee b)$; that is, $a\wedge b \leq a \leq a\vee b$. Similarly, $a\theta_2 b$ implies $a\vee b \leq a \leq a\wedge b$. Moreover, if $a \leq b$, then $a\theta_1 u\theta_2 b$ and $a \leq^* u \geq^* b$ for some $u \in P$. Therefore, $a \leq^* a \vee b \leq^* u$ and $b \leq^* a \vee b \leq^* u$ yield $(a \vee b, u) \in \theta_1 \cap \theta_2$; that is, $u = a \vee b$.

COROLLARY 4. *Let* $\leq$ *be a compatible ordering of a lattice* $\mathbf{L} = (P; \leq^*)$ *and let* θ_1 *and* θ_2 *be defined as in* (2.1). *Then for* $a, b \in P$,

(i) $a\,\theta_1\, b$ *implies* $a \wedge b \leq a, b \leq a \vee b$,
(ii) $a\,\theta_2\, b$ *implies* $a \vee b \leq a, b \leq a \wedge b$,
(iii) $a \leq b$ *implies* $a\,\theta_1\,(a \vee b)\,\theta_2\, b$ *and* $a\,\theta_2\,(a \wedge b)\,\theta_1\, b$,
(iv) $a \prec b$ *implies* $a\,\theta_1\, b$ *or* $a\,\theta_2\, b$.

THEOREM 2. *Let* $\mathbf{P} = (P; \leq)$ *be an ordered set and* $\mathbf{L} = (P; \leq^*)$ *be a lattice. Then* $\mathbf{P}\#\mathbf{L}$ *if and only if there are lattices* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ (*with underlying sets* L_1 *and* L_2) *and a map* $\psi\colon P \to L_1 \times L_2$ *such that*

(i) ψ *is a lattice embedding of* $\mathbf{L}$ *into* $\mathbf{L}_1 \times \mathbf{L}_2$,
(ii) ψ *is an order embedding of* $\mathbf{P}$ *into* $\mathbf{L}_1^{\partial} \times \mathbf{L}_2$.

Proof. The forward implication follows from Theorem 1 since θ_1 and θ_2 defined as in (2.1) are congruence relations of L with $\theta_1 \cap \theta_2 = \omega$. Conversely, let θ_1 and θ_2 be congruence relations of $\mathbf{L}$ corresponding to $\mathbf{L}_1$ and $\mathbf{L}_2$, and identify $\mathbf{L}_1$ with $\mathbf{L}/\theta_1$ and $\mathbf{L}_2$ with $\mathbf{L}/\theta_2$ respectively. Then $\theta_1 \cap \theta_2 = \omega$. It remains to show that the order relation $\leq$ of $\mathbf{L}_1^{\partial} \times \mathbf{L}_2$ is the compatible ordering $(\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$ of $\mathbf{L}$. Let $a \leq b$. Then $\psi(a) \leq \psi(b)$. So $[a]\theta_1 \geq^* [b]\theta_1$ and $[a]\theta_2 \leq^* [b]\theta_2$ yield $a\,\theta_1\,(a \vee b)\,\theta_2\, b$; that is, $(a,b) \in (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. If $(a,b) \in (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$, it follows from the argument above Corollary 4 that $a\,\theta_1\,(a\vee b)\,\theta_2\,b$ which yields $\big([a]\theta_1,[b]\theta_2\big) = \psi(a\vee b)\in \operatorname{Im}\psi$. Therefore $\psi(a)\le\psi(a\vee b)$. Since ψ is an order-embedding of $\mathbf{P}$ into $\mathbf{L}_1^{\partial}\times\mathbf{L}_2$, we have $a\le a\vee b$. Analogously $a\vee b\le b$. Hence $a\le b$. □

3. Connected ordered sets having amenable lattice orders

Let $\le$ be an order on a set P and let $\le^c$ denote the equivalence closure of $\le$; that is, the smallest equivalence relation on P containing $\le$. Then for $a_1,a_2,\dots,a_n\in P$, $a_1\le a_2\ge a_3\le\cdots\le a_n$ implies $a_1\le^c a_n$. Hence, if θ is the set of all pairs $(a,b)\in P^2$ such that a and b are in the same component, then θ is a subset of $\le^c$. But, in fact, θ is an equivalence relation on P containing $\le$. Therefore, $\theta = \le^c$. Moreover, if $\le$ is a compatible ordering of a lattice $\mathbf{L}=(P;\le^*)$, then θ is a congruence relation of $\mathbf{L}$. Let $a\in P$ and $x,y,z\in[a]\theta$ with $x\le y$. Since $\le$ and θ are compatible with $\wedge$ and $\vee$ of $\mathbf{L}$, we have $x\wedge z\le y\wedge z$, $x\vee z\le y\vee z$, $(x\wedge z)\,\theta\,a\,\theta\,(y\wedge z)$ and $(x\vee z)\,\theta\,a\,\theta\,(y\vee z)$. This shows that each block of θ is amenable with the corresponding order-component.

LEMMA 3. *Let $\mathbf{L}$ be an amenable lattice order of an ordered set $\mathbf{P}$. Then each order-component of $\mathbf{P}$ has an amenable lattice order which is a convex sublattice of $\mathbf{L}$.*

Conversely, let $\mathbf{P}=\Big(\bigcup_{i\in I}P_i;\le\Big)$ be an ordered set where $P_i\cap P_j=\emptyset$ if $i\neq j$, let $\mathbf{L}_i=(P_i;\le_i^*)$ be an amenable lattice order of $\mathbf{P}_i=(P_i;\le)$ for all $i\in I$ and let $<$ be a strict total order of I. Define a binary relation $\le^*$ on $P=\bigcup_{i\in I}P_i$ as follows:

(i) $a\le^* b \iff a\le_i^* b$ whenever $a,b\in P_i$ for some $i\in I$,

or

(ii) $a\le^* b \iff a\in P_i,\ b\in P_j$ for $i<j$.

Then, clearly, $\le^*$ is a lattice order on P such that $\le$ is a compatible ordering of $(P;\le_i^*)$.

THEOREM 3. *An ordered set has an amenable lattice order just if each its order components has.*

We shall now prove that the compatible orders of a lattice arising from complementary pairs of congruences are precisely the connected compatible orders. Moreover, connected compatible orders of a lattice satisfy the upper and lower bound properties (LBP and UBP) definded below.

LEMMA 4. *Let $\le$ be a connected compatible order of a lattice $\mathbf{L}=(P;\wedge,\vee,\le^*)$ and let θ_1 and θ_2 be as in* (2.1). *Then θ_1 is the complement of θ_2 in* $\operatorname{Con}\mathbf{L}$.

Proof. It remains to show that $\theta_1 \vee \theta_2 = P \times P$. Let $a, b \in P$. Since $\leq$ is connected, there are elements $a = a_0, a_1, \ldots, a_n = b$ such that $a_i \leq a_{i+1}$ or $a_{i+1} \leq a_i$ for all $i = 0, 1, \ldots, n-1$ which yields from Corollary 4(iii) that either $a_i \, \theta_1 \, (a_i \vee a_{i+1}) \, \theta_2 \, a_{i+1}$ or $a_{i+1} \, \theta_1 \, (a_i \vee a_{i+1}) \, \theta_2 \, a_i$. In either cases, we have $(a_i, a_{i+1}) \in \theta_1 \vee \theta_2$ for all $i = 0, 1, \ldots, n-1$. Hence, by the transitivity of $\theta_1 \vee \theta_2$, we have $(a, b) \in \theta_1 \vee \theta_2$. □

Remark. Let $\mathbf{L}$ be a lattice and let θ_1 and θ_2 be congruence relations of $\mathbf{L}$. It is known ([3]) that $(a, b) \in \theta_1 \vee \theta_2$ if and only if there is a sequence $a \wedge b = z_0 \leq^* z_1 \leq^* \cdots \leq^* z_n = a \vee b$ such that $z_0 \, \theta_1 \, z_1 \, \theta_2 \, z_2 \ldots \theta_2 \, z_n$.

Lemma 5. *Let $\mathbf{L} = (P; \wedge, \vee, \leq^*)$ be a lattice and let θ_1 and θ_2 be a complementary pair of congruences of $\mathbf{L}$. Then the compatible ordering $(\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$ is connected.*

Proof. Denote the compatible order $(\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$ by $\leq$. Let $a, b \in P$. Then $(a, b) \in \theta_1 \vee \theta_2$ and hence there is a sequence $a \wedge b = z_0 \leq^* z_1 \leq^* \cdots \leq^* z_n = a \vee b$ such that for each $i = 0, 1, \ldots, n-1$ we have either $z_i \, \theta_1 \, z_{i+1}$ or $z_i \, \theta_2 \, z_{i+1}$. It follows from Corollary 3 that $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$ for all $i = 0, 1, \ldots, n-1$.

By using $a = a \vee (a \wedge b)$ and either $(a \vee z_{i+1}) \theta_1 (a \vee z_i)$ or $(a \vee z_{i+1}) \theta_2 (a \vee z_i)$ we have either $a \vee z_i \leq a \vee z_{i+1}$ or $a \vee z_{i+1} \leq a \vee z_i$. By a symmetric proof we obtain either $b \wedge z_i \leq b \wedge z_{i+1}$ or $b \wedge z_{i+1} \leq b \wedge z_i$ for all $i = 0, 1, \ldots, n-1$. Therefore, we have a sequence $a = c_0 = a \vee z_0$, $c_1 = a \vee z_1$, $\ldots$, $c_n = a \vee z_n = a \vee b = z_n$, $c_{n+1} = z_{n-1}$, $\ldots$, $c_{2n} = z_0 = a \wedge b = z_0 \wedge b$, $c_{2n+1} = z_1 \wedge b$, $\ldots$, $c_{3n} = z_n \wedge b = b$ such that either $c_i \leq c_{i+1}$ or $c_{i+1} \leq c_i$ for all $i = 0, 1, \ldots, 3n$. Hence, $\leq$ is connected. □

The following corollaries follow directly from Corollary 4, Lemma 4 and Lemma 5.

Corollary 5. *The map $\leq \mapsto (\theta_1, \theta_2)$ induced a bijection between the connected compatible orderings of a lattice and the pairs of complementary congruence relations on the lattice.*

Corollary 6. *If $\mathbf{L}$ is a subdirectly irreducible lattice, then $\leq$ and $\geq$ are the only connected compatible order of $\mathbf{L}$.*

We say that an ordered set $\mathbf{P}$ satisfies the *lower bound property* (*LBP*) if any pairs of elements of $\mathbf{P}$ which have a lower bound have a greatest lower bound. Dually, $\mathbf{P}$ satisfies the *upper bound property* (*UBP*) if any pairs of elements of $\mathbf{P}$ which have an upper bound have a least upper bound.

We shall now show that a connected compatible order of a lattice satisfies the lower bound property and the upper bound property.

LEMMA 6. *Let* $\mathbf{P}$ *be a connected ordered set having an amenable lattice order. Then* $\mathbf{P}$ *satisfies LBP and UBP.*

P r o o f. Let $\mathbf{P} = (P; \leq)$ be a connected ordered set and let $\mathbf{L} = (P; \wedge, \vee, \leq^*)$ be an amenable lattice order of $\mathbf{P}$. Let $\mu(x,y,z)$ denote a ternary function $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ in $\mathbf{L}$. Since $\leq$ is compatible with $\wedge$ and $\vee$, the function μ is monotone with respect to both $\leq$ and $\leq^*$. Let $a, b, u, l \in P$ with $a \leq u$, $b \leq u$, $l \leq a$ and $l \leq b$. It is easily seen that the upper bound $\mu(a,b,u)$ and the lower bound $\mu(a,b,l)$ of a and b are minimal and maximal respectively. By Theorem 2, since the order relation of $\mathbf{L}_1^{\partial} \times \mathbf{L}_2$ is compatible with the operations of $\mathbf{L}_1 \times \mathbf{L}_2$, a minimal upper bound $\mu(a,b,u)$ and a maximal lower bound $\mu(a,b,l)$ are unique. □

Let $\mathbf{C} = \langle C; \leq^* \rangle$ be an infinite chain, $P = C \cup \{a, b, c\}$ where $a, b, c \notin C$. Define an order relation $\leq$ on P as follows:

(i) $x \leq y \iff x \leq^* y$ for all $x, y \in C$,
(ii) $x \leq y$ for all $x \in C$ and $y \in \{a, b, c\}$,
(iii) $a \leq c \geq b$.

Then, $\mathbf{P} = (P; \leq)$ is an example of ordered sets which does not satisfy the lower bound property and hence it has no amenable lattice order.

Let $\mathbf{P} = \langle P; \leq \rangle$ be an ordered set and θ be an equivalence relation on P. Define a binary relation $\leq_\theta$ on P/θ by

$$[a]\theta \leq_\theta [b]\theta \iff a\,\theta\, c \leq d\,\theta\, b \qquad \text{for some} \quad c, d \in P\,.$$

Then $\leq_\theta$ need not be transitive. Let $\leq_\theta^t$ denote the transitive closure of $\leq_\theta$. It was proved in [2] that if $\leq$ is a compatible ordering of a lattice $\mathbf{L}$ and θ is also a congruence relation of $\mathbf{L}$, then $\leq_\theta^t$ is an order on P/θ.

LEMMA 7. ([2]) *Let* $\mathbf{L} = (P; \wedge, \vee, \leq^*)$ *be a lattice and let* θ *be a congruence relation of* $\mathbf{L}$. *If* $\leq$ *is a compatible ordering of a lattice* $\mathbf{L}$, *then* $\leq_\theta^t$ *is an order on* P/θ. *Moreover,* $\leq_\theta^t$ *is a compatible ordering of* $(P/\theta; \leq^*)$.

We shall now characterize all ordered sets which have an amenable lattice order.

THEOREM 4. *Let* $\mathbf{P} = \langle P; \leq \rangle$ *be a connected ordered set,* $\leq^*$ *be an amenable lattice order of* $\mathbf{P}$ *and let* θ_1 *and* θ_2 *be defined as in* (2.1). *Then*

(i) $(P/\theta_1; \leq_{\theta_1}^t)$ *and* $(P/\theta_2; \leq_{\theta_2}^t)$ *are lattices;*
(ii) *denote the join and meet on* $(P/\theta_1; \leq_{\theta_1}^t)$ *and on* $(P/\theta_2; \leq_{\theta_2}^t)$ *by* $+$, $\cdot$ *and* $\cup$, $\cap$ *respectively; then for* $a, b \in P$ *there are unique* $c, d \in P$ *such that* $c \in [a]\theta_1 \cdot [b]\theta_1$, $c \in [a]\theta_2 \cup [b]\theta_2$, $d \in [a]\theta_1 + [b]\theta_1$ *and* $d \in [a]\theta_2 \cap [b]\theta_2$;
(iii) *if* a *and* b *are noncomparable, then* $(a, c) \in \theta_1$ *implies* $(b, c) \notin \theta_2$ *and* $(b, d) \in \theta_2$ *implies* $(a, d) \notin \theta_1$.

Proof. Let $\vee$ and $\wedge$ denote the join and meet operations of the lattice $\mathbf{L} = (P; \leq^*)$. By the assumption and an application of Theorem 1, we have $\theta_1 \cap \theta_2 = \omega$, $\leq = (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$, and $(P/\theta_1; \wedge, \vee, \leq^*)$ and $(P/\theta_2; \wedge, \vee, \leq^*)$ are lattices. Hence the natural map $\psi \colon P \to P/\theta_1 \times P/\theta_2$ is a lattice embedding of $\mathbf{L}$ into $\mathbf{P}/\theta_1 \times \mathbf{P}/\theta_2$ and an order embedding of $\mathbf{P}$ into $(\mathbf{P}/\theta_1)^{\partial} \times \mathbf{P}/\theta_2$ respectively.

(i) It remains to show that $\leq^t_{\theta_1}$ and $\leq^t_{\theta_2}$ are the restrictions of $\geq^*$ to P/θ_1 and of $\leq^*$ to P/θ_2 respectively. It is clear that $a \leq b$ implies $[a]\theta_1 \geq^* [b]\theta_1$ and $[a]\theta_2 \leq^* [b]\theta_2$. This shows that $\leq_{\theta_1}$ is a subset of $\geq^*$ restricted to P/θ_1 and $\leq_{\theta_2}$ is a subset of $\leq^*$ restricted to P/θ_2; so are $\leq^t_{\theta_1}$ and $\leq^t_{\theta_2}$.

Now, let $a, b \in P$ with $[a]\theta_1 \geq^* [b]\theta_1$. Then $a \; \theta_1 \; (a \vee b)$ and $b \; \theta_1 \; (a \wedge b)$. According to Lemma 3 and the remark, we have a sequence $a \wedge b = z_0 \leq^* z_1 \leq^* \cdots \leq^* z_n = a \vee b$ such that $z_0 \; \theta_2 \; z_1 \; \theta_1 \; z_2 \ldots \theta_2 \; z_n$. It follows from Corollary 2 with $z_{2m} \; \theta_2 \; z_{2m+1}$ and $z_{2m} \leq^* z_{2m+1}$ for $0 \leq m < n$ that $z_{2m+1} \leq z_{2m}$ for all $0 \leq m < n$. Therefore $[b]\theta_1 = [z_0]\theta_1 \geq [z_1]\theta_1 \geq \cdots \geq [z_n]\theta_1 = [a]\theta_1$; that is, $[a]\theta_1 \leq^t_{\theta_1} [b]\theta_1$. Hence the restriction of $\geq^*$ to P/θ_1 is a subset of $\leq^t_{\theta_1}$. Analogously, the restriction of $\leq^*$ to P/θ_2 is a subset of $\leq^t_{\theta_2}$.

Denote the join and meet on the lattices $\mathbf{P}/\theta_1 = (P/\theta_1; \leq^t_{\theta_1})$ and $\mathbf{P}/\theta_2 = (P/\theta_2; \leq^t_{\theta_2})$ by $+$, $\cdot$ and $\cup$, $\cap$; respectively.

(ii) Since $\psi(a \vee b) = ([a]\theta_1 \vee [b]\theta_1, [a]\theta_2 \vee [b]\theta_2) = ([a]\theta_1 \cdot [b]\theta_1, [a]\theta_2 \cup [b]\theta_2)$ and $\psi(a \wedge b) = ([a]\theta_1 \wedge [b]\theta_1, [a]\theta_2 \wedge [b]\theta_2) = ([a]\theta_1 + [b]\theta_1, [a]\theta_2 \cap [b]\theta_2)$, we have $a \vee b$ and $a \wedge b$ corresponding to c and d in condition (ii).

Condition (iii) is obvious from (ii) since $\leq$ is equal to $(\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. □

We shall now use Theorem 4 to give an example of ordered sets having no amenable lattice orders. Let $\mathrm{P} = \{0, 1, 2, 3, 4, 5, 6\}$ and $\leq$ be an order on P defined by $0 \leq 1 \leq 2$, $0 \leq 4 \leq 3 \leq 2$, $5 \leq 4$, $5 \leq 6 \leq 3$. Suppose that $\mathbf{L} = (P; \leq^*)$ is an amenable lattice order of $\mathbf{P} = (P; \leq)$. According to Corollary 1, we have that $(A = \{0, 1, 2, 3, 4\}; \leq^*)$ is a cover-preserving sublattice of $\mathbf{L}$ isomorphic to the subdirectly irreducible lattice $\mathbf{N}_5$; hence, Corollary 6 implies that $(A; \leq^*)$ is either $(A; \leq)$ or $(A; \leq)^{\partial}$.

Let θ_1 and θ_2 be defined as in (2.1) and denote the restriction of θ_1 and θ_2 to A by $\theta_1|_A$ and $\theta_2|_A$ respectively. Then one of $\theta_1|_A$ or $\theta_2|_A$ is the identity relation ω and the other is the universal relation $\iota = A \times A$. We may assume that $\theta_1|_A = \omega$ and $\theta_2|_A = \iota$. One can show by using Theorem 1 and Corollary 4 that $3 \; \theta_1 \; 6$, $4 \; \theta_1 \; 5$ and $6 \; \theta_2 \; 5$. Hence, $(P/\theta_1, \leq^t_{\theta_1})$ is $\mathbf{N}_5$ and $(P/\theta_2, \leq^t_{\theta_2})$ is a 2-element chain. For $1, 5 \in P$, we have $[1]\theta_1 + [5]\theta_1 = [2]\theta_1 = \{2\}$ and $[1]\theta_2 \cap [5]\theta_2 = [5]\theta_2 = \{5, 6\}$ which have an empty intersection which contradicts condition (ii) of Theorem 4. If we assume that $\theta_1|_A = \iota$ and $\theta_2|_A = \omega$, then we get a similar contradiction. Hence, $\mathbf{P}$ has no amenable lattice order.

THEOREM 5. *Let $\mathbf{P} = \langle P; \leq \rangle$ be an ordered set and let θ_1 and θ_2 be equivalence relations on P satisfying conditions* (i), (ii), *and* (iii) *of Theorem* 4. *Then there is a lattice $\mathbf{L}$ such that $\mathbf{P}\#\mathbf{L}$.*

P r o o f. Define a binary relation $\leq^*$ on P as follows:

$$a \leq^* b \iff b \in [a]\theta_1 \cdot [b]\theta_1 \ \ \& \ \ b \in [a]\theta_2 \cup [b]\theta_2 .$$

Let $a, b \in P$. Then there exists an element $c \in P$ such that $[c]\theta_1 = [a]\theta_1 \cdot [b]\theta_1$ and $[c]\theta_2 = [a]\theta_2 \cup [b]\theta_2$. Hence, $[c]\theta_1 \leq^t_{\theta_1} [a]\theta_1$ and $[a]\theta_2 \leq^t_{\theta_2} [c]\theta_2$; which show that $[a]\theta_1 \cdot [c]\theta_1 = [c]\theta_1$ and $[a]\theta_2 \cup [c]\theta_2 = [c]\theta_2$; or equivalently, $c \in [a]\theta_1 \cdot [c]\theta_1$ and $c \in [a]\theta_2 \cup [c]\theta_2$. Thus $a \leq^* c$. By analogy, we have $b \leq^* c$. Now let $u \in P$ be such that $a \leq^* u$ and $b \leq^* u$. Then $[u]\theta_1 = [a]\theta_1 \cdot [b]\theta_1 \cdot [u]\theta_1 = [c]\theta_1 \cdot [u]\theta_1$ and $[u]\theta_2 = [a]\theta_2 \cup [b]\theta_2 \cup [u]\theta_2 = [c]\theta_2 \cup [u]\theta_2$; that is, $c \leq^* u$. Therefore, c is the least upper bound of a and b with respect to $\leq^*$; and we can prove analogously that every pair of elements in P has the greatest lower bound with respect to $\leq^*$. Hence, we have that $\leq^*$ is a lattice order on P. Let $\vee$ and $\wedge$ denote the join and meet operations of the lattice $\mathbf{L} = (P; \leq^*)$. To show that θ_1 and θ_2 are congruence relations of $\mathbf{L}$, let $a, b, c \in P$ with $a\,\theta_1\,b$. Then $a \vee c \in [a]\theta_1 \cdot [c]\theta_1 = [b]\theta_1 \cdot [c]\theta_1$ and $b \vee c \in [b]\theta_1 \cdot [c]\theta_1$ imply $(a \vee c)\,\theta_1\,(b \vee c)$. Analogously, we have $(a \wedge c)\,\theta_1\,(b \wedge c)$. A similar argument yields $(a \vee c)\,\theta_2\,(b \vee c)$ and $(a \wedge c)\,\theta_2\,(b \wedge c)$.

Since condition (ii) implies $\theta_1 \cap \theta_2 = \omega$, the natural map $\psi\colon P \to P/\theta_1 \times P/\theta_2$ is a lattice embedding of $\mathbf{L}$ into $\mathbf{L}/\theta_1 \times \mathbf{L}/\theta_2$. Now $[a]\theta_1 \leq^t_{\theta_1} [b]\theta_1$ if and only if $[a \vee b]\theta_1 = [a]\theta_1 \cdot [b]\theta_1 = [a]\theta_1$ if and only if $[a]\theta_1 = [a \vee b]\theta_1 \geq^* [b]\theta_1$. Thus $(P/\theta_1; \leq^t_{\theta_1}) \cong (P/\theta_1; \geq^*) \cong (\mathbf{L}/\theta_1)^{\partial}$. Similarly, $(P/\theta_2; \leq^t_{\theta_2}) \cong (P/\theta_2; \leq^*) \cong \mathbf{L}/\theta_2$. Finally, we will show that ψ is an order embedding of $\mathbf{P}$ into $(\mathbf{L}/\theta_1)^{\partial} \times \mathbf{L}/\theta_2$; this is equivalent to prove that $\leq$ is equal to $(\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. If $a \leq b$, then $[a]\theta_1 \leq^t_{\theta_1} [b]\theta_1$ and $[a]\theta_2 \leq^t_{\theta_2} [b]\theta_2$ imply that $[a]\theta_1 = [a \vee b]\theta_1$ and $[a \vee b]\theta_2 = [b]\theta_2$; that is, $a\,\theta_1\,(a \vee b)\,\theta_2\,b$ which together with $a \leq^* a \vee b \geq^* b$ yields $(a, b) \in (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. Now let $(a, b) \in (\theta_1 \cap \leq^*) \circ (\theta_2 \cap \geq^*)$. Then $a\,\theta_1\,u\,\theta_2\,b$ and $a \leq^* u \geq^* b$ for some $u \in P$. Hence $[u]\theta_1 \geq^* [b]\theta_1$ and $[a]\theta_2 \leq^* [u]\theta_2$; or equivalently, $[u]\theta_1 \leq^t_{\theta_1} [b]\theta_1$ and $[a]\theta_2 \leq^t_{\theta_2} [u]\theta_2$. Thus $a \vee b \in [a]\theta_1 \cdot [b]\theta_1 = [u]\theta_1 \cdot [b]\theta_1 = [u]\theta_1$ and $a \vee b \in [a]\theta_2 \cup [b]\theta_2 = [a]\theta_2 \cup [u]\theta_2 = [u]\theta_2$; that is, $(a \vee b, u) \in \theta_1 \cap \theta_2$. So $a \vee b = u$. By condition (iii), since $a\,\theta_1\,(a \vee b)\,\theta_2\,b$, we have $a \leq b$ or $b \leq a$. But, $b \leq a$ implies $a = b$, we conclude that $a \leq b$.

It follows from Theorem 2 that $\leq^*$ is an amenble lattice order of $\mathbf{P}$. □

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