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COVARIANCE MATRIX ELEMENTS ESTIMATION: SPECIAL LINEAR MODEL WITHOUT AND WITH REPEATED MEASUREMENT

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ABSTRACT. The paper shows locally best linear-quadratic unbiased estimators of the covariance matrix elements in a special structure of the linear model, where the dispersions depend on mean value parameters. The existence, uniqueness and the dispersion of these estimators are investigated. Their asymptotic behaviour is determined in a special case. The existence of these estimators and their dispersion in the case of increasing repetitions of one measurement is also investigated.

1. Introduction

Let us measure linear dependence (2 variable case) with a measuring device whose dispersion characteristic is linear-quadratically dependent on the actual measured value, i.e., we have the model

$$(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}), \tag{1.1}$$

where $\boldsymbol{Y}_{n,1}$ is a normally distributed random vector. Its realization $\boldsymbol{y}_{n,1}$ is the result of measurement. The mean value $\mathcal{E}(\boldsymbol{Y}) = \mathbf{X}_{n,2}\boldsymbol{\beta}_{2,1}$ ($\mathbf{X}$ is a known design matrix of order $n \times 2$ with ith row $(1, x_i)$, $x_i \neq x_j$ for $i \neq j$; $\boldsymbol{\beta} \in \mathcal{R}^2$ is the unknown parameter, $n \geqq 4$). The covariance matrix of the vector $\boldsymbol{Y}$ is

$$\boldsymbol{\Sigma} = \sigma^2 \boldsymbol{\Sigma}(\boldsymbol{\beta}) = \sigma^2 \begin{pmatrix} (a + b|e_1'\mathbf{X}\boldsymbol{\beta}|)^2 & 0 & \dots & 0 \\ 0 & (a + b|e_2'\mathbf{X}\boldsymbol{\beta}|)^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (a + b|e_n'\mathbf{X}\boldsymbol{\beta}|)^2 \end{pmatrix},$$

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where a, b and σ^2 are known positive constants (the characteristics of the measuring device, for more details, see, e.g., [2; p. 28], [5; p. 456, 914]), and e_i' is the transpose of the ith unit vector.

Under the assumption that one (say the jth) measurement is repeated several times independently, we have the model

$$(\boldsymbol{Y}, \mathbf{X}^R\boldsymbol{\beta}, \boldsymbol{\Sigma}_R), \tag{1.2}$$

where $\boldsymbol{Y}_{n+J-1,1}$ is again a normally distributed random vector, and its realization $\boldsymbol{y}_{n+J-1,1}$ is the result of measurement. The mean value $\mathcal{E}(\boldsymbol{Y}) = \mathbf{X}^R_{n+J-1,2}\boldsymbol{\beta}_{2,1}$, where

$$\mathbf{X}^R_{n+J-1,2} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_{j-1} \\ 1 & x_j \\ 1 & x_{j+1} \\ \vdots & \vdots \\ 1 & x_n \\ 1 & x_j \\ \vdots & \vdots \\ 1 & x_j \end{pmatrix},$$

and $x_i \neq x_j$ for $i \neq j$. $J \geqq 1$ is the number of repeated measurements at the point x_j. The covariance matrix of the vector $\boldsymbol{Y}$ is now

$$\boldsymbol{\Sigma}_R = \sigma^2\boldsymbol{\Sigma}_R(\boldsymbol{\beta})$$

$$= \sigma^2 \begin{pmatrix} (a+b|\boldsymbol{e}_1'\mathbf{X}^R\boldsymbol{\beta}|)^2 & 0 & \dots & 0 \\ 0 & (a+b|\boldsymbol{e}_2'\mathbf{X}^R\boldsymbol{\beta}|)^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (a+b|\boldsymbol{e}_{n+J-1}'\mathbf{X}^R\boldsymbol{\beta}|)^2 \end{pmatrix}$$

$$= \sigma^2 \begin{pmatrix} (a+b|\boldsymbol{e}_1'\mathbf{X}^R\boldsymbol{\beta}|)^2 \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & & & & & \\ 0 & (a+b|\boldsymbol{e}_j'\mathbf{X}^R\boldsymbol{\beta}|)^2 & & & & \dots & 0 \\ \vdots & & \ddots & & & & \\ 0 & & & (a+b|\boldsymbol{e}_n'\mathbf{X}^R\boldsymbol{\beta}|)^2 & 0 & \dots & 0 \\ 0 & & & & (a+b|\boldsymbol{e}_j'\mathbf{X}^R\boldsymbol{\beta}|)^2 & \dots & 0 \\ \vdots & & & & & \ddots & 0 \\ 0 & & \dots & & & & (a+b|\boldsymbol{e}_j'\mathbf{X}^R\boldsymbol{\beta}|)^2 \end{pmatrix},$$

where a, b and σ^2 are again known positive constants. It is evident that we consider independent measurements.

In Section 2 of the paper, we look for the $\boldsymbol{\beta}_{\circ}$-LBLQUE (locally best linear-quadratic unbiased estimator, see, e.g., [7]) of the functional $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ ($j = 1, 2, \ldots, n$) of the parameter $\boldsymbol{\beta}$ (i.e., of the element of the covariance matrix $\boldsymbol{\Sigma}$) in the model (1.1). This is of a great importance for an observer; he has to know whether the variability of the variances of the measurements needs or does not need to be taken into account in processing the measured data.

Further, we investigate the existence of the above mentioned estimator and its uniqueness (in Section 3), we derive the form of this estimator (in Section 4) and also determine its dispersion (in Section 5).

The existence of the $\boldsymbol{\beta}_{\circ}$-LBLQUE of the functional $\sigma^2\big(a + b|\mathbf{e}_i'\mathbf{X}^R\boldsymbol{\beta}|\big)^2$ ($i = 1, 2, \ldots, n+J-1$) of parameter $\boldsymbol{\beta}$ in the model (1.2) is investigated in Section 6. *We also determine the dispersion of the* $\boldsymbol{\beta}_{\circ}$-LBLQUE in the model (1.2) (Section 7) and its asymptotic behaviour in the case of increasing J (the number of repetitions of the jth measurement) (in Section 8). So this paper forms a base for investigation of the general (2 variable) model with repetitions of an arbitrary measurement. The above mentioned investigations are of a great importance for an observer for the reasons stated above. The observer is also interested in the influence of repeating the jth measurement on the dispersion of the estimator of $\sigma^2\big(a + b|\mathbf{e}_i'\mathbf{X}^R\boldsymbol{\beta}|\big)^2$ for $i \neq j$ and for $i = j$.

Some propositions needed for our investigations in Section 2, Section 3 and Section 4 are in Appendix 1.

The asymptotic behavior of the dispersion in the model (1.1) in the special case of increasing x_i with $a = \sigma^2 = 1$ and $\boldsymbol{\beta}_{\circ} = (0, 1)'$ is investigated in Appendix 2.

How to verify the correctness of the numerical value of the estimate in the model (1.1) is shown in Appendix 3.

This paper is a continuation of [6], [7] and [8] (see also [9]).

2. The $\boldsymbol{\beta}_{\circ}$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}_j\mathbf{X}\boldsymbol{\beta}|\big)^2$ in the model (1.1)

Let us denote by $\mathcal{D}$ the following class of $n \times n$ matrices:

$$\mathcal{D} = \left\{ \mathbf{D}_{n,n} : \operatorname{Tr}\mathbf{D} \begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}| & 0 & \ldots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\boldsymbol{\beta}| & \ldots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \ldots & |\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}| \end{pmatrix} = 0 \quad \forall\{\boldsymbol{\beta} \in \mathcal{R}^2\}, \right.$$

$$\left. \operatorname{Tr}\mathbf{D} = 0\,, \quad \mathbf{X}'\Big(\mathbf{D} + \sigma^2 b^2 \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i'\mathbf{D}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X} = \mathbf{O} \right\}.$$

($\operatorname{Tr}\mathbf{D}$ is the trace of the matrix $\mathbf{D}$.)

LEMMA 2.1. *In the model* (1.1), $\mathbf{B} \in \mathcal{D}$ *if and only if*

$$e_i'\mathbf{B}e_i = 0\,, \qquad i = 1, 2, \dots, n\,, \tag{2.1}$$

and

$$\mathbf{X}'\mathbf{B}\mathbf{X} = \mathbf{O}\,. \tag{2.2}$$

P r o o f . According to Lemma 9.1, the condition

$$\operatorname{Tr}\mathbf{B}\begin{pmatrix} |e_1'\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0 \\ 0 & |e_2'\mathbf{X}\boldsymbol{\beta}| & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & |e_n'\mathbf{X}\boldsymbol{\beta}| \end{pmatrix} = \sum_{i=1}^{n} e_i'\mathbf{B}e_i|e_i'\mathbf{X}\boldsymbol{\beta}| = 0 \quad \forall\{\boldsymbol{\beta}\in\mathcal{R}^2\}$$

is equivalent to (2.1). Now, we can easily finish the proof of the lemma. □

COROLLARY 2.2. *In the model* (1.1),

$$\mathbf{B} \in \mathcal{D} \iff \operatorname{vec}\mathbf{B} \in \operatorname{Ker}\mathbb{X}'\,,$$

where $\operatorname{Ker}\mathbb{X}' = \{\boldsymbol{\xi} \in \mathcal{R}^{n^2} :\ \mathbb{X}'\boldsymbol{\xi} = \boldsymbol{O}\}$ *(for the definition of the matrix* $\mathbb{X}'$ *and also the definition of* $\operatorname{vec}\mathbf{B}$, *see Appendix* 1).

LEMMA 2.3. *In the model* (1.1), $\mathbf{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ *is an unbiased estimator of* $\sigma^2(a + b|e_j'\mathbf{X}\boldsymbol{\beta}|)^2$, $j \in \{1, 2, \dots, n\}$, *if and only if*

$$\boldsymbol{a} \in \operatorname{Ker}\mathbf{X}'\,, \tag{2.3}$$

and

$$\mathbb{X}'(\mathbf{I} + \mathbf{I}^*)\operatorname{vec}\mathbf{A} = \begin{pmatrix} \mathbf{O}_{4,1} \\ 2e_j \end{pmatrix}, \tag{2.4}$$

where $\mathbf{I}^*$ *is defined in Appendix* 1.

P r o o f . The random variable $\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ is an unbiased estimator of $\sigma^2(a + b|e_j'\mathbf{X}\boldsymbol{\beta}|)^2$ if and only if

$$\begin{aligned}
&\forall\{\boldsymbol{\beta}\in\mathcal{R}^2\} \\
&\quad \mathcal{E}_{\boldsymbol{\beta}}(\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}) \\
&= \boldsymbol{a}'\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta} + \operatorname{Tr}\mathbf{A}\boldsymbol{\Sigma} \\
&= \boldsymbol{a}'\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta} + \sigma^2 a^2 \operatorname{Tr}\mathbf{A} + 2ab\sigma^2 \operatorname{Tr}\mathbf{A}\begin{pmatrix} |e_1'\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0 \\ 0 & |e_2'\mathbf{X}\boldsymbol{\beta}| & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & |e_n'\mathbf{X}\boldsymbol{\beta}| \end{pmatrix} \\
&\qquad + \sigma^2 b^2 \sum_{i=1}^{n} e_i'\mathbf{A}e_i(e_i'\mathbf{X}\boldsymbol{\beta})^2 = \sigma^2\big(a + b|e_j'\mathbf{X}\boldsymbol{\beta}|\big)^2\,,
\end{aligned}$$

which is equivalent (for details, see [7; Lemma 2.1]) to

$$e_i'\mathbf{A}e_i = 0\,, \quad i = 1,2,\dots,j-1,j+1,\dots,n\,, \qquad e_j'\mathbf{A}e_j = 1\,, \tag{2.5}$$

$$\mathbf{X}'(\mathbf{A}+\mathbf{A}')\mathbf{X} = \mathbf{O}\,, \tag{2.6}$$

and (2.3). As (2.5) and (2.6) are equivalent to (2.4), the lemma is proved. □

Proofs of the next two lemmas can be found in [7] and are omitted.

Lemma 2.4. *In the model* (1.1), *the random variable* $a'Y + Y'\mathbf{A}Y$ *is the* $\beta_\circ$*-LBLQUE of its mean value (in the class of linear-quadratic estimators) if and only if there exists a vector* $z \in \mathcal{R}^n$ *that*

$$a = -(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_\circ + (\mathbf{X}')^-_{m(\Sigma(\beta_\circ))}\mathbf{X}'z \tag{2.7}$$

and

$$\forall\{\mathbf{D}\in\mathcal{D}\}$$
$$\operatorname{Tr}(\mathbf{D}+\mathbf{D}')\{\sigma^2\Sigma(\beta_\circ)(\mathbf{A}+\mathbf{A}')\Sigma(\beta_\circ) + 2\mathbf{X}\beta_\circ z'\mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma(\beta_\circ))}\big]'\Sigma(\beta_\circ)\} = 0\,. \tag{2.8}$$

($(\mathbf{X}')^-_{m(\Sigma(\beta_\circ))}$ *is an arbitrary but fixed minimum* $\Sigma(\beta_\circ)$*-norm g-inverse of the matrix* $\mathbf{X}'$, *i.e., a matrix satisfying the relations* $\mathbf{X}'(\mathbf{X}')^-_{m(\Sigma(\beta_\circ))}\mathbf{X}' = \mathbf{X}'$ *and* $((\mathbf{X}')^-_{m(\Sigma(\beta_\circ))}\mathbf{X}')'\Sigma(\beta_\circ) = \Sigma(\beta_\circ)(\mathbf{X}')^-_{m(\Sigma(\beta_\circ))}\mathbf{X}'$.*)*

Lemma 2.5. *For the arbitrary matrices* $\mathbf{A}$, $\mathbf{X}$ *and a vector* $\beta_\circ$,

$$\exists\{z\in\mathcal{R}^n\} \quad -(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_\circ + (\mathbf{X}')^-_{m(\Sigma(\beta_\circ))}\mathbf{X}'z = a$$

and

$$a \in \operatorname{Ker}\mathbf{X}'$$

if and only if

$$a = \big(\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma(\beta_\circ))}\mathbf{X}'\big)\big(-(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_\circ\big)\,.$$

Lemma 2.6. *In the model* (1.1), *the random variable* $a'Y + Y'\mathbf{A}Y$ *is the* $\beta_\circ$*-LBLQUE of* $\sigma^2\big(a + b|e_j'\mathbf{X}\beta|\big)^2$, $j \in \{1,2,\dots,n\}$, *if and only if*

$$a = -(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_\circ\,, \tag{2.9}$$

$$\forall\{\mathbf{D}\in\mathcal{D}\} \quad \operatorname{Tr}(\mathbf{D}+\mathbf{D}')\Sigma(\beta_\circ)(\mathbf{A}+\mathbf{A}')\Sigma(\beta_\circ) = 0\,, \tag{2.10}$$

and (2.5) *and* (2.6) *hold.*

Proof. The random variable $a'Y + Y'\mathbf{A}Y$ is the $\beta_\circ$-LBLQUE of $\sigma^2\big(a + b|e_j'\mathbf{X}\beta|\big)^2$ if and only if

(i) $a'Y + Y'\mathbf{A}Y$ is an unbiased estimator of $\sigma^2\big(a + b|e_j'\mathbf{X}\beta|\big)^2$,
(ii) $a'Y + Y'\mathbf{A}Y$ is the $\beta_\circ$-LBLQUE of its mean value.

According to Lemma 2.3 and Lemma 2.4, (i) and (ii) are satisfied if and only if (2.3), (2.5), (2.6), (2.7) and (2.8) hold. As (2.3), (2.6) and (2.7) are equivalent to (2.9) and (2.6) (Lemma 2.5), and (2.8) is equivalent to (2.10) (because of $\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$ for all $\mathbf{D} \in \mathcal{D}$ (see Lemma 2.1)), the lemma is proved. □

Lemma 2.7. *In the model* (1.1),

$$\begin{aligned} &\forall\{\mathbf{D}\in\mathcal{D}\} \qquad \operatorname{Tr}\sigma^2(\mathbf{D}+\mathbf{D}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) = 0 \\ \iff &\exists\{\gamma\in\mathcal{R}^{4+n}\} \qquad \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A}+\mathbf{A}') = \mathbb{X}\gamma\,. \end{aligned}$$

Proof.

$$\begin{aligned} &\forall\{\mathbf{D}\in\mathcal{D}\} \qquad \operatorname{Tr}\sigma^2(\mathbf{D}+\mathbf{D}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) = 0 \\ \iff &\forall\{\mathbf{D}\in\mathcal{D}\} \qquad \big[\operatorname{vec}\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big]'\operatorname{vec}\mathbf{D} = 0 \\ \iff &\exists\{\gamma\in\mathcal{R}^{4+n}\} \qquad \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A}+\mathbf{A}') = \mathbb{X}\gamma \end{aligned}$$

(using Corollary 1.2 and the well-known relations $\operatorname{Tr}\mathbf{AB} = (\operatorname{vec}\mathbf{B}')'\operatorname{vec}\mathbf{A}$, $\operatorname{vec}\mathbf{ABC} = (\mathbf{C}'\otimes\mathbf{A})\operatorname{vec}\mathbf{B}$, where $\otimes$ means the Kronecker product, see, e.g., [4; p. 11]). □

As a consequence of Lemma 2.6 and Lemma 2.7 we have the following:

Lemma 2.8. *In the model* (1.1), *the random variable* $\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE of* $\sigma^2\big(a + b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, $j \in \{1,2,\dots,n\}$, *if and only if*

$$\boldsymbol{a} = -(\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ$$

and $\exists\gamma\in\mathcal{R}^{4+n}$ *such that*

$$\big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A}+\mathbf{A}') = \mathbb{X}\gamma\,, \tag{2.11}$$

$$\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\operatorname{vec}\mathbf{A} = \begin{pmatrix}\mathbf{O}_{4,1}\\ 2\boldsymbol{e}_j\end{pmatrix}. \tag{2.12}$$

We simply note that (2.5) and (2.6) are evidently equivalent to (2.12).

As (2.11) is equivalent to

$$\operatorname{vec}(\mathbf{A}+\mathbf{A}') = \big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\gamma\,,$$

we see that

$$\mathbf{I}^*\operatorname{vec}(\mathbf{A}+\mathbf{A}') = \operatorname{vec}(\mathbf{A}+\mathbf{A}')$$

and so, according to Lemma 9.3,

$$\begin{aligned} &\mathbf{I}^*\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\gamma \\ &= \big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\big((\mathbf{X}\otimes\mathbf{X})\mathbf{I}^*, \boldsymbol{e}_1\otimes\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\otimes\boldsymbol{e}_n\big)\gamma \\ &= \big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\big((\mathbf{X}\otimes\mathbf{X}), \boldsymbol{e}_1\otimes\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\otimes\boldsymbol{e}_n\big)\gamma\,. \end{aligned} \tag{2.13}$$

As $\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)$ is a nonsingular matrix, and $(\mathbf{X}\otimes\mathbf{X})$ is of full column rank (see Lemma 9.2), we obtain from (2.13) that

$$e_2'\gamma = e_3'\gamma\,.$$

We can modify Lemma 2.8 as follows:

Lemma 2.9. *In the model (1.1), the random variable* $\boldsymbol{a}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ *is the* $\beta_\circ$*-LBLQUE of* $\sigma^2\left(a+b|e_j'\mathbf{X}\beta|\right)^2$, $j\in\{1,2,\dots,n\}$, *if and only if*

$$\exists\{\gamma\in\mathcal{R}^{4+n},\ e_2'\gamma=e_3'\gamma\}:$$

$$\mathbb{X}'\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)\mathbb{X}\gamma=\begin{pmatrix}\mathbf{O}_{4,1}\\ 2e_j\end{pmatrix}. \qquad (2.14)$$

In this case,

$$\operatorname{vec}(\mathbf{A}+\mathbf{A}')=\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)\mathbb{X}\gamma\,,$$
$$\boldsymbol{a}=-(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_\circ\,.$$

Lemma 2.10.

$$\exists\{\gamma\in\mathcal{R}^{4+n},\ e_2'\gamma=e_3'\gamma\}:$$
$$\mathbb{X}'\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)\mathbb{X}\gamma=\begin{pmatrix}\mathbf{O}_{4,1}\\ 2e_j\end{pmatrix}$$
$$\iff \exists\{\delta\in\mathcal{R}^{4+n}\}:$$
$$\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\delta=\begin{pmatrix}\mathbf{O}_{4,1}\\ 8e_j\end{pmatrix}.$$

Proof. With $\delta=\gamma$ (see (9.3))

$$\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\delta$$
$$=2\begin{pmatrix}\mathbf{I}+\mathbf{I}^* & \mathbf{O}\\ \mathbf{O} & 2\mathbf{I}\end{pmatrix}\mathbb{X}'\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)\mathbb{X}\delta=\begin{pmatrix}\mathbf{O}\\ 8e_j\end{pmatrix}.$$

Now let

$$\gamma=\begin{pmatrix}\mathbf{I}+\mathbf{I}^* & \mathbf{O}\\ \mathbf{O} & 2\mathbf{I}\end{pmatrix}\frac{\delta}{2}.$$

So $e_2'\gamma=e_3'\gamma$ and

$$\mathbb{X}'\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)\mathbb{X}\gamma$$
$$=2\mathbb{X}'\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)\mathbb{X}\begin{pmatrix}\mathbf{I}+\mathbf{I}^* & \mathbf{O}\\ \mathbf{O} & 2\mathbf{I}\end{pmatrix}\frac{\delta}{4}$$
$$=\frac{1}{4}\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\left(\Sigma^{-1}(\beta_\circ)\otimes\Sigma^{-1}(\beta_\circ)\right)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\delta=\begin{pmatrix}\mathbf{O}\\ 2e_j\end{pmatrix}.$$

□

From Lemma 2.10, we finally have the last modification of Lemma 2.9.

THEOREM 2.11. *In the model* (1.1), *the random variable* $\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE of* $\sigma^2\big(a + b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, $j \in \{1, 2, \dots, n\}$, *if and only if* $\exists\{\boldsymbol{\delta}\in\mathcal{R}^{4+n}\}$:

$$\mathbb{X}'(\mathbf{I} + \mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I} + \mathbf{I}^*)\mathbb{X}\boldsymbol{\delta} = \begin{pmatrix} \mathbf{O} \\ 8\boldsymbol{e}_j \end{pmatrix}. \tag{2.15}$$

In this case,

$$\begin{aligned} \operatorname{vec}(\mathbf{A} + \mathbf{A}') &= (\mathbf{I} + \mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\frac{\boldsymbol{\delta}}{2}, \\ \boldsymbol{a} &= -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ . \end{aligned} \tag{2.16}$$

3. Existence of the $\boldsymbol{\beta}_\circ$-LBLQUE in the model (1.1)

As (2.15) is a n.a.s. condition for the existence of the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, $j \in \{1, 2, \dots, n\}$, we see that this estimator exists if and only if

$$\begin{aligned} \begin{pmatrix} \mathbf{O} \\ 8\boldsymbol{e}_j \end{pmatrix} &\in \mu\big(\mathbb{X}'(\mathbf{I} + \mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I} + \mathbf{I}^*)\mathbb{X}\big) \\ &= \mu\big(\mathbb{X}'(\mathbf{I} + \mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ)\big)\big) \\ &= \mu\begin{pmatrix} (\mathbf{I} + \mathbf{I}^*)(\mathbf{X}' \otimes \mathbf{X}') \\ 2\boldsymbol{e}_1' \otimes \boldsymbol{e}_1' \\ \vdots \\ 2\boldsymbol{e}_n' \otimes \boldsymbol{e}_n' \end{pmatrix}, \end{aligned} \tag{3.1}$$

where

$$\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) = \begin{pmatrix} (a+b|\boldsymbol{e}_1'\mathbf{X}_1\boldsymbol{\beta}_\circ|)^{-1} & 0 & \dots & 0 \\ 0 & (a+b|\boldsymbol{e}_2'\mathbf{X}_1\boldsymbol{\beta}_\circ|)^{-1} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & (a+b|\boldsymbol{e}_n'\mathbf{X}_1\boldsymbol{\beta}_\circ|)^{-1} \end{pmatrix},$$

and $\mu\big(\mathbb{X}'(\mathbf{I} + \mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ)\big)\big)$ is the vector space spanned by the columns of the matrix $\mathbb{X}'(\mathbf{I} + \mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ)\big)$ (see also Lemma 9.3 and Lemma 9.4).

As

$$(\mathbf{I} + \mathbf{I}^*)(\mathbf{X}' \otimes \mathbf{X}') = \begin{pmatrix} 2 & 2 & \dots & 2 & 2 & 2 & \dots & 2 & \dots & 2 & \dots & 2 \\ 2x_1 & x_1+x_2 & \dots & x_1+x_n & x_2+x_1 & 2x_2 & \dots & x_2+x_n & \dots & x_n+x_1 & \dots & 2x_n \\ 2x_1 & x_1+x_2 & \dots & x_1+x_n & x_2+x_1 & 2x_2 & \dots & x_2+x_n & \dots & x_n+x_1 & \dots & 2x_n \\ 2x_1^2 & 2x_1x_2 & \dots & 2x_1x_n & 2x_2x_1 & 2x_2^2 & \dots & 2x_2x_n & \dots & 2x_nx_1 & \dots & 2x_n^2 \end{pmatrix},$$

let us look for such a vector

$$d_o = (d_o^{11}, d_o^{21}, \dots, d_o^{n1}, d_o^{12}, \dots, d_o^{n2}, \dots, d_o^{nn})' \tag{3.2}$$

for which

$$\begin{pmatrix} (\mathbf{I}+\mathbf{I}^*)(\mathbf{X}'\otimes\mathbf{X}') \\ 2e_1'\otimes e_1' \\ \vdots \\ 2e_n'\otimes e_n' \end{pmatrix} d_o = \begin{pmatrix} \mathbf{O} \\ 8e_j \end{pmatrix}. \tag{3.3}$$

It is obvious that

$$d_o^{11} = d_o^{22} = \dots = d_o^{j-1,j-1} = d_o^{j+1,j+1} = \dots = d_o^{nn} = 0 \tag{3.4}$$

and

$$d_o^{jj} = 4\,. \tag{3.5}$$

From the assumptions of the model (1.1), there exist three different x_i, x_j and x_k. Then

$$\det\begin{pmatrix} 2 & 2 & 2 \\ x_i+x_j & x_i+x_k & x_j+x_k \\ 2x_ix_j & 2x_ix_k & 2x_jx_k \end{pmatrix} = 4(x_i-x_k)(x_j-x_k)(x_j-x_i) \neq 0\,,$$

and so, the matrix

$$\begin{pmatrix} 2 & 2 & 2 \\ x_i+x_j & x_i+x_k & x_j+x_k \\ 2x_ix_j & 2x_ix_k & 2x_jx_k \end{pmatrix}$$

is nonsingular. The equation

$$\begin{pmatrix} 2 & 2 & 2 \\ x_i+x_j & x_i+x_k & x_j+x_k \\ 2x_ix_j & 2x_ix_k & 2x_jx_k \end{pmatrix}\begin{pmatrix} d_o^{ij} \\ d_o^{ik} \\ d_o^{jk} \end{pmatrix} = \begin{pmatrix} -8 \\ -8x_j \\ -8x_j^2 \end{pmatrix} \tag{3.6}$$

is always soluble (for every x_j), and d_o for which (3.4), (3.5), (3.6) hold and the rest of coordinates are zeros satisfies (3.3).

We have proved the next theorem:

THEOREM 3.1. *There always exists d_o satisfying (3.3), and so, the β_o-LBLQUE of $\sigma^2(a+b|e_j'\mathbf{X}\beta|)^2$ exists for each $j\in\{1,2,\dots,n\}$.*

A n.a.s. condition for $a'Y + Y'AY$ to be the β_o-LBLQUE of $\sigma^2(a+b|e_j'\mathbf{X}\beta|)^2$ is (2.15) (in which case (2.16) is also valid). We see that for every solution δ of (2.15) (i.e.,

$$\begin{aligned}\delta &= [\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)(\Sigma^{-1}(\beta_o)\otimes\Sigma^{-1}(\beta_o))(\mathbf{I}+\mathbf{I}^*)\mathbb{X}]^- \begin{pmatrix} \mathbf{O} \\ 8e_j \end{pmatrix} \\ &\quad + \{\mathbf{I} - [\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)(\Sigma^{-1}(\beta_o)\otimes\Sigma^{-1}(\beta_o))(\mathbf{I}+\mathbf{I}^*)\mathbb{X}]^- \cdot \\ &\qquad \cdot [\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)(\Sigma^{-1}(\beta_o)\otimes\Sigma^{-1}(\beta_o))(\mathbf{I}+\mathbf{I}^*)\mathbb{X}]\}\eta\,)\end{aligned}$$

we have that

$$\begin{aligned}\operatorname{vec}(\mathbf{A}+\mathbf{A}') &= (\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\frac{\boldsymbol{\delta}}{2}\\ &= \frac{1}{2}(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\cdot\\ &\quad\cdot\big[\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\big]^-\begin{pmatrix}\mathbf{O}\\ 8\boldsymbol{e}_j\end{pmatrix}\end{aligned} \tag{3.7}$$

is unique (and does not depend on the choice of the g-inverse as (3.1) is always satisfied).

COROLLARY 3.2. *There always exists a unique $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a+b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for each $j\in\{1,2,\dots,n\}$.*

4. Formula for the $\boldsymbol{\beta}_\circ$-LBLQUE in the model (1.1)

As the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a+b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ is unique and given by (3.7) (for arbitrary choice of the g-inverse), we obtain the formula for it as

$$\begin{aligned}\boldsymbol{a}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y} &= -\boldsymbol{\beta}_\circ'\mathbf{X}'(\mathbf{A}+\mathbf{A}')\boldsymbol{Y}+\boldsymbol{Y}'\frac{\mathbf{A}+\mathbf{A}'}{2}\boldsymbol{Y}\\ &= \frac{1}{4}\big(\boldsymbol{Y}'\otimes(\boldsymbol{Y}-2\mathbf{X}\boldsymbol{\beta}_\circ)'\big)(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\cdot\\ &\quad\cdot\big[\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\big]^-\begin{pmatrix}\mathbf{O}\\ 8\boldsymbol{e}_j\end{pmatrix} \qquad (4.1)\\ &= \frac{1}{2}\big(\boldsymbol{Y}'\otimes(\boldsymbol{Y}-2\mathbf{X}\boldsymbol{\beta}_\circ)'\big)(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\cdot\\ &\quad\cdot\big\{\mathbb{X}'\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\big\}^{-1}\begin{pmatrix}\mathbf{O}\\ \boldsymbol{e}_j\end{pmatrix}\end{aligned}$$

(see Lemma 9.4).

Let us denote

$$\big\{\mathbb{X}'\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\big\}^{-1}=\begin{pmatrix}\mathbf{M}_{4,4} & \mathbf{N}_{4,n}\\ \mathbf{N}'_{n,4} & \mathbf{P}_{n,n}\end{pmatrix}.$$

According to Lemma 9.5, we now have

$$\begin{aligned}&\boldsymbol{a}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}\\ &= \frac{1}{2}\big(\boldsymbol{Y}'\otimes(\boldsymbol{Y}-2\mathbf{X}\boldsymbol{\beta}_\circ)'\big)(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\begin{pmatrix}\mathbf{N}\boldsymbol{e}_j\\ \mathbf{P}\boldsymbol{e}_j\end{pmatrix}\\ &= \frac{1}{2}\big(\boldsymbol{Y}'\otimes(\boldsymbol{Y}-2\mathbf{X}\boldsymbol{\beta}_\circ)'\big)(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\cdot\\ &\quad\cdot\Big((\mathbf{X}\otimes\mathbf{X})\mathbf{N}\boldsymbol{e}_j+\sum_{i=1}^{n}(\boldsymbol{e}_i\otimes\boldsymbol{e}_i)_{n^2,1}\boldsymbol{e}_i'\mathbf{P}\boldsymbol{e}_j\Big).\end{aligned} \tag{4.2}$$

Since for $i \neq j$

$$(\boldsymbol{e}_i' \otimes \boldsymbol{e}_i')\Big[(\mathbf{X} \otimes \mathbf{X})\mathbf{N}\boldsymbol{e}_j + \sum_{l=1}^{n}(\boldsymbol{e}_l \otimes \boldsymbol{e}_l)_{n^2,1}\boldsymbol{e}_l'\mathbf{P}\boldsymbol{e}_j\Big] = 0$$

and

$$(\boldsymbol{e}_j' \otimes \boldsymbol{e}_j')\Big[(\mathbf{X} \otimes \mathbf{X})\mathbf{N}\boldsymbol{e}_j + \sum_{l=1}^{n}(\boldsymbol{e}_l \otimes \boldsymbol{e}_l)_{n^2,1}\boldsymbol{e}_l'\mathbf{P}\boldsymbol{e}_j\Big] = \big(a + b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^4 ,$$

we obtain from (4.2) after a short computation

$$\begin{aligned}
&\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y} \\
&= (Y_j^2 - 2Y_j\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ) - \sum_{r=1}^{n}\sum_{\substack{s=1\\ s\neq r}}^{n} \frac{\big(Y_rY_s - (Y_r\boldsymbol{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ + Y_s\boldsymbol{e}_r'\mathbf{X}\boldsymbol{\beta}_\circ)\big)}{\big(a + b|\boldsymbol{e}_r'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\big(a + b|\boldsymbol{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} \cdot \\
&\quad \cdot \big\{w^{11} + x_j(w^{12} + w^{13}) + x_j^2w^{14} + x_s\big[w^{21} + x_j(w^{22} + w^{23}) + x_j^2w^{24}\big] \\
&\quad + x_r\big[w^{31} + x_j(w^{32} + w^{33}) + x_j^2w^{34}\big] + x_rx_s\big[w^{41} + x_j(w^{42} + w^{43}) + x_j^2w^{44}\big]\big\} ,
\end{aligned}$$

where the w^{ij} are elements of the matrix $\mathbf{W}^{-1}$ ($\mathbf{W}$ is given by (9.7), see also Lemma 9.5).

From Lemma 9.6, we finally obtain the formula for the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ as

$$\begin{aligned}
&\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y} \\
&= (Y_j^2 - 2Y_j\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ) - \sum_{r=1}^{n}\sum_{\substack{s=1\\ s\neq r}}^{n} \frac{\big(Y_rY_s - (Y_r\boldsymbol{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ + Y_s\boldsymbol{e}_r'\mathbf{X}\boldsymbol{\beta}_\circ)\big)}{\big(a + b|\boldsymbol{e}_r'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\big(a + b|\boldsymbol{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} \cdot \\
&\quad \cdot \big\{w^{11} + w^{12}(2x_j + x_r + x_s) + w^{14}(x_j^2 + x_rx_s) + (w^{22} + w^{23})x_j(x_r + x_s) \\
&\quad + w^{24}x_j(x_rx_j + x_sx_j + 2x_rx_s) + w^{44}x_rx_sx_j^2\big\} ,
\end{aligned} \tag{4.3}$$

where the w^{ij} are given by (9.12) – (9.18).

5. Dispersion of the $\boldsymbol{\beta}_\circ$-LBLQUE in the model (1.1)

Let us now calculate the dispersion of the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$

at $\boldsymbol{\beta}_\circ$.

$$\begin{aligned}
&\mathcal{D}_{\boldsymbol{\beta}_\circ}(\boldsymbol{a}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y})\\
&=2\sigma^4 \operatorname{Tr}\frac{\mathbf{A}+\mathbf{A}'}{2}\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\frac{\mathbf{A}+\mathbf{A}'}{2}\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\\
&\quad+\sigma^2\big\{(\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ-(\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ\big\}'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big\{(\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ-(\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ\big\}\\
&=\frac{\sigma^4}{2}\operatorname{Tr}(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\\
&=\frac{\sigma^4}{2}\big[\operatorname{vec}(\mathbf{A}+\mathbf{A}')\big]'\operatorname{vec}\big[\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big]\\
&=\frac{\sigma^4}{2}\big[\operatorname{vec}(\mathbf{A}+\mathbf{A}')\big]'\big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A}+\mathbf{A}')\\
&=\frac{\sigma^4}{8}\boldsymbol{\delta}'\mathbb{X}'\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\cdot\\
&\quad\cdot(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\boldsymbol{\delta}\\
&=\frac{\sigma^4}{8}\boldsymbol{\delta}'\big[\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\big]\boldsymbol{\delta}\,, \qquad (5.1)
\end{aligned}$$

where $\boldsymbol{\delta}$ is a solution of (2.15). As (3.1) holds, we have

$$\begin{aligned}
&\mathcal{D}_{\boldsymbol{\beta}_\circ}(\boldsymbol{a}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y})\\
&=\frac{\sigma^4}{8}\boldsymbol{\delta}'\begin{pmatrix}\mathbf{O}\\ 8\boldsymbol{e}_j\end{pmatrix}=\sigma^4\boldsymbol{\delta}'\begin{pmatrix}\mathbf{O}\\ \boldsymbol{e}_j\end{pmatrix}\\
&=8\sigma^4(\mathbf{O}_{1,4}\boldsymbol{e}_j')\big[\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\big]^-\begin{pmatrix}\mathbf{O}_{4,1}\\ 8\boldsymbol{e}_j\end{pmatrix},
\end{aligned}$$

and this dispersion does not depend on the choice of the g-inverse

$$\big[\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\big]^-.$$

According to Lemma 9.4, we have

$$\begin{aligned}
&\mathcal{D}_{\boldsymbol{\beta}_\circ}(\boldsymbol{a}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y})\\
&=\sigma^4(\mathbf{O}\boldsymbol{e}_j')\big[\mathbb{X}'\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\big]^{-1}\begin{pmatrix}\mathbf{I}+\mathbf{I}^* & \mathbf{O}\\ \mathbf{O} & 2\mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{O}\\ \boldsymbol{e}_j\end{pmatrix}\\
&=2\sigma^4\boldsymbol{e}_j'\mathbf{P}\boldsymbol{e}_j\\
&=2\sigma^4\big[(a+b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ|)^4+w^{11}+x_j(w^{12}+w^{13}+w^{21}+w^{31})\\
&\qquad+x_j^2(w^{14}+w^{22}+w^{23}+w^{32}+w^{33}+w^{41})\\
&\qquad+x_j^3(w^{24}+w^{34}+w^{42}+w^{43})+x_j^4w^{44}\big]\,. \qquad (5.2)
\end{aligned}$$

After a short computation, we obtain from (5.2) (see Lemma 9.6)

$$\begin{aligned}&\mathcal{D}_{\beta_o}(\boldsymbol{a}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y})\\&=2\sigma^4\big[(a+b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}_o|)^4+w^{11}+4x_jw^{12}+2x_j^2(w^{14}+w^{22}+w^{23})\\&\quad+4x_j^3w^{24}+x_j^4w^{44}\big]\,,\end{aligned}\tag{5.3}$$

where the w^{ij} are given by (9.12) – (9.18).

6. Existence of the $\boldsymbol{\beta}_o$-LBLQUE in the model (1.2)

Let us denote

$$(\mathbb{X}^R)'_{n+4,(n+J-1)^2}=\begin{pmatrix}(\mathbf{X}^R)'\otimes(\mathbf{X}^R)'\\ \boldsymbol{e}_1'\otimes\boldsymbol{e}_1'\\ \vdots\\ \boldsymbol{e}_{j-1}'\otimes\boldsymbol{e}_{j-1}'\\ \boldsymbol{e}_j'\otimes\boldsymbol{e}_j'+\sum\limits_{q=n+1}^{n+J-1}\boldsymbol{e}_q'\otimes\boldsymbol{e}_q'\\ \boldsymbol{e}_{j+1}'\otimes\boldsymbol{e}_{j+1}'\\ \vdots\\ \boldsymbol{e}_n'\otimes\boldsymbol{e}_n'\end{pmatrix}.$$

The next Theorem can be proved (see [3]):

THEOREM 6.1. *In the model* (1.2), *the random variable* $\boldsymbol{a}_{(i)}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}_{(i)}\boldsymbol{Y}$ *is the* $\boldsymbol{\beta}_o$*-LBLQUE of* $\sigma^2\big(a+b|\boldsymbol{e}_i'\mathbf{X}^R\boldsymbol{\beta}|\big)^2$, $i\in\{1,2,\dots,n+J-1\}$, *if and only if* $\exists\{\boldsymbol{\delta}_{(i)}\in\mathcal{R}^{4+n}\}$:

$$(\mathbb{X}^R)'(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}_R^{-1}(\boldsymbol{\beta}_o)\otimes\boldsymbol{\Sigma}_R^{-1}(\boldsymbol{\beta}_o)\big)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}^R\boldsymbol{\delta}_{(i)}=\begin{pmatrix}\mathbf{O}_{4,0}\\ 8\boldsymbol{e}_q\end{pmatrix},\tag{6.1}$$

where

$$\begin{aligned}q&=i\qquad \textit{for}\quad i\in\big\{\{1,2,\dots,n\}-\{j\}\big\}\,,\\ q&=j\qquad \textit{for}\quad i\in\{j,n+1,\dots,n+J-1\}\,.\end{aligned}$$

In this case,

$$\operatorname{vec}\big(\mathbf{A}_{(i)}+\mathbf{A}_{(i)}'\big)=\frac{1}{4}(\mathbf{I}+\mathbf{I}^*)\big(\boldsymbol{\Sigma}_R^{-1}(\boldsymbol{\beta}_o)\otimes\boldsymbol{\Sigma}_R^{-1}(\boldsymbol{\beta}_o)\big)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}^R\boldsymbol{\delta}_{(i)}$$

and

$$a = -(\mathbf{A}_{(i)} + \mathbf{A}'_{(i)})\mathbf{X}^R \boldsymbol{\beta}_\circ .$$

From the n.a.s. condition (6.1), we see that the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\boldsymbol{e}_i'\mathbf{X}^R\boldsymbol{\beta}|\big)^2$, $i \in \{1,2,\dots,n+J-1\}$, exists if and only if

$$\begin{pmatrix} \mathbf{O} \\ 8\boldsymbol{e}_q \end{pmatrix} \in \mu\big((\mathbb{X}^R)'(\mathbf{I}+\mathbf{I}^*)(\boldsymbol{\Sigma}_R{}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}_R{}^{-1}(\boldsymbol{\beta}_\circ))(\mathbf{I}+\mathbf{I}^*)\mathbb{X}^R\big)$$
$$= \mu\big((\mathbb{X}^R)'(\mathbf{I}+\mathbf{I}^*)(\boldsymbol{\Sigma}_R{}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}_R{}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ))\big) = \mu\big((\mathbb{X}^R)'(\mathbf{I}+\mathbf{I}^*)\big), \tag{6.2}$$

where

$$\boldsymbol{\Sigma}_R^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) = \begin{pmatrix} (a+b|\boldsymbol{e}_1'\mathbf{X}_1^R\boldsymbol{\beta}|)^{-1} & 0 & \dots & 0 \\ 0 & (a+b|\boldsymbol{e}_2'\mathbf{X}_1^R\boldsymbol{\beta}|)^{-1} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & (a+b|\boldsymbol{e}_{n+J-1}'\mathbf{X}_1^R\boldsymbol{\beta}|)^{-1} \end{pmatrix}$$

($\mu(\mathbf{A})$ is the vector space spanned by the columns of the matrix $\mathbf{A}$),

$$q = i \qquad \text{for} \quad i \in \big\{\{1,2,\dots,n\} - \{j\}\big\},$$
$$q = j \qquad \text{for} \quad i \in \{j, n+1, \dots, n+J-1\}.$$

Lemma 6.2. *Let $i \in \{1,2,\dots,n+J-1\}$. The vector*

$$\boldsymbol{d}_{(i)} = \big(d_{(i)}^{1,1}, d_{(i)}^{2,1}, \dots, d_{(i)}^{n+J-1,1}, d_{(i)}^{1,2}, \dots, d_{(i)}^{n+J-1,2}, \dots, d_{(i)}^{n+J-1,n+J-1}\big)'$$

with

$$d_{(i)}^{i,i} = 4,$$

$d_{(i)}^{i,k}$, $d_{(i)}^{i,t}$ and $d_{(i)}^{k,t}$ determined by

$$\begin{pmatrix} 2 & 2 & 2 \\ x_q + x_k & x_q + x_t & x_k + x_t \\ 2x_q x_k & 2x_q x_t & 2x_k x_t \end{pmatrix}^{-1} \begin{pmatrix} -8 \\ -8x_q \\ -8x_q^2 \end{pmatrix} = \begin{pmatrix} d_{(i)}^{i,k} \\ d_{(i)}^{i,t} \\ d_{(i)}^{k,t} \end{pmatrix},$$

where

$$q = i \qquad \text{for} \quad i \in \big\{\{1,2,\dots,n\} - \{j\}\big\},$$
$$q = j \qquad \text{for} \quad i \in \{j, n+1, \dots, n+J-1\},$$

and the remaining of coordinates zero, satisfies the equation $(\mathbb{X}^R)'(\mathbf{I}+\mathbf{I}^*)\boldsymbol{d}_{(i)} = \begin{pmatrix} \mathbf{O} \\ 8\boldsymbol{e}_q \end{pmatrix}$.

P r o o f . The proof is based on the fact that in the model (1.2), there exist three different x_q, x_k and x_t and is omitted (see also Section 3). □

So we have proved the next theorem:

THEOREM 6.3. *In the model* (1.2), *the* $\boldsymbol{\beta}_\circ$*-LBLQUE of* $\sigma^2\big(a+b|\boldsymbol{e}_i'\mathbf{X}^R\boldsymbol{\beta}|\big)^2$ *exists for every* $i\in\{1,2,\dots,n+J-1\}$.

We only note that this $\boldsymbol{\beta}_\circ$-LBLQUE is unique for every $i\in\{1,2,\dots,n+J-1\}$ (see Section 3).

7. Dispersion of the $\boldsymbol{\beta}_\circ$-LBLQUE in the model (1.2)

It can easily be shown (see Lemma 9.2) that the matrix $(\mathbb{X}^R)'$ has rank $4+n$ (i.e., it has full row rank). From this follows the existence of the matrix

$$\big\{(\mathbb{X}^R)'\big(\boldsymbol{\Sigma}_R{}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}_R{}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}^R\big\}^{-1}=\begin{pmatrix}\mathbf{M}^R_{4,4} & \mathbf{N}^R_{4,n}\\ (\mathbf{N}^R)'_{n,4} & \mathbf{P}^R_{n,n}\end{pmatrix}.$$

In the same way as in Section 5, we obtain that the dispersion of the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a+b|\boldsymbol{e}_i'\mathbf{X}^R\boldsymbol{\beta}|\big)^2$, $i\in\{1,2,\dots,n+J-1\}$, at $\boldsymbol{\beta}_\circ$ is

$$\mathcal{D}_{\boldsymbol{\beta}_\circ}\big(\boldsymbol{a}_{(i)}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}_{(i)}\boldsymbol{Y}\big)=2\sigma^4\boldsymbol{e}_q'\mathbf{P}^R\boldsymbol{e}_q\,, \tag{7.1}$$

where

$$q=i \qquad \text{for} \quad i\in\big\{\{1,2,\dots,n\}-\{j\}\big\}\,,$$

and

$$q=j \qquad \text{for} \quad i\in\{j,n+1,\dots,n+J-1\}\,.$$

Using the notation

$$(\mathbb{X}^R)'\big(\boldsymbol{\Sigma}_R{}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}_R{}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}^R=\begin{pmatrix}\mathbf{V}^R_{4,4} & \mathbf{B}^R_{4,n}\\ (\mathbf{B}^R)'_{n,4} & \mathbf{Z}^R_{n,n}\end{pmatrix},$$

$$\mathbf{V}_{4,4}^{R} =$$

$$= \big((\mathbf{X}^R)'\boldsymbol{\Sigma}_R^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}^R\big) \otimes \big((\mathbf{X}^R)'\boldsymbol{\Sigma}_R^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}^R\big)$$

$$= \begin{pmatrix}
\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} \varphi_i + J\varphi_j\Big)^2 & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} \varphi_i + J\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} \varphi_i + J\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big) \\
\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} \varphi_i + J\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} \varphi_i + J\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i + Jx_j^2\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i + Jx_j^2\varphi_j\Big) \\
\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} \varphi_i + J\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i + Jx_j^2\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} \varphi_i + J\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i + Jx_j^2\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big) \\
\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i + Jx_j^2\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i + Jx_j^2\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i + Jx_j\varphi_j\Big) & \Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i + Jx_j^2\varphi_j\Big)\Big(\sum\limits_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i + Jx_j^2\varphi_j\Big)
\end{pmatrix},$$

with $\varphi_i = \big(a + b|\boldsymbol{e}_i'\mathbf{X}^R\boldsymbol{\beta}_\circ|\big)^{-2}$,

$$\mathbf{B}^R = \begin{pmatrix}
\varphi_1^2 & \varphi_2^2 & \dots & J\varphi_j^2 & \dots & \varphi_n^2 \\
x_1\varphi_1^2 & x_2\varphi_2^2 & \dots & Jx_j\varphi_j^2 & \dots & x_n\varphi_n^2 \\
x_1\varphi_1^2 & x_2\varphi_2^2 & \dots & Jx_j\varphi_j^2 & \dots & x_n\varphi_n^2 \\
x_1^2\varphi_1^2 & x_2^2\varphi_2^2 & \dots & Jx_j^2\varphi_j^2 & \dots & x_n^2\varphi_n^2
\end{pmatrix}$$

and

$$\mathbf{Z}^R = \begin{pmatrix}
\varphi_1^2 & 0 & \dots & & & 0 \\
0 & \varphi_2^2 & & & & \\
\vdots & & \ddots & & & \\
0 & & & J\varphi_j^2 & & \\
\vdots & & & & \ddots & \\
0 & & & & & \varphi_n^2
\end{pmatrix}.$$

Then for $\mathbf{P}^R$ we have (see, e.g., [1; p. 66])

$$\mathbf{P}^R = (\mathbf{Z}^R)^{-1} + (\mathbf{Z}^R)^{-1}(\mathbf{B}^R)'(\mathbf{W}^R)^{-1}\mathbf{B}^R(\mathbf{Z}^R)^{-1},$$

where

$$\mathbf{W}^R = \mathbf{V}^R - \mathbf{B}^R(\mathbf{Z}^R)^{-1}(\mathbf{B}^R)'.$$

If we denote

$$\alpha = \sum_{\substack{i=1\\ i\neq j}}^{n} \varphi_i, \qquad \beta = \sum_{\substack{i=1\\ i\neq j}}^{n} \varphi_i^2, \qquad \gamma = \sum_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i, \qquad \delta = \sum_{\substack{i=1\\ i\neq j}}^{n} x_i\varphi_i^2,$$

$$\varepsilon = \sum_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i, \qquad \xi = \sum_{\substack{i=1\\ i\neq j}}^{n} x_i^2\varphi_i^2, \qquad \eta = \sum_{\substack{i=1\\ i\neq j}}^{n} x_i^3\varphi_i^2, \qquad \rho = \sum_{\substack{i=1\\ i\neq j}}^{n} x_i^4\varphi_i^2,$$

then

$$\mathbf{W}^R = \begin{pmatrix} (\alpha+J\varphi_j)^2-(\beta+J\varphi_j^2) & (\alpha+J\varphi_j)(\gamma+Jx_j\varphi_j)-(\delta+Jx_j\varphi_j^2) & (\alpha+J\varphi_j)(\gamma+Jx_j\varphi_j)-(\delta+Jx_j\varphi_j^2) & (\gamma+Jx_j\varphi_j)^2-(\xi+Jx_j^2\varphi_j^2) \\ (\gamma+Jx_j\varphi_j)(\alpha+J\varphi_j)-(\delta+Jx_j\varphi_j^2) & (\alpha+J\varphi_j)(\varepsilon+Jx_j^2\varphi_j)-(\xi+Jx_j^2\varphi_j^2) & (\gamma+Jx_j\varphi_j)^2-(\xi+Jx_j^2\varphi_j^2) & (\gamma+Jx_j\varphi_j)(\varepsilon+Jx_j^2\varphi_j)-(\eta+Jx_j^3\varphi_j^2) \\ (\gamma+Jx_j\varphi_j)(\alpha+J\varphi_j)-(\delta+Jx_j\varphi_j^2) & (\gamma+Jx_j\varphi_j)^2-(\xi+Jx_j^2\varphi_j^2) & (\alpha+J\varphi_j)(\varepsilon+Jx_j^2\varphi_j)-(\xi+Jx_j^2\varphi_j^2) & (\gamma+Jx_j\varphi_j)(\varepsilon+Jx_j^2\varphi_j)-(\eta+Jx_j^3\varphi_j^2) \\ (\gamma+Jx_j\varphi_j)^2-(\xi+Jx_j^2\varphi_j^2) & (\gamma+Jx_j\varphi_j)(\varepsilon+Jx_j^2\varphi_j)-(\eta+Jx_j^3\varphi_j^2) & (\gamma+Jx_j\varphi_j)(\varepsilon+Jx_j^2\varphi_j)-(\eta+Jx_j^3\varphi_j^2) & (\varepsilon+Jx_j^2\varphi_j)^2-(\rho+Jx_j^4\varphi_j^2) \end{pmatrix}.$$

Now it is easy to obtain (see also Section 5) for $i \in \{\{1,2,\dots,n\} - \{j\}\}$:

$$\begin{aligned} &\mathcal{D}_{\boldsymbol{\beta}_\circ}\left(\boldsymbol{a}'_{(i)}\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}_{(i)}\boldsymbol{Y}\right) \\ &= 2\sigma^4\left[\left(a + b|\boldsymbol{e}_i'\mathbf{X}^R\boldsymbol{\beta}_\circ|\right)^4 + w^{11} + 4x_i w^{12} \right. \\ &\qquad \left. + 2x_i^2(w^{14} + w^{22} + w^{23}) + 4x_i^3 w^{24} + x_i^4 w^{44}\right], \end{aligned} \tag{7.2}$$

and for $i \in \{j, n+1, \dots, n+J-1\}$:

$$\begin{aligned} &\mathcal{D}_{\boldsymbol{\beta}_\circ}\left(\boldsymbol{a}'_{(i)}\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}_{(i)}\boldsymbol{Y}\right) \\ &= 2\sigma^4\left[\frac{(a + b|\boldsymbol{e}_j'\mathbf{X}^R\boldsymbol{\beta}_\circ|)^4}{J} + w^{11} + 4x_j w^{12} \right. \\ &\qquad \left. + 2x_j^2(w^{14} + w^{22} + w^{23}) + 4x_j^3 w^{24} + x_j^4 w^{44}\right], \end{aligned} \tag{7.3}$$

where the w^{ij} are elements of the matrix $(\mathbf{W}^R)^{-1}$.

8. Asymptotic behaviour of the β_o-LBLQUE in the model (1.2)

Let us now investigate the asymptotic behaviour of the β_o-LBLQUE of $\sigma^2(a+b|e_i'\mathbf{X}^R\beta|)^2$ at β_o for increasing J.

Using Lemma 9.6, after a little tedious computation, we obtain

$$w^{11}+4x_iw^{12}+2x_i^2(w^{14}+w^{22}+w^{23})+4x_i^3w^{24}+x_i^4w^{44} = \frac{J^3\varphi_j^3(x_j^2\alpha-2x_j\gamma+\varepsilon)(x_i-x_j)^4+J^2a_2+Ja_1+a_0}{J^3\alpha_3+J^2\alpha_2+J\alpha_1+\alpha_0}, \tag{8.1}$$

where

$$\begin{aligned}\alpha_3=\frac{1}{\varphi^3}\{\varepsilon^3-\varepsilon\rho+x_j[2\alpha\rho-6\varepsilon^2\gamma+4\varepsilon\eta]\\ +x_j^2[3\alpha\varepsilon^2-\alpha\rho-6\varepsilon\xi+12\gamma^2\varepsilon-8\gamma\eta]\\ +x_j^3[-8\gamma^3+12\gamma\xi-12\alpha\gamma\varepsilon+4\alpha\eta+4\varepsilon\delta]\\ +x_j^4[3\alpha^3\varepsilon-\beta\varepsilon-6\alpha\xi+12\alpha\gamma^2-8\gamma\delta]\\ +x_j^5[2\beta\gamma-6\alpha^2\gamma+4\alpha\delta]+x_j^6[\alpha^3-\alpha\beta]\}\end{aligned} \tag{8.2}$$

and a_2, a_1, a_0, α_2, α_1, α_0 are constants. (We note only that the term at J^3 in the numerator in (8.1) is always positive.)

Let us now calculate

$$\lim_{J\to\infty}\mathcal{D}_{\beta_o}(a_{(i)}'Y+Y'\mathbf{A}_{(i)}Y),$$

where $a_{(i)}'Y+Y'\mathbf{A}_{(i)}Y$ is the β_o-LBLQUE of $\sigma^2(a+b|e_i'\mathbf{X}^R\beta_o|)^2$.

(i) If $i\in\{\{1,2,\dots,n\}-\{j\}\}$, we obtain from (7.2), (8.1) and (8.2):

$$\lim_{J\to\infty}\mathcal{D}_{\beta_o}(a_{(i)}'Y+Y'\mathbf{A}_{(i)}Y)=2\sigma^4(a+b|e_i'\mathbf{X}^R\beta_o|)^4+2\sigma^4\frac{(x_j-x_i)^4}{b_1} \tag{8.3}$$

with

$$\begin{aligned}b_1&=\{\varepsilon^2-\rho+x_j[4\eta-4\varepsilon\gamma]+x_j^2[2\alpha\varepsilon+4\gamma^2-6\xi]+x_j^3[4\delta-4\alpha\gamma]+x_j^4[\alpha^2-\beta]\}\\ &=(x_j^2\alpha-2x_j\gamma+\varepsilon)^2-(x_j^4\beta-4x_j^3\delta+6x_j^2\xi-4x_j\eta+\rho)\\ &=\left(\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{(x_j-x_i)^2}{(a+b|e_i'\mathbf{X}^R\beta_o|)^2}\right)^2-\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{(x_j-x_i)^4}{(a+b|e_i'\mathbf{X}^R\beta_o|)^4}\end{aligned} \tag{8.4}$$

(which is always positive).

(ii) If $i \in \{j, n+1, \dots, n+J-1\}$, we obtain from (7.3), (8.1) and (8.2) (taking into account (8.4)):

$$\lim_{J\to\infty} \mathcal{D}_{\boldsymbol{\beta}_\circ}\left(\boldsymbol{a}'_{(i)}\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}_{(i)}\boldsymbol{Y}\right) = 0\,, \tag{8.5}$$

i.e., $\boldsymbol{a}'_{(j)}\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}_{(j)}\boldsymbol{Y}$ is a consistent estimator of $\sigma^2\left(a + b|\boldsymbol{e}'_i\mathbf{X}^R\boldsymbol{\beta}|\right)^2$ at $\boldsymbol{\beta}_\circ$.

Appendix 1

Lemma 9.1. *If $x_i \neq x_j$ for $i \neq j$, then*

$$\forall\{\boldsymbol{\beta}=(\beta_1,\beta_2)'\in\mathcal{R}^2\} \qquad \sum_{i=1}^{n} b_i|\beta_1 + \beta_2 x_i| = 0 \tag{9.1}$$

if and only if

$$b_1 = b_2 = \dots = b_n = 0 \tag{9.2}$$

is valid.

Proof. Without loss of generality, we can assume that

$$x_1 < x_2 < \dots < x_n\,.$$

Let us denote

$$|x_{i+1} - x_i| = a_i\,, \qquad i = 1,2,\dots,n-1\,.$$

It is obvious that for $n \geq l > r \geq 1$ we have

$$|x_l - x_r| = |x_r - x_l| = a_r + a_{r+1} + \dots + a_{l-1} > 0\,.$$

If (9.1) is valid, then for $\beta_1^{(j)} = -x_j$, $\beta_2^{(j)} = 1$ $(j = 1,2,\dots,n)$ we have

$$\sum_{i=1}^{n} b_i|-x_1 + x_i| = 0\,,$$

$$\vdots$$

$$\sum_{i=1}^{n} b_i|-x_n + x_i| = 0\,,$$

which can be rewritten as

$$\mathbf{A}\boldsymbol{b} = \begin{pmatrix} 0 & a_1 & a_1 + a_2 & \dots & \sum_{i=1}^{n-1} a_i \\ a_1 & 0 & a_2 & \dots & \sum_{i=2}^{n-1} a_i \\ a_1 + a_2 & a_2 & 0 & \dots & \sum_{i=3}^{n-1} a_i \\ \vdots & & & & \vdots \\ \sum_{i=1}^{n-2} a_i & \dots & \dots & \dots & a_{n-1} \\ \sum_{i=1}^{n-1} a_i & \dots & \dots & \dots & 0 \end{pmatrix} \boldsymbol{b} = \boldsymbol{O},$$

where $\boldsymbol{b}' = (b_1, \dots, b_n)$. If we subtract the $(j-1)$st row of the matrix $\mathbf{A}$ from its jth row $(j = n, n-1, \dots, 2)$ and successively add the first to the jth column $(j = n, n-1, \dots, 2)$, we obtain the matrix

$$\begin{pmatrix} 0 & a_1 & a_1 + a_2 & a_1 + a_2 + a_3 & \dots & \sum_{i=1}^{n-2} a_i & \sum_{i=1}^{n-1} a_i \\ a_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ a_2 & 2a_2 & 0 & 0 & \dots & 0 & 0 \\ a_3 & 2a_3 & 2a_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & 2a_{n-1} & 2a_{n-1} & 2a_{n-1} & \dots & 2a_{n-1} & 0 \end{pmatrix},$$

whose determinant is

$$(-1)^{n+1} 2^{n-2} \sum_{i=1}^{n-1} a_i \prod_{i=1}^{n-1} a_i > 0 .$$

The determinant of the matrix $\mathbf{A}$ has the same value. So (9.2) is valid. The opposite implication is trivial, and we have proved the lemma. □

By $\operatorname{vec} \mathbf{A}_{p,q}$, we denote the vector

$$\operatorname{vec} \mathbf{A} = (a_{11}, a_{21}, \dots, a_{p1}, a_{12}, a_{22}, \dots, a_{p2}, \dots, a_{1q}, a_{2q}, \dots, a_{pq})'$$

for an arbitrary $p \times q$ matrix with elements $a_{ij} = \boldsymbol{e}_i' \mathbf{A} \boldsymbol{e}_j$. $\mathbf{I}^*$ is a symmetric nonsingular matrix for which the next assertion is valid:

$$\forall \{\mathbf{A}_{t,t}\} \qquad \mathbf{I}^* \operatorname{vec} \mathbf{A} = \operatorname{vec} \mathbf{A}' .$$

LEMMA 9.2. *Let $n \geqq 4$. The matrix $\mathbb{X}'$ of order $(4+n)\times n^2$*

$$\mathbb{X}' = \begin{pmatrix} \mathbf{X}'\otimes\mathbf{X}' \\ e_1'\otimes e_1' \\ \vdots \\ e_n'\otimes e_n' \end{pmatrix}$$

is of rank $4+n$.

Proof. The 3rd, 4th, $(n+3)$rd and $(n+4)$th column of the matrix $\mathbb{X}'$ are

$$\begin{aligned}&(1, x_3, x_1, x_1x_3, 0, \dots, 0)',\\ &(1, x_4, x_1, x_1x_4, 0, \dots, 0)',\\ &(1, x_3, x_2, x_2x_3, 0, \dots, 0)',\\ &(1, x_4, x_2, x_2x_4, 0, \dots, 0)'.\end{aligned}$$

Since

$$\begin{aligned}
&\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_3 & x_4 & x_3 & x_4 \\ x_1 & x_1 & x_2 & x_2 \\ x_1x_3 & x_1x_4 & x_2x_3 & x_2x_4 \end{pmatrix}\\
&= \det\begin{pmatrix} 1 & 1 & 0 & 0 \\ x_3 & x_4 & 0 & 0 \\ x_1 & x_1 & x_2-x_1 & x_2-x_1 \\ x_1x_3 & x_1x_4 & x_3(x_2-x_1) & x_4(x_2-x_1) \end{pmatrix}\\
&= \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_3 & x_4-x_3 & 0 & 0 \\ x_1 & 0 & x_2-x_1 & 0 \\ x_1x_3 & x_1(x_4-x_3) & x_3(x_2-x_1) & (x_2-x_1)(x_4-x_3) \end{pmatrix}\\
&= (x_2-x_1)^2(x_4-x_3)^2\,,
\end{aligned}$$

it is obvious that the rank of the matrix $\mathbb{X}'$ is $4+n$. □

LEMMA 9.3. *We have*

$$(\mathbf{X}'\otimes\mathbf{X}')\mathbf{I}^*_{n^2,n^2} = \mathbf{I}^*_{4,4}(\mathbf{X}'\otimes\mathbf{X}')\,. \tag{9.3}$$

Proof. For an arbitrary matrix $\mathbf{Z}_{n,n}$ is

$$\begin{aligned}\mathbf{I}^*_{4,4}(\mathbf{X}'\otimes\mathbf{X}')\operatorname{vec}\mathbf{Z} &= \mathbf{I}^*\operatorname{vec}\mathbf{X}'\mathbf{Z}\mathbf{X} = \operatorname{vec}\mathbf{X}'\mathbf{Z}'\mathbf{X}\\ &= (\mathbf{X}'\otimes\mathbf{X}')\operatorname{vec}\mathbf{Z}' = (\mathbf{X}'\otimes\mathbf{X}')\mathbf{I}^*_{n^2,n^2}\operatorname{vec}\mathbf{Z}\end{aligned}$$

(using the formula $\operatorname{vec}\mathbf{ABC} = (\mathbf{C}'\otimes\mathbf{A})\operatorname{vec}\mathbf{B}$). □

LEMMA 9.4. *One choice of*

$$\left[\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\right]^-$$

is

$$\frac{1}{8}\{\mathbb{X}'\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\}^{-1}\begin{pmatrix}(\mathbf{I}+\mathbf{I}^*)_{4,4} & \mathbf{O}_{4,n}\\ \mathbf{O}_{n,4} & 2\mathbf{I}_{n,n}\end{pmatrix}.$$

P r o o f . First we note that according to Lemma 9.2, is $\mathbb{X}'$ of full row rank, so the inverse of the matrix $\mathbb{X}'\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}$ exists.

Since

$$\mathbf{I}^*\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)=\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbf{I}^*$$

(see, e.g., [7; Lemma 3.8]) and also (9.3) is valid, we have

$$\begin{aligned}
&\mathbb{X}'(\mathbf{I}+\mathbf{I}^*)\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I}+\mathbf{I}^*)\mathbb{X}\\
&=2\begin{pmatrix}(\mathbf{X}'\otimes\mathbf{X}')(\mathbf{I}+\mathbf{I}^*)\\ 2e_1'\otimes e_1'\\ \vdots\\ 2e_n'\otimes e_n'\end{pmatrix}\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\\
&=2\begin{pmatrix}(\mathbf{I}+\mathbf{I}^*)_{4,4} & \mathbf{O}_{4,n}\\ \mathbf{O}_{n,4} & 2\mathbf{I}_{n,n}\end{pmatrix}\mathbb{X}'\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X} \qquad (9.4)\\
&=2\mathbb{X}'\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\begin{pmatrix}(\mathbf{I}+\mathbf{I}^*) & \mathbf{O}\\ \mathbf{O} & 2\mathbf{I}\end{pmatrix}. \qquad (9.5)
\end{aligned}$$

So we have

$$\begin{aligned}
&2\begin{pmatrix}(\mathbf{I}+\mathbf{I}^*)_{4,4} & \mathbf{O}_{4,n}\\ \mathbf{O}_{n,4} & 2\mathbf{I}_{n,n}\end{pmatrix}\mathbb{X}'\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\cdot\\
&\cdot\frac{1}{8}\{\mathbb{X}'\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\}^{-1}\begin{pmatrix}(\mathbf{I}+\mathbf{I}^*)_{4,4} & \mathbf{O}_{4,n}\\ \mathbf{O}_{n,4} & 2\mathbf{I}_{n,n}\end{pmatrix}\cdot\\
&\cdot 2\begin{pmatrix}(\mathbf{I}+\mathbf{I}^*)_{4,4} & \mathbf{O}_{4,n}\\ \mathbf{O}_{n,4} & 2\mathbf{I}_{n,n}\end{pmatrix}\mathbb{X}'\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\\
&\qquad\qquad=2\begin{pmatrix}(\mathbf{I}+\mathbf{I}^*)_{4,4} & \mathbf{O}_{4,n}\\ \mathbf{O}_{n,4} & 2\mathbf{I}_{n,n}\end{pmatrix}\mathbb{X}'\big(\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}.
\end{aligned}$$

□

Let us denote

$$\mathbf{V}_{4,4} = \big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\mathrm{o}})\mathbf{X}\big) \otimes \big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\mathrm{o}})\mathbf{X}\big)\,,$$

$$\mathbf{T}_{4,4} = \begin{pmatrix} \sum\limits_{i=1}^{n} \varphi_i^2 & \sum\limits_{i=1}^{n} x_i\varphi_i^2 & \sum\limits_{i=1}^{n} x_i\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^2\varphi_i^2 \\ \sum\limits_{i=1}^{n} x_i\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^2\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^2\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^3\varphi_i^2 \\ \sum\limits_{i=1}^{n} x_i\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^2\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^2\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^3\varphi_i^2 \\ \sum\limits_{i=1}^{n} x_i^2\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^3\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^3\varphi_i^2 & \sum\limits_{i=1}^{n} x_i^4\varphi_i^2 \end{pmatrix}, \tag{9.6}$$

where $\varphi_i = \big(a + b|\boldsymbol{e}_i'\mathbf{X}\boldsymbol{\beta}_{\mathrm{o}}|\big)^{-2}$ and

$$\mathbf{W}_{4,4} = \mathbf{V} - \mathbf{T}\,.$$

It can be easily seen that

$$\mathbf{W} = \begin{pmatrix} \Big(\sum\limits_{i=1}^{n}\varphi_i\Big)^2 - \sum\limits_{i=1}\varphi_i^2 & \sum\limits_{i=1}^{n}\varphi_i\sum\limits_{i=1}^{n}x_i\varphi_i - \sum\limits_{i=1}^{n}x_i\varphi_i^2 & \sum\limits_{i=1}^{n}\varphi_i\sum\limits_{i=1}^{n}x_i\varphi_i - \sum\limits_{i=1}^{n}x_i\varphi_i^2 & \Big(\sum\limits_{i=1}^{n}x_i\varphi_i\Big)^2 - \sum\limits_{i=1}^{n}x_i^2\varphi_i^2 \\ \sum\limits_{i=1}^{n}\varphi_i\sum\limits_{i=1}^{n}x_i\varphi_i - \sum\limits_{i=1}^{n}x_i\varphi_i^2 & \sum\limits_{i=1}^{n}\varphi_i\sum\limits_{i=1}^{n}x_i^2\varphi_i - \sum\limits_{i=1}^{n}x_i^2\varphi_i^2 & \Big(\sum\limits_{i=1}^{n}x_i\varphi_i\Big)^2 - \sum\limits_{i=1}^{n}x_i^2\varphi_i^2 & \sum\limits_{i=1}^{n}x_i\varphi_i\sum\limits_{i=1}^{n}x_i^2\varphi_i - \sum\limits_{i=1}^{n}x_i^3\varphi_i^2 \\ \sum\limits_{i=1}^{n}\varphi_i\sum\limits_{i=1}^{n}x_i\varphi_i - \sum\limits_{i=1}^{n}x_i\varphi_i^2 & \Big(\sum\limits_{i=1}^{n}x_i\varphi_i\Big)^2 - \sum\limits_{i=1}^{n}x_i^2\varphi_i^2 & \sum\limits_{i=1}^{n}\varphi_i\sum\limits_{i=1}^{n}x_i^2\varphi_i - \sum\limits_{i=1}^{n}x_i^2\varphi_i^2 & \sum\limits_{i=1}^{n}x_i\varphi_i\sum\limits_{i=1}^{n}x_i^2\varphi_i - \sum\limits_{i=1}^{n}x_i^3\varphi_i^2 \\ \Big(\sum\limits_{i=1}^{n}x_i\varphi_i\Big)^2 - \sum\limits_{i=1}^{n}x_i^2\varphi_i^2 & \sum\limits_{i=1}^{n}x_i\varphi_i\sum\limits_{i=1}^{n}x_i^2\varphi_i - \sum\limits_{i=1}^{n}x_i^3\varphi_i^2 & \sum\limits_{i=1}^{n}x_i\varphi_i\sum\limits_{i=1}^{n}x_i^2\varphi_i - \sum\limits_{i=1}^{n}x_i^3\varphi_i^2 & \Big(\sum\limits_{i=1}^{n}x_i^2\varphi_i\Big)^2 - \sum\limits_{i=1}^{n}x_i^4\varphi_i^2 \end{pmatrix}. \tag{9.7}$$

We denote the elements of the matrix $\mathbf{W}^{-1}$ by $\{w^{ij}\}$; its regularity follows from the regularity of the matrix $\mathbb{X}'\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\mathrm{o}}) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\mathrm{o}})\big)\mathbb{X}$ (cf. the proof of Lemma 9.5).

LEMMA 9.5. *If*

$$\big\{\mathbb{X}'\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\mathrm{o}}) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\mathrm{o}})\big)\mathbb{X}\big\}^{-1} = \begin{pmatrix} \mathbf{M}_{4,4} & \mathbf{N}_{4,n} \\ \mathbf{N}'_{n,4} & \mathbf{P}_{n,n} \end{pmatrix}, \tag{9.8}$$

then for $j \in \{1, 2, \dots, n\}$ *we have*

$$\mathbf{N}\boldsymbol{e}_j = -\begin{pmatrix} w^{11} + x_j(w^{12} + w^{13}) + x_j^2 w^{14} \\ w^{21} + x_j(w^{22} + w^{23}) + x_j^2 w^{24} \\ w^{31} + x_j(w^{32} + w^{33}) + x_j^2 w^{34} \\ w^{41} + x_j(w^{42} + w^{43}) + x_j^2 w^{44} \end{pmatrix}, \tag{9.9}$$

and for $l \in \{1,2,\dots,n\}$ *is*

$$e_l'\mathbf{P}e_j = \delta_{jl}\big(a + b|e_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^4 - (1\ x_l\ x_l\ x_l^2)\mathbf{N}e_j\,,$$

where δ_{jl} *is* 0 *for* $j \neq l$ *and* 1 *for* $l = j$.

P r o o f . Since

$$\mathbb{X}'\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X} = \begin{pmatrix} \mathbf{V} & (\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ))(e_1\otimes e_1,\dots,e_n\otimes e_n) \\ \begin{pmatrix} e_1'\otimes e_1' \\ \vdots \\ e_n'\otimes e_n' \end{pmatrix}(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}) & \boldsymbol{\Sigma}^{-2}(\boldsymbol{\beta}_\circ) \end{pmatrix},$$

we have

$$\mathbf{P} = \boldsymbol{\Sigma}^2(\boldsymbol{\beta}_\circ) + \boldsymbol{\Sigma}^2(\boldsymbol{\beta}_\circ)\begin{pmatrix} e_1'\otimes e_1' \\ \vdots \\ e_n'\otimes e_n' \end{pmatrix}\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\big)\mathbf{W}^{-1}\cdot \tag{9.10}$$
$$\cdot\big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(e_1\otimes e_1,\dots,e_n\otimes e_n)\boldsymbol{\Sigma}^2(\boldsymbol{\beta}_\circ)$$

(see, e.g., [1; p. 66]) because of the identity

$$\mathbf{T} = \big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(e_1\otimes e_1,\dots,e_n\otimes e_n)\boldsymbol{\Sigma}^2(\boldsymbol{\beta}_\circ)\cdot$$
$$\cdot\begin{pmatrix} e_1'\otimes e_1' \\ \vdots \\ e_n'\otimes e_n' \end{pmatrix}\big((\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\big)\,.$$

Similarly, we have

$$\mathbf{N} = -\mathbf{W}^{-1}\big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(e_1\otimes e_1,\dots,e_n\otimes e_n)\boldsymbol{\Sigma}^2(\boldsymbol{\beta}_\circ)\,. \tag{9.11}$$

From (9.10), we obtain for $l \in \{1,2,\dots,n\}$

$$e_l'\mathbf{P}e_j = \delta_{jl}\big(a + b|e_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^4 + (1\quad x_l\quad x_l\quad x_l^2)\,\mathbf{W}^{-1}\begin{pmatrix} 1 \\ x_j \\ x_j \\ x_j^2 \end{pmatrix}$$
$$= \delta_{jl}\big(a + b|e_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^4 - (1\quad x_l\quad x_l\quad x_l^2)\,\mathbf{N}e_j\,.$$

Since

$$\big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(e_1\otimes e_1,\dots,e_n\otimes e_n)\boldsymbol{\Sigma}^2(\boldsymbol{\beta}_\circ)e_j = (1\quad x_j\quad x_j\quad x_j^2)\,,$$

from (9.11) it is easy to obtain (9.9). □

Lemma 9.6. *The nonsingular matrix*

$$\begin{pmatrix} a & b & b & d \\ b & c & d & e \\ b & d & c & e \\ d & e & e & f \end{pmatrix}$$

has

$$\frac{1}{\Delta}\begin{pmatrix} c^2f+2de^2-2ce^2-d^2f & -bcf-d^2e+dce+bdf & -bcf-d^2e+cde+bdf & -2bde-c^2d+d^3+2bce \\ -bcf-d^2e+cde+bdf & acf+2bde-cd^2-b^2f-ae^2 & -adf-2bde+d^3+b^2f+ae^2 & ade+bcd-bd^2-ace \\ -bcf-d^2e+cde+bdf & -adf-2bde+d^3+b^2f+ae^2 & acf+2bde-cd^2-b^2f-ae^2 & ade+bcd-bd^2-ace \\ -2bde-c^2d+d^3+2bce & ade+bcd-bd^2-ace & ade+bcd-bd^2-ace & ac^2+2b^2d-2b^2c-ad^2 \end{pmatrix},$$

where

$$\Delta = (c-d)\Delta_1 = (c-d)\big[(c+d)(af-d^2)+2e(bd-ae)+2b(de-bf)\big]$$

as its inverse.

P r o o f. The lemma can be proved by an easy computation. □

From the previous lemma, we obtain the following formulas for the elements of the matrix $\mathbf{W}^{-1}$:

$$\begin{aligned} w^{11} = \frac{1}{\Delta_1}\Bigg\{ & \Bigg[\Bigg(\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_\circ|)^2}\Bigg)^2 - \sum_{i=1}^{n}\frac{x_i^4}{(a+b|e_i'\mathbf{X}\beta_\circ|)^4}\Bigg]\cdot \\ & \cdot\Bigg[\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_\circ|)^2}\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_\circ|)^2} \\ & \quad -2\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_\circ|)^4}+\Bigg(\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_\circ|)^2}\Bigg)^2\Bigg] \\ & -2\Bigg[\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_\circ|)^2}\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_\circ|)^2} \\ & \quad -\sum_{i=1}^{n}\frac{x_i^3}{(a+b|e_i'\mathbf{X}\beta_\circ|)^4 x_i^3}\Bigg]^2\Bigg\}, \end{aligned} \tag{9.12}$$

$$w^{12} =$$

$$= \frac{1}{\Delta_1}\Bigg\{ \Bigg[\Bigg(\sum_{i=1}^{n} \frac{x_i}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \Bigg)^2 - \sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} \Bigg] \cdot$$

$$\cdot \Bigg[\sum_{i=1}^{n} \frac{x_i}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} - \sum_{i=1}^{n} \frac{x_i^3}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} \Bigg]$$

$$- \Bigg[\Bigg(\sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \Bigg)^2 - \sum_{i=1}^{n} \frac{x_i^4}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} \Bigg] \cdot$$

$$\cdot \Bigg[\sum_{i=1}^{n} \frac{1}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \sum_{i=1}^{n} \frac{x_i}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} - \sum_{i=1}^{n} \frac{x_i}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} \Bigg] \Bigg\}, \tag{9.13}$$

$$w^{22} + w^{23} = \frac{1}{\Delta_1}\Bigg\{ \Bigg[\Bigg(\sum_{i=1}^{n} \frac{1}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \Bigg)^2 - \sum_{i=1}^{n} \frac{1}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} \Bigg] \cdot$$

$$\cdot \Bigg[\Bigg(\sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \Bigg)^2 - \sum_{i=1}^{n} \frac{x_i^4}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} \Bigg]$$

$$- \Bigg[\Bigg(\sum_{i=1}^{n} \frac{x_i}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \Bigg)^2 - \sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} \Bigg]^2 \Bigg\}, \tag{9.14}$$

$$w^{14} =$$

$$= \frac{1}{\Delta_1}\Bigg\{ 2 \Bigg[\sum_{i=1}^{n} \frac{1}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \sum_{i=1}^{n} \frac{x_i}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} - \sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} \Bigg] \cdot$$

$$\cdot \Bigg[\sum_{i=1}^{n} \frac{x_i}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} - \sum_{i=1}^{n} \frac{x_i^3}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} x_i^3 \Bigg]$$

$$- \Bigg[\Bigg(\sum_{i=1}^{n} \frac{x_i}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \Bigg)^2 - \sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4} \Bigg] \cdot$$

$$\cdot \Bigg[\sum_{i=1}^{n} \frac{1}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} - 2 \sum_{i=1}^{n} \frac{x_i^2}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^4}$$

$$+ \Bigg(\sum_{i=1}^{n} \frac{x_i}{\big(a+b|e_i'\mathbf{X}\beta_o|\big)^2} \Bigg)^2 \Bigg] \Bigg\}, \tag{9.15}$$

$$w^{24} =$$

$$= \frac{1}{\Delta_1}\left\{\left[\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}-\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\cdot\right.$$

$$\cdot\left[\left(\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]$$

$$-\left[\left(\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\cdot$$

$$\left.\cdot\left[\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^2}-\sum_{i=1}^{n}\frac{x_i^3}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\right\}, \tag{9.16}$$

$$w^{44} =$$

$$= \frac{1}{\Delta_1}\left\{\left[\left(\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\cdot\right.$$

$$\cdot\left[\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right.$$

$$\left.-2\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^4}+\left(\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2\right]$$

$$\left.-2\left[\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}-\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]^2\right\} \tag{9.17}$$

with

$$\Delta_1 = \left[\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right.$$

$$\left.-2\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^4}+\left(\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2\right]\cdot$$

$$\cdot\left\{\left[\left(\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\cdot\right.$$

$$\cdot\left[\left(\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{x_i^4}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]$$

$$
-\left[\left(\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]^2\Bigg\}
$$

$$
+2\left[\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^2}-\sum_{i=1}^{n}\frac{x_i^3}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\cdot
$$

$$
\cdot\Bigg\{\left[\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}-\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\cdot
$$

$$
\cdot\left[\left(\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^4}x_i^2\right]
$$

$$
-\left[\left(\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\cdot
$$

$$
\cdot\left[\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^2}-\sum_{i=1}^{n}\frac{x_i^3}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\Bigg\}
$$

$$
+2\left[\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}-\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\cdot
$$

$$
\cdot\Bigg\{\left[\left(\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\cdot
$$

$$
\cdot\left[\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^2}-\sum_{i=1}^{n}\frac{x_i^3}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]
$$

$$
-\left[\sum_{i=1}^{n}\frac{1}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^2}-\sum_{i=1}^{n}\frac{x_i}{(a+b|e_i'\mathbf{X}\beta_o|)^4}x_i\right]\cdot
$$

$$
\cdot\left[\left(\sum_{i=1}^{n}\frac{x_i^2}{(a+b|e_i'\mathbf{X}\beta_o|)^2}\right)^2-\sum_{i=1}^{n}\frac{x_i^4}{(a+b|e_i'\mathbf{X}\beta_o|)^4}\right]\Bigg\}. \tag{9.18}
$$

Appendix 2

Let us now investigate in the model (1.1) the dispersion of the $\boldsymbol{\beta}_o$-LBLQUE of $\sigma^2\left(a+b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}|\right)^2$ at $\boldsymbol{\beta}_o$ in the special case of increasing the number of measuring points x_i.

Let

$$x_i = 2^i + \frac{2^i - 1}{b} i, \qquad i = 1, 2, \dots,$$
$$a = \sigma^2 = 1, \qquad b > 0,$$

and

$$\boldsymbol{\beta}_\circ = (0, 1)'.$$

We obtain

$$\left(a + b|\boldsymbol{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\right)^2 = 4^i(b+1)^2, \qquad i = 1, 2, \dots,$$

and so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\left(a + b|\boldsymbol{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\right)^2} = \frac{1}{3(b+1)^2}, \tag{10.1}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i}{\left(a + b|\boldsymbol{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\right)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{1}{b}\left(2^i(b+1) - 1\right)}{4^i(b+1)^2}$$
$$= \frac{1}{b(b+1)} - \frac{1}{3b(b+1)^2} = \frac{3b+2}{3b(b+1)^2}. \tag{10.2}$$

Similarly, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i^2}{\left(a + b|\boldsymbol{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\right)^2} = \frac{1}{b^2}, \tag{10.3}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\left(a + b|\boldsymbol{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\right)^4} = \frac{1}{15(b+1)^4}, \tag{10.4}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i}{\left(a + b|\boldsymbol{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\right)^4} = \frac{1}{7b(b+1)^3} - \frac{1}{15b(b+1)^4}, \tag{10.5}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^2}{\left(a + b|\boldsymbol{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\right)^4} = \frac{1}{3b^2(b+1)^2} - \frac{2}{7b^2(b+1)^3}$$
$$+ \frac{1}{15b^2(b+1)^4}, \tag{10.6}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^3}{\left(a + b|\boldsymbol{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\right)^4} = \frac{1}{b^3(b+1)} - \frac{1}{b^3(b+1)^2}$$
$$+ \frac{3}{7b^3(b+1)^3} - \frac{1}{15b^3(b+1)^4}, \tag{10.7}$$

and finally

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{\left(a+b|e_i'\mathbf{X}\boldsymbol{\beta}_{\rm o}|\right)^4}=\frac{1}{b^4}\,. \tag{10.8}$$

If we denote the matrix $\mathbf{W}$ (defined in (9.7)) by

$$\mathbf{W}=\begin{pmatrix} w_1 & w_2 & w_2 & w_4\\ w_2 & w_3 & w_4 & w_5\\ w_2 & w_4 & w_3 & w_5\\ w_4 & w_5 & w_5 & w_6\end{pmatrix},$$

we have (see Lemma 9.6)

$$\Delta_1=w_1w_3w_6+w_1w_4w_6-w_3w_4^2-w_4^3+4w_2w_4w_5-2w_1w_5^2-2w_2^2w_6\,.$$

From (10.1) – (10.8), we have

$$\lim_{n\to\infty}\frac{1}{n}w_3=\frac{1}{3b^2(b+1)^2}\,,$$
$$\lim_{n\to\infty}\frac{1}{n}w_5=\frac{3b+2}{3b^3(b+1)^2}\,,$$

and

$$\lim_{n\to\infty}\frac{1}{n^2}w_6=\frac{1}{b^4}\,.$$

The other elements of the matrix $\mathbf{W}$ tend to constants with increasing n. It is easy to see (using Lemma 9.6 and (5.3)) that

$$\lim_{n\to\infty}w^{11}=\frac{45}{2}(b+1)^4\,,$$
$$\lim_{n\to\infty}w^{12}=\lim_{n\to\infty}(w^{14}+w^{22}+w^{23})=\lim_{n\to\infty}w^{24}=\lim_{n\to\infty}w^{44}=0\,,$$

and the dispersion of the $\boldsymbol{\beta}_{\rm o}$-LBLQUE of $\sigma^2\left(a+b|e_i'\mathbf{X}\boldsymbol{\beta}|\right)^2$ at $\boldsymbol{\beta}_{\rm o}$ in the model (1.1) tends to $2\sigma^4\left[\left(a+b|e_i'\mathbf{X}\boldsymbol{\beta}_{\rm o}|\right)^4+\frac{45}{2}(b+1)^4\right]=2\sigma^4\left[(1+bx_i)^4+\frac{45}{2}(b+1)^4\right]$ for $i=1,2,\dots$.

Appendix 3

A very serious question is proving the calculated value of the estimate. Here we give an alternative method of computing the $\boldsymbol{\beta}_{\rm o}$-LBLQUE of $\sigma^2\left(a+b|e_j'\mathbf{X}\boldsymbol{\beta}|\right)^2$ in the model (1.1). This method can be used for proving the numerical value of the estimate (as we know that the estimator is unique (see Section 3)).

From (4.1), we obtained that $\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ is the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\boldsymbol{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ in (1.1) if and only if

$$\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y} = \frac{1}{2}\big(\boldsymbol{Y}' \otimes (\boldsymbol{Y} - 2\mathbf{X}\boldsymbol{\beta}_\circ)'\big)(\mathbf{I} + \mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbb{X}\cdot \\ \cdot \{\mathbb{X}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ))\mathbb{X}\}^{-1}\begin{pmatrix}\mathbf{O}\\ \boldsymbol{e}_j\end{pmatrix}.$$

Using the notation

$$\{\mathbb{X}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ))\mathbb{X}\}^{-1} = \begin{pmatrix}\mathbf{M}_{4,4} & \mathbf{N}_{4,n}\\ \mathbf{N}'_{n,4} & \mathbf{P}_{n,n}\end{pmatrix}$$

and using, e.g., [1; p. 66, (15)] we obtain

$$\begin{aligned}
&\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}\\
&= \frac{1}{2}\big(\boldsymbol{Y}' \otimes (\boldsymbol{Y} - 2\mathbf{X}\boldsymbol{\beta}_\circ)'\big)(\mathbf{I} + \mathbf{I}^*)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\cdot\\
&\quad \cdot (\mathbf{X} \otimes \mathbf{X}, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \dots, \boldsymbol{e}_n \otimes \boldsymbol{e}_n)\cdot\\
&\quad \cdot \begin{pmatrix} -\mathbf{V}^{-1}\big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \dots, \boldsymbol{e}_n \otimes \boldsymbol{e}_n) \\ \mathbf{I}\end{pmatrix}\cdot\\
&\quad \cdot \Bigg[\boldsymbol{\Sigma}^{-2}(\boldsymbol{\beta}_\circ) - \begin{pmatrix}\boldsymbol{e}_1' \otimes \boldsymbol{e}_1'\\ \vdots \\ \boldsymbol{e}_n' \otimes \boldsymbol{e}_n'\end{pmatrix}\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X} \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\big)\mathbf{V}^{-1}\cdot\\
&\qquad \cdot \big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \dots, \boldsymbol{e}_n \otimes \boldsymbol{e}_n)\Bigg]^{-1}\boldsymbol{e}_j\\
&= \frac{1}{2}\big(\boldsymbol{Y}' \otimes (\boldsymbol{Y} - 2\mathbf{X}\boldsymbol{\beta}_\circ)'\big)(\mathbf{I} + \mathbf{I}^*)\cdot\\
&\quad \cdot \Big\{-\Big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\big(\mathbf{X}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\big)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\\
&\qquad \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\big(\mathbf{X}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\mathbf{X}\big)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\Big)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \dots, \boldsymbol{e}_n \otimes \boldsymbol{e}_n)\\
&\qquad + \sum_{u=1}^{n}\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\boldsymbol{e}_u \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\boldsymbol{e}_u\big)\boldsymbol{e}_u'\Big\}\mathbf{P}\boldsymbol{e}_j
\end{aligned}$$

$$= \frac{1}{2}\big(\boldsymbol{Y}' \otimes (\boldsymbol{Y} - 2\mathbf{X}\boldsymbol{\beta}_{\circ})'\big)(\mathbf{I} + \mathbf{I}^*) \cdot$$
$$\cdot \Bigg\{ \sum_{u=1}^{n} \Big[(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\boldsymbol{e}_u \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\boldsymbol{e}_u)$$
$$- \Big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}(\mathbf{X}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\boldsymbol{e}_u$$
$$\otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big(\mathbf{X}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\boldsymbol{e}_u\Big)\Big] \boldsymbol{e}'_u \Bigg\} \mathbf{P}\boldsymbol{e}_j\,, \tag{11.1}$$

where

$$\mathbf{P} = \Bigg[\boldsymbol{\Sigma}^{-2}(\boldsymbol{\beta}_{\circ}) - \begin{pmatrix} \boldsymbol{e}'_1 \otimes \boldsymbol{e}'_1 \\ \vdots \\ \boldsymbol{e}'_n \otimes \boldsymbol{e}'_n \end{pmatrix} \Big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big(\mathbf{X}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})$$
$$\otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big(\mathbf{X}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\Big)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \dots, \boldsymbol{e}_n \otimes \boldsymbol{e}_n)\Bigg]^{-1}.$$

We see that

$$\{\mathbf{P}^{-1}\}_{i,j} = \delta_{i,j}\big(a + b|\boldsymbol{e}_j\mathbf{X}\boldsymbol{\beta}|\big)^4 - \big\{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big(\mathbf{X}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big\}^2_{i,j} \tag{11.2}$$

($\delta_{i,j}$ is defined in Lemma 9.5).

As $\mathbf{I}^*(\mathbf{D}\boldsymbol{e}_t \otimes \mathbf{D}\boldsymbol{e}_t) = \mathbf{D}\boldsymbol{e}_t \otimes \mathbf{D}\boldsymbol{e}_t$ for any square matrix $\mathbf{D}$, we obtain from (11.1) the alternative formula for the $\boldsymbol{\beta}_{\circ}$-LBLQUE of $\sigma^2\big(a + b|\boldsymbol{e}'_j\mathbf{X}\boldsymbol{\beta}|\big)^2$ as

$$\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$$
$$= \sum_{s=1}^{n}\sum_{t=1}^{n} Y_s(Y_t - 2\boldsymbol{e}'_t\mathbf{X}\boldsymbol{\beta}_{\circ}) \cdot$$
$$\cdot \Bigg\{ \sum_{u=1}^{n} \Big[\delta_{su}\delta_{tu}\big(a + b|\boldsymbol{e}'_j\mathbf{X}\boldsymbol{\beta}_{\circ}|\big)^{-4} \tag{11.3}$$
$$- \big\{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}(\mathbf{X}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big\}_{s,u} \cdot$$
$$\cdot \big\{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big(\mathbf{X}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\mathbf{X}\big)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big\}_{t,u}\Big]\{\mathbf{P}\}_{u,j}\Bigg\}\,,$$

where $\{\mathbf{P}\}_{u,j}$ can be obtained from (11.2).

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