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# ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A CERTAIN TYPE OF THE THIRD ORDER DIFFERENTIAL EQUATIONS

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In [1] M. Greguš among other things investigates the asymptotic properties of solutions without zero-points and also asymptotic properties of oscillatory solutions of the third order linear differential equation in the normal form

$$y'''(\tau) + 2A(\tau)y'(\tau) + [A'(\tau) + b(\tau)]y(\tau) = 0. \tag{$A_0$}$$

The functions $A, A', b \in C^0(\langle \tau_0, \beta))$, $-\infty \leqq \tau_0 < \beta \leqq +\infty$. He simultaneously investigates the adjoint differential equation to the equation ($A_0$) in the form

$$z'''(\tau) + 2A(\tau)z'(\tau) + [A'(\tau) - b(\tau)]z(\tau) = 0. \tag{$B_0$}$$

The function $b(\tau)$ (Laguerre invariant) is supposed to have the property (V): $b(\tau) \geqq 0$ *for all* $\tau \in \langle \tau_0, \beta)$ *and* $b(\tau) \equiv 0$ *does not hold in any interval.*

In this paper we investigate asymptotic behaviour of solutions without zero-points and the asymptotic behaviour of the oscillatory solutions of a more general third order linear ordinary differential equation of the form

$$[r(t)[r(t)x'(t)]']' + 2a_1(t)x'(t) + [a_1'(t) + b_1(t)]x(t) = 0,\ t \geqq t_0, \tag{$A_1$}$$

*where* $r(t) > 0$, $rx' \in C^0(\langle t_0, \infty))$, $[rx']' \in C^0(\langle t_0, \infty)$, $[r[rx']']' \in C^0(\langle t_0, \infty))$; $a_1 \in C^1(\langle t_0, \infty))$, $b_1 \in C^0(\langle t_0, \infty))$, $-\infty < t_0 < +\infty$.

The adjoint equation to ($A_1$) is of the form

$$[r(t)[r(t)\xi'(t)]']' + 2a_1(t)\xi'(t) + [a_1'(t) - b_1(t)]\xi(t) = 0. \tag{$B_1$}$$

Two cases will be considered:

**I.** (1) $$\int^{\infty} \mathrm{d}t/r(t) = \infty \quad \text{and}$$

**II.** (2) $$\int^{\infty} \mathrm{d}t/r(t) < \infty.$$

In both cases the solutions of $(A_1)$ will be transformed into solutions of equations which will have the form $(A_0)$, and conversely. On the basis of these transformations we shall apply the results from [1] and [2] for the solutions of $(A_1)$. The idea to apply the transformations stems from the paper [3] by Philos.

**I.** If $\int^\infty \mathrm{d}t/r(t) = \infty$, then we denote

$$R(t) = \int_{t_0}^{t} \mathrm{d}s/r(s) \quad \text{for } t \geqq t_0. \tag{3}$$

The function $R \in C^4(\langle t_0, \infty))$, is increasing and maps the interval $\langle t_0, \infty)$ onto the interval $\langle 0, \infty)$. Its inverse function $R^{-1}(\tau)$ is increasing on $\langle 0, \infty)$, and the latter will be mapped onto the interval $\langle t_0, \infty)$.

**Lemma I.** *Suppose that* (1) *holds. Let $x(t)$ be a solution of* $(A_1)$ *in the interval* $\langle t_0, \infty)$. *Then the function*

$$y(\tau) = x[R^{-1}(\tau)] \quad \textit{for all } \tau \in \langle 0, \infty) \tag{4}$$

*is a solution of*

$$y''' + 2a_1[R^{-1}(\tau)]y' + [a_1'[R^{-1}(\tau)] + b_1[R^{-1}(\tau)]]\, r[R^{-1}(\tau)] \quad y = 0 \tag{$A_{01}$}$$

*in* $\langle 0, \infty)$.

*Conversely, if $y(\tau)$ is a solution of* $(A_{01})$ *in* $\langle 0, \infty)$, *and the function $x(t)$ is determined by relation* (4), *i.e.*

$$x(t) = y[R(t)], \quad t \in \langle t_0, \infty), \tag{4,1}$$

*then the function* (4,1) *satisfies the equation* $(A_1)$ *in* $\langle t_0, \infty)$.

Proof. Differentiating the relation (4), and considering $R'(t) = 1/r(t)$ for all $t \in \langle t_0, \infty)$, we obtain the equalities

$$y'(\tau) = r(t)\,x'(t) \tag{5}$$

$$y''(\tau) = r(t)\,[r(t)\,x'(t)]' \tag{6}$$

$$y'''(\tau) = r(t)\,[r(t)\,[r(t)\,x'(t)]']', \quad \tau = R(t),\ \tau \in \langle 0, \infty), \tag{7}$$

for the function, $y(\tau)$ given by (4). From the equalities (5)—(7) and from $t = R^{-1}(\tau)$ it follows on the basis of $(A_1)$ that

$$\begin{aligned} y'''(\tau) &= -2a_1(t)\,r(t)\,x'(t) - [a_1'(t) + b_1(t)]\,r(t)\,x(t) = \\ &= -2a_1[R^{-1}(\tau)]\,y' - [a_1'[R^{-1}(\tau)] + b_1[R^{-1}(\tau)]]\,r[R^{-1}(\tau)]\,y, \end{aligned}$$

and so $y(\tau)$ is a solution of $(A_{01})$ from the corresponding interval.

The relation (4) means the bijective mapping of the space of solutions of $(A_1)$.

onto the space of solutions of $(A_{01})$. Indeed, for arbitrary values $y_0^{(i)}$, $i = 0, 1, 2$ the system of conditions

$$y_0 = x(t_0)$$
$$y_0' = r(t_0)\,x'(t_0)$$
$$y_0'' = r(t_0)\,[r(t_0)\,x''(t_0) + r'(t_0)\,x'(t_0)]$$

has the only solution in the variables $x(t_0)$, $x'(t_0)$, $x''(t_0)$. From that the second part of the assertion of Lemma I follows.

Remark 1. Between the solutions of the adjoint equation

$$(B_1) \qquad [r(t)\,[r(t)\,\xi'(t)]']' + 2a_1(t)\,\xi'(t) + [a_1'(t) - b_1(t)]\,\xi(t) = 0$$

and those of the equation

$$(B_{01}) \qquad z''' + 2a_1[R^{-1}(\tau)]\,z' + [a_1'[R^{-1}(\tau)] - b_1[R^{-1}(\tau)]]\,r[R^{-1}(\tau)]\,z = 0$$

there is a similar relation as for the equations $(A_1)$ and $(A_{01})$. If $\xi(t)$ is the solution of $(B_1)$ in $\langle t_0, \infty)$, then the function

$$(4{,}2) \qquad z(\tau) = \xi[R^{-1}(\tau)] \quad \text{for all } \tau \in \langle 0, \infty)$$

is the solution of $(B_{01})$.

Remark 2. If we put

$$(8) \qquad A_{01}(\tau) = a_1[R^{-1}(\tau)]; \quad b_{01}(\tau) = b_1[R^{-1}(\tau)]\,r[R^{-1}(\tau)]$$

for all $\tau \in \langle 0, \infty)$, so on the basis of the equality

$$[a_1[R^{-1}(\tau)]]' = a_1'[R^{-1}(\tau)]\,r[R^{-1}(\tau)]$$

the equations $(A_{01})$ and $(B_{01})$ can be written in the form

$$y''' + 2A_{01}(\tau)\,y' + [A_{01}'(\tau) + b_{01}(\tau)]\,y = 0,$$
$$z''' + 2A_{01}(\tau)\,z' + [A_{01}'(\tau) - b_{01}(\tau)]\,z = 0.$$

**II.** If $\int^{\infty} \mathrm{d}t/r(t) < \infty$, then we define the function

$$(9) \qquad \varrho(t) = \int_{t_0}^{\infty} \mathrm{d}s/r(s), \quad t \geqq t_0$$

which belongs to the class $C^4(\langle t_0, \infty))$, is decreasing and maps the interval $\langle t_0, \infty)$ onto the interval $(0, \varrho(t_0))$. Denote $\varrho(t_0) = \varrho_0$. Its inverse function $\varrho^{-1}$ is decreasing in the interval $(0, \varrho_0\rangle$, and it maps this interval onto the interval $\langle t_0, \infty)$. Then a composite function $\varrho^{-1}(1/\tau)$ and $\varrho^{-1}(\mathrm{e}^{-\tau})$ belongs to the class $C^4(\langle \tau_0, \infty))$, where $\tau_0 = 1/\varrho_0$ and $\tau_0 = -\ln \varrho_0$ respectively, is increasing in this interval, and maps the interval $\langle \tau_0, \infty)$ onto $\langle t_0, \infty)$.

**Lemma II.** *Assume that* (2) *holds. Let* $x(t)$ *be the solution of* ($A_1$) *in the interval* $\langle t_0, \infty)$. *Then the function*

$$(10)\qquad y(\tau) = \tau^2 x[\varrho^{-1}(1/\tau)] \quad \textit{for all } \tau \in \langle \tau_0, \infty)$$

*is the solution of* ($A_{02}$) *in the interval* $\langle t_0, \infty)$, *where*

$$(A_{02})\qquad y''' + 2\tau^{-4} a_1[\varrho^{-1}(1/\tau)] y' + \tau^{-6}\{r[\varrho^{-1}(1/\tau)][a_1'[\varrho^{-1}(1/\tau)] + b_1[\varrho^{-1}(1/\tau)]] - 4\tau a_1[\varrho^{-1}(1/\tau)]\} y = 0.$$

*Conversely, if* $y(\tau)$ *is the solution of equation* ($A_{02}$) *in* $\langle \tau_0, \infty)$, *then the function* $x(t)$ *determined by the relation* (10), *i.e.*

$$(10{,}1)\qquad x(t) = \varrho^2(t) y[\varrho^{-1}(t)], \quad t \in \langle t_0, \infty)$$

*satisfies* ($A_1$) *in* $\langle t_0, \infty)$.

The proof can be obtained in a similar way as in the case of Lemma I.

Remark 3. By relation

$$(10{,}2)\qquad z(\tau) = \tau^2 \xi[\varrho^{-1}(1/\tau)]$$

the space of the solutions of ($B_1$) is transformed bijectively onto the space of the solutions of

$$(B_{02})\qquad z'''(\tau) + 2\tau^{-4} a_1[\varrho^{-1}(1/\tau)] z' + \tau^{-6}\{r[\varrho^{-1}(1/\tau)][a_1'[\varrho^{-1}(1/\tau)] - b_1[\varrho^{-1}(1/\tau)] - 4\tau a_1[\varrho^{-1}(1/\tau)]\} z = 0.$$

Remark 4. If we denote

$$(11)\qquad A_{02}(\tau) = \tau^{-4} a_1[\varrho^{-1}(1/\tau)]; \quad b_{02}(\tau) = \tau^{-6} b_1[\varrho^{-1}(1/\tau)] r[\varrho^{-1}(1/\tau)],$$

so on the basis of equality

$$A_{02}'(\tau) = -4\tau^{-5} a_1[\varrho^{-1}(1/\tau)] + \tau^{-6} a_1'[\varrho^{-1}(1/\tau)] r[\varrho^{-1}(1/\tau)]$$

the equations ($A_{02}$) and ($B_{02}$) can be written in the form

$$y''' + 2A_{02}(\tau) y' + [A_{02}'(\tau) + b_{02}(\tau)] y = 0,$$
$$z''' + 2A_{02}(\tau) z' + [A_{02}'(\tau) - b_{02}(\tau)] z = 0.$$

**Lemma III.** *Suppose that* (2) *is satisfied. Let* $x(t)$ *be the solution of* ($A_1$) *in* $\langle t_0, \infty)$. *Then the function*

$$(12)\qquad y(\tau) = e^{\tau} x[\varrho^{-1}(e^{-\tau})] \quad \textit{for all } \tau \geqq \tau_0 = -\ln \varrho(t_0)$$

*is the solution of* ($A_{03}$) *in* $\langle \tau_0, \infty)$, *where*

$$(A_{03})\qquad y''' + 2[e^{-2\tau} a_1[\varrho^{-1}(e^{-\tau})] - 1/2] y' + e^{-3\tau}\{r[\varrho^{-1}(e^{-\tau})][a_1'[\varrho^{-1}(e^{-\tau})] + b_1[\varrho^{-1}(e^{-\tau})] - 2e^{-2\tau} a_1[\varrho^{-1}(e^{-\tau})]]\} y = 0.$$

*Conversely, if $y(\tau)$ is the solution of* $(A_{03})$ *in* $\langle \tau_0, \infty)$, *and* $x(t)$ *is determined by* (12), *i.e.*

$$x(t) = \varrho(t)\, y[-\ln \varrho(t)] \quad t \in \langle t_0, \infty), \tag{12,1}$$

*then the function* (12,1) *satisfies an equation* $(A_1)$ *in* $\langle t_0, \infty)$.

The proof can be obtained in a similar way as in Lemma I.

Remark 5. By relation

$$z(\tau) = e^{-\tau} \xi[\varrho^{-1}(e^{-\tau})] \tag{12,2}$$

the space of solutions of $(B_1)$ is transformed bijectively onto the space of solutions of

$$(B_{03}) \qquad z''' + 2[e^{-2\tau} a_1 [\varrho^{-1}(e^{-\tau})] - 1/2]\, z' + e^{-3\tau} \{[a_1' [\varrho^{-1}(e^{-\tau})] - b_1 [\varrho^{-1}(e^{-\tau})]]\, r[\varrho^{-1}(e^{-\tau})] - 2e^{\tau} a_1 [\varrho^{-1}(e^{-\tau})]\}\, z = 0.$$

Remark 6. By designating

$$A_{03}(\tau) = e^{-2\tau} a_1 [\varrho^{-1}(e^{-\tau})] - 1/2;$$
$$b_{03}(\tau) = e^{-3\tau} b_1 [\varrho^{-1}(e^{-\tau})]\, r[\varrho^{-1}(e^{-\tau})]$$

and from the equality

$$A_{03}'(\tau) = e^{-3\tau} r[\varrho^{-1}(e^{-\tau})]\, a_1' [\varrho^{-1}(e^{-\tau})] - 2e^{-2\tau} a_1 [\varrho^{-1}(e^{-\tau})]$$

the equations $(A_{03})$ and $(B_{03})$ can be written in the form

$$y''' + 2A_{03}(\tau)\, y' + [A_{03}'(\tau) + b_{03}(\tau)]\, y = 0$$
$$z''' + 2A_{03}(\tau)\, z' + [A_{03}'(\tau) - b_{03}(\tau)]\, z = 0.$$

Next we shall present the results, which by using Lemma I and some lemmas and theorems proved by M. Greguš, for the solutions of $(A_1)$.

**Lemma 1.1.** *Let* (1) *hold. Let* $a_1(t) \leqq 0$, $a_1'(t) + b_1(t) \geqq 0$ *in* $\langle t_0, \infty)$ *and let the function* $b_1(t)$ *have the property*

$(V_1)$: $b_1(t) \geqq 0$ *for all* $t \in \langle t_0, \infty)$ *while* $b_1(t) \not\equiv 0$ *in any subinterval.*

*Let* $0 < r(t) \leqq M$ *in* $\langle t_0, \infty)$. *Let* $\xi$ *be the solution of the differential equation* $(B_1)$ *with* $\xi(\alpha_1) = \xi'(\alpha_1) = 0$, $\xi''(\alpha_1) > 0$ *for* $\alpha_1 \in \langle t_0, \infty)$. *Then* $\xi(t) \to \infty$, $\xi'(t) \to \infty$ *as* $t \to \infty$.

Proof. If we transform the equation $(A_1)$ into $(A_{01})$ by using Lemma I., we can see that $(A_{01})$ satisfies all assumptions of Lemma 3.1 in [2, p. 120] and moreover, in this transformation [relation (4,1)] there corresponds to the solution $\xi(t)$ of $(B_1)$ with a double zero-point $\alpha_1$ the solution $z(\tau)$ of $(B_{01})$ having a double zero-point in $\tau_1 = R[\alpha_1]$. According to Lemma 3.1 in [2] $z(\tau) \to \infty$,

$z'(\tau) \to \infty$ as $\tau \to \infty$. From (4,1), (1) and (4,2) we have: $\xi(t) = z[R(t)] \to \infty$, $\xi'(t) = r^{-1}(t) z'[R(t)] \to \infty$ as $t \to \infty$, because $1/r(t) \geqq 1/M > 0$ for all $t \geqq t_0$.

**Lemma 1.2.** *Let the hypotheses of Lemma* 1.1 *be satisfied and, moreover, let*

$$\int_{t_0}^{\infty} R^2(t) [a_1'(t) + b_1(t)] \, dt = \infty. \tag{13}$$

*Then the solution $\xi$ mentioned in Lemma* 1.1 *satisfies also*

$$r(t) [r(t) \xi'(t)]' + 2a_1(t) \xi(t) \to \infty \quad \text{as } t \to \infty. \tag{14}$$

Proof. With respect to both Lemma 1.1, from which we take the notation, and Lemma 3.2 in [2, p. 120] it is sufficient to prove that the condition

$$\int_0^{\infty} \tau^2 [a_1'[R^{-1}(\tau)] + b_1[R^{-1}(\tau)]] \, r[R^{-1}(\tau)] \, d\tau = \infty$$

is equivalent to (13) and the property of the solution $z$ of $(B_{01})$ $\lim_{\tau \to \infty} [z''(\tau) + 2A_{01}(\tau) z(\tau)] = \infty$ means the relation (14). However,

$$\int_0^{\infty} \tau^2 [a_1'[R^{-1}(\tau)] + b_1[R^{-1}(\tau)]] \, r[R^{-1}(\tau)] \, d\tau = \int_{t_0}^{\infty} R^2(t) [a_1'(t) + b_1(t)] \, dt$$

and on the basis of (4,1), (4,2) we get the statement of Lemma 1.2.

By using (3), (4), (5), (6) and the monotonicity of $R(t)$ we obtain from Theorem 4 in [1] the following:

**Theorem 1.1.** *Let the assumptions of Lemma* 1.2 *be satisfied in* $\langle t_0, \infty)$. *Then there exists exactly one solution $x$ of the differential equation* $(A_1)$ [*up to linear dependence*] *with the following properties*: $x(t) \neq 0$, $\operatorname{sgn} x(t) = \operatorname{sgn} [r(t) x'(t)]' \neq \operatorname{sgn} x'(t)$ *for* $t \in \langle t_0, \infty)$; $x(t)$, $r(t) x'(t)$, $r(t) [r(t) x'(t)]'$ *are monotonic functions of* $t \in \langle t_0, \infty)$ *and* $x(t) \to 0$, $r(t) x'(t) \to 0$, $r(t) [r(t) x'(t)]' \to 0$ *as* $t \to \infty$.

From Theorem 5 in [1] there follows

**Theorem 1.2.** *Let* $a_1(t) \leqq 0$, $a_1'(t) + b_1(t) \geqq 0$ *and* $b_1(t)$ *have the property* $(V_1)$ *for* $t \in \langle t_0, \infty)$. *If the differential equation* $(A_1)$ *has an oscillatory solution on* $\langle t_0, \infty)$, *then all solutions of* $(A_1)$ *are oscillatory on* $\langle t_0, \infty)$ *with one exception of the solution $x$* [*up to linear dependence*] *with the following properties*: $x(t) \neq 0$, $\operatorname{sgn} x(t) = \operatorname{sgn} [r(t) x'(t)]' \neq \operatorname{sgn} x'(t)$ *for* $t \in \langle t_0, \infty)$; $x(t)$, $r(t) x'(t)$, $r(t) [r(t) x'(t)]'$ *are monotonic functions for* $t \in \langle t_0, \infty)$ *and* $r(t) x'(t) \to 0$, $r(t) [r(t) x'(t)]' \to 0$ *as* $t \to \infty$.

Example. Consider the equation

$$[t^{\varkappa} [t^{\varkappa} x'(t)]']' + \varkappa t^{2(\varkappa - 1)} x'(t) + [\varkappa(\varkappa - 1) + (\varkappa - 2)(\varkappa - 3)] \, t^{2\varkappa - 3} x(t) = 0 \tag{15}$$

for $t \geqq t_0 > 0$. $\varkappa$ is a non-positive number. As easily verified, all the hypotheses of Theorem 1.2 are fulfilled, and the solution $x(t) = 1/t$ of (15) has all the properties satisfying the conclusions of Theorem 1.2.

Since on the basis of (3) and (8),

$$\int_0^\infty b_{01}(\tau)\,\mathrm{d}\tau = \int_{t_0}^\infty b_1(t)\,\mathrm{d}t,$$

from Theorem 3.4 in [2, p. 122] it follows

**Theorem 1.3.** *Let $b_1(t)$ have the property* ($V_1$) *in* $\langle t_0, \infty)$ *and let*

$$\int_{t_0}^\infty b_1(t)\,\mathrm{d}t = \infty.$$

*Then the equation* ($A_1$) *has at least one solution $x$ with no zeros in* $\langle t_0, \infty)$ *and satisfying* $\liminf_{t\to\infty} x(t) = 0$.

By using Lemma II we can obtain the following results for the equation ($A_1$).

**Lemma 2.1.** *Let* (2) *hold. Let $a_1(t) \leqq 0$, $\varrho(t)\,r(t)\,(a_1'(t) + b_1(t)) \geqq 4a_1(t)$ for $t \in \langle t_0, \infty)$. Let $b_1(t)$ have the property* ($V_1$). *Let $\xi$ be the solution of the equation* ($B_1$) *with the properties*: $\xi(\alpha_2) = \xi'(\alpha_2) = 0$, $\xi''(\alpha_2) > 0$, $\alpha_2 \in \langle t_0, \infty)$. *Then* $\varrho^{-2}(t)\,\xi(t) \to \infty$, $[r(t)\,\xi'(t) + 2\varrho^{-1}(t)\,\xi(t)] \to \infty$ *as $t \to \infty$, where $\varrho(t)$ is determined by* (9).

**Lemma 2.2.** *Let the assumptions of Lemma 2.1 be fulfilled and, moreover,*

$$\int_{t_0}^\infty \varrho^2(t)\,[a_1'(t) + b_1(t) - 4\varrho^{-1}(t)\,r^{-1}(t)\,a_1(t)]\,\mathrm{d}t = \infty.$$

*Then the solution $\xi$ mentioned in Lemma 2.1 satisfies also* $[\varrho^2(t)\,r(t)\,[r(t)\,\xi'(t)]' + 2\varrho(t)\,r(t)\,\xi'(t) + 2(1 + \varrho^2(t)\,a_1(t))\,\xi(t)] \to \infty$ *as $t \to \infty$.*

Further, we can prove a theorem where we use the formula (9) and derivatives of the relation (10).

**Theorem 2.1.** *Let the assumptions of Lemma 2.2 be satisfied in* $\langle t_0, \infty)$. *Then there exists exactly one solution $x$ of* ($A_1$) [*up to linear dependence*] *with the following properties*: $x(t) \neq 0$, $\operatorname{sgn} x(t) = \operatorname{sgn}[\varrho^2(t)\,r(t)\,[r(t)\,x'(t)]' + 2\varrho(t)\,r(t)\cdot x'(t) + 2x(t)] \neq \operatorname{sgn}[r(t)\,x'(t) + 2\varrho^{-1}(t)\,x(t)]$ *for* $t \in \langle t_0, \infty)$; $\varrho^{-2}(t)\,x(t)$, $r(t)\cdot x'(t) + 2\varrho^{-1}(t)\,x(t)$, $\varrho^2(t)\,r(t)\,[r(t)\,x'(t)]' + 2\varrho(t)\,r(t)\,x'(t) + 2x(t)$ *are monotonic functions and* $\varrho^{-2}(t)\,x(t) \to 0$, $[r(t)\,x'(t) + 2\varrho^{-1}(t)\,x(t)] \to 0$, $[\varrho^2(t)\,r(t)\cdot[r(t)\,x'(t)]' + 2\varrho(t)\,x'(t)\,r(t) + 2x(t)] \to 0$ *as $t \to \infty$.*

**Theorem 2.2.** *Let* (2) *hold. Let $a_1(t) \leqq 0$, $\varrho(t)\,r(t)\,[a_1'(t) + b_1(t)] \geqq 4a_1(t)$ for $t \in \langle t_0, \infty)$ and $b_1(t)$ have the property* ($V_1$). *If the differential equation* ($A_1$) *has an oscillatory solution on* $\langle t_0, \infty)$, *then all solutions of* ($A_1$) *are oscillatory with one* *exception of the solution x [up to linear dependence] with the following properties:* $x(t) \neq 0$, $\operatorname{sgn} x(t) = \operatorname{sgn}[\varrho^2(t) r(t) [r(t) x'(t)]' + 2\varrho(t) r(t) x'(t) + 2x(t)] \neq \operatorname{sgn} \cdot [r(t) x'(t) + 2\varrho^{-1}(t) x(t)]$ *for* $t \in \langle t_0, \infty)$; $\varrho^{-2}(t) x(t)$, $r(t) x'(t) + 2\varrho^{-1}(t) x(t)$, $\varrho^2(t) r(t) [r(t) x'(t)]' + 2\varrho(t) r(t) x'(t) + 2x(t)$ *are monotonic functions on* $\langle t_0, \infty)$ *and* $[r(t) x'(t) + 2\varrho^{-1}(t) x(t)] \to 0$, $[\varrho^2(t) r(t) [r(t) x'(t)]' + 2\varrho(t) r(t) x'(t) + 2x(t)] \to 0$ *as* $t \to \infty$.

**Theorem 2.3.** *Let* $b_1(t)$ *have the property* $(V_1)$ *in* $\langle t_0, \infty)$ *and let*

$$\int_{t_0}^{\infty} \varrho^4(t) b_1(t) r^{-1}(t) \, dt = \infty.$$

*Then the equation* $(A_1)$ *has at least one solution x with no zeros in* $\langle t_0, \infty)$ *and satisfying* $\liminf_{t \to \infty} \varrho^{-2}(t) x(t) = 0$.

Further assertions can be derived by using Lemma III.

**Lemma 3.1.** *Suppose that* (2) *holds and* $\varrho(t)$ *is determined by* (9). *Let* $\varrho^2(t) a_1(t) \leqq 1/2$, $\varrho(t) r(t) [a_1'(t) + b_1(t)] \geqq 2a_1(t)$ *and let* $b_1(t)$ *have the property* $(V_1)$ *in* $\langle t_0, \infty)$. *Let* $\xi$ *be such a solution of* $(B_1)$ *that* $\xi(\alpha_3) = \xi'(\alpha_3) = 0$, $\xi''(\alpha_3) > 0$, $\alpha_3 \in \langle t_0, \infty)$. *Then* $\varrho^{-1}(t) \xi(t) \to \infty$, $[r(t) \xi'(t) + \varrho^{-1}(t) \xi(t)] \to \infty$ *as* $t \to \infty$.

**Lemma 3.2.** *Let the assumptions of Lemma 3.1 be satisfied and, moreover, let*

$$\int_{t_0}^{\infty} [\ln \varrho(t)]^2 [\varrho^2(t) [a_1'(t) + b_1(t)] - 2\varrho(t) r^{-1}(t) a_1(t)] \, dt$$

*diverge. Then the solution* $\xi$ *mentioned in Lemma 3.1 satisfies also* $[\varrho(t) r(t) \cdot [r(t) \xi'(t)]' + r(t) \xi(t) + 2\varrho(t) a_1(t) \xi(t)] \to \infty$ *as* $t \to \infty$.

**Theorem 3.1.** *Let the hypotheses of Lemma 3.2 be fulfilled in* $\langle t_0, \infty)$. *Then there exists exactly one solution x of* $(A_1)$ *[up to linear dependence] with the following properties:* $x(t) \neq 0$ *for* $t \in \langle t_0, \infty)$, $\varrho^{-1}(t) x(t)$, $r(t) x'(t) + \varrho^{-1}(t) x(t)$, $\varrho(t) r(t) [r(t) x'(t)]' + r(t) x'(t) + \varrho^{-1}(t) x(t)$ *are monotonic functions in* $\langle t_0, \infty)$; $\operatorname{sgn} x(t) = \operatorname{sgn}[\varrho(t) r(t) [r(t) x'(t)]' + r(t) x'(t) + \varrho^{-1}(t) x(t)] \neq \operatorname{sgn}[r(t) x'(t) + \varrho^{-1}(t) x(t)]$ *for* $t \in \langle t_0, \infty)$ *and* $\varrho^{-1}(t) x(t) \to 0$, $[r(t) x'(t) + \varrho^{-1}(t) x(t)] \to 0$, $\varrho(t) r(t) [r(t) x'(t)]' \to 0$ *as* $t \to \infty$.

**Theorem 3.2.** *Let* (2) *hold and let* $\varrho^2(t) a_1(t) \leqq 1/2$, $\varrho(t) r(t) [a_1'(t) + b_1(t)] \geqq 2a_1(t)$ *and let* $b_1(t)$ *have the property* $(V_1)$ *in* $\langle t_0, \infty)$. *If the differential equation* $(A_1)$ *has an oscillatory solution in* $\langle t_0, \infty)$, *then all solutions of* $(A_1)$ *are oscillatory with one exception of the solution x [up to the linear dependence] with the following properties:* $x(t) \neq 0$, $\operatorname{sgn} x(t) = \operatorname{sgn}[\varrho(t) r(t) [r(t) x'(t)]' + r(t) x'(t)' + \varrho^{-1}(t) \cdot x(t)] \neq \operatorname{sgn}[r(t) x'(t) + \varrho^{-1}(t) x(t)]$ *for* $t \in \langle t_0, \infty)$; $x(t) \varrho^{-1}(t)$, $r(t) x'(t) + \varrho^{-1}(t) x(t)$, $\varrho(t) r(t) [r(t) x'(t)]' + r(t) x'(t) + \varrho^{-1}(t) x(t)$ *are monotonic functions and* $[r(t) x'(t) + \varrho^{-1}(t) x(t)] \to 0$, $\varrho(t) r(t) [r(t) x'(t)]' \to 0$ *as* $t \to \infty$.

**Theorem 3.3.** *Let* $b_1(t)$ *have the property* $(V_1)$ *and let*

$$\int_{t_0}^{\infty} \varrho^2(t)\, b_1(t)\, dt = \infty.$$

*Then the differential equation* $(A_1)$ *has at least one solution* $x$ *with no zeros in* $\langle t_0, \infty)$ *and satisfying* $\liminf_{t \to \infty} \varrho^{-1}(t)\, x(t) = 0$.

Finally the following theorem which generalizes Theorem 1.15 in [2, p. 20] can be proved by using the Lemmas I—III.

**Theorem 4.** *If the function* $b_1(t) \geqq 0$ *for* $t \in \langle t_0, \infty)$, *then there exists a solution of the equation* $(A_1)$ *without zeros in* $\langle t_0, \infty)$.

Example. In the equation

$$[t[tx'(t)]']' + (1 - 3\varkappa^2) \ln^{-2}(t)\, x'(t) + (t^{-1} \ln^{-3}(t))(3\varkappa^2 + 2\varkappa^3 - 1)\, x(t) = 0,$$

where $\varkappa \geqq 0$ is a constant, the Laguerre invariant $b_1(t) = 2(t^{-1} \ln^{-3}(t))\, \varkappa^3$ is nonnegative on an interval $\langle t_0, \infty)$, $t_0 > e$. This equation has by Theorem 4 a solution without zeros. Such solutions, for example, are $x_1(t) = \ln^{1+\varkappa}(t)$, $x_2(t) = \ln^{1+\varkappa}(t) \cdot \ln(\ln t)$.

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# АСИМПТОТИЧЕСКИЕ СВОЙСТВА РЕШЕНИЙ ОПРЕДЕЛЕННОГО ТИПА ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТРЕТЬЕГО ПОРЯДКА

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## Резюме

В работах М. Грегуша [1] и [2] приведены асимптотические свойства решений без нулевых точек и также асимптотические свойства осциллирующих решений дифференциального уравнения третьего порядка в форме $(A_0)$. В этой статье исследованы выше приведенные свойства для дифференциального уравнения в форме $(A_1)$.

Предполагается, что условия (1) или (2) выполены. В обоих случаях преобразуются решения уравнения $(A_1)$ в решения уравнения $(A_{0i})$ $i = 1, 2, 3$ (которые в форме $(A_0)$) и наоборот. На основании преобразования применяются результаты работ [1] и [2] к решению уравнения $(A_1)$.