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Dedicated to Professor Tibor Katriňák

WEAK DIRECT FACTORS OF LATTICES

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(*Communicated by Sylvia Pulmannová*)

ABSTRACT. A sublattice A of a lattice L is said to be a weak direct factor of L if there exist a lattice B with a least element 0 and an embedding $\varphi\colon L \to A \times B$ with $\varphi(A) = A \times \{0\}$. There are given necessary and sufficient conditions for a sublattice A of a lattice L to be a weak direct factor of L. Further, the ordered system of all weak direct factors of a lattice is investigated.

1. Introduction and preliminaries

For groups and some other algebraic structures there is a natural way for defining an internal direct product decomposition. This notion can be easily transferred to partially ordered sets with a least element. J. Jakubík and M. Csontóová [J-Cs] introduced the notion of an internal direct product decomposition of a partially ordered set (not necessarily with a least element) with a central element. Let us recall the definition of a two-factor internal direct product decomposition of a partially ordered set $(P, \leq)$ with the central element $s \in P$.

Let A, B be partially ordered sets and let φ be an isomorphism of P onto $A \times B$. For $x \in P$ we denote $\varphi(x) = \big(x(A), x(B)\big)$. Put $A(s) = \big\{x \in P : x(B) = s(B)\big\}$, $B(s) = \big\{x \in P : x(A) = s(A)\big\}$. Then $\varphi^s\colon P \to A(s) \times B(s)$ defined by $\varphi^s(x) = \Big(\varphi^{-1}\big((x(A), s(B))\big), \varphi^{-1}\big((s(A), x(B))\big)\Big)$ is an isomorphism and it is said to be the *internal direct product decomposition of P with the central element s*. The sets $A(s)$, $B(s)$ are convex subsets of P such that $A(s) \cap B(s) = \{s\}$; they are called *internal direct factors of P through s*.

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The internal direct product decompositions of some special partially ordered sets and direct factors were investigated also in [J1]–[J5].

In the present paper we introduce the notion of a weak direct factor of a lattice (2.1). As to the relation of this notion to the above mentioned one of an internal direct factor, the situation is as follows. If L is a lattice, then the direct factor $A(s)$ (or $B(s)$) is a weak direct factor of L if and only if $s(B)$ (or $s(A)$) is the least element of B (or A). The main result is a characterization of weak direct factors of lattices (2.8). As a consequence (2.12), weak direct factors of a distributive lattice are just its almost principal ideals, which play an important role in the connection with the notion of the affine completeness (cf. [P]). We also investigate the ordered system $\mathcal{W}(L)$ of all weak direct factors of a lattice L. We show that if L is a distributive lattice or a lattice having a greatest element, then $\mathcal{W}(L)$ is a sublattice of the lattice $\operatorname{Id} L$ of all ideals of L (3.4 and 3.5). Whether the latter holds for any lattice L, is an open question.

We will use the standard lattice-theoretical terminology and notations (see e.g. [B] or [G]). Let us remind some notions which will play a key role. Let L be a lattice. By an *ideal* of L, a nonempty subset A satisfying the following two conditions will be meant:

(i) $a_1, a_2 \in A \implies a_1 \vee a_2 \in A$;
(ii) $a \in A,\ x \in L,\ x \leq a \implies x \in A$.

If $a \in L$, then $(a\rangle = \{x \in L : \ x \leq a\}$ is an ideal of L; it will be called the *principal ideal* generated by a. The notion of a *congruence relation* and of a *homomorphism* will be supposed to be known. We will use the following statement (cf. [G-Sch] and [M]).

(ω) *A reflexive binary relation θ on a lattice L is a congruence relation if and only if the following three conditions are satisfied for $x, y, z, t \in L$:*
(i) $x\,\theta\,y \iff x \wedge y\,\theta\,x \vee y$;
(ii) $x \leq y \leq z,\ x\,\theta\,y,\ y\,\theta\,z \implies x\,\theta\,z$;
(iii) $x \leq y,\ x\,\theta\,y \implies x \wedge t\,\theta\,y \wedge t,\ x \vee t\,\theta\,y \vee t$.

2. Weak direct factors

DEFINITION 2.1. Let L be a lattice, A a sublattice of L. We will say that A is a *weak direct factor of* L if there exist a lattice B with a least element 0 and a one-to-one homomorphism $\varphi\colon L \to A \times B$ such that $\varphi(A) = A \times \{0\}$.

We will give necessary and sufficient conditions for a sublattice A of a lattice L to be a weak direct factor of L.

LEMMA 2.2. *Let A be a weak direct factor of a lattice L. Then A is an ideal of L.*

Proof. Let $x \leq a \in A$. We have to show that $x \in A$. It holds that $\varphi(x) \leq \varphi(a) = (a', 0)$, $\varphi(x) = (a_1, b_1)$ for some $a', a_1 \in A$, $b_1 \in B$, where φ and B are as in 2.1. Then evidently $b_1 = 0$ and hence $x \in A$. □

DEFINITION 2.3. (cf. [P]) An ideal A of a lattice L will be said to be *almost principal* if for each $x \in L$, the set $\{a \in A : \ a \leq x\}$ has a greatest element. This element will be denoted by $x(A)$.

It is easy to see that each principal ideal in a lattice L is almost principal; if $a \in L$, $A = (a\rangle$, then $x(A) = x \wedge a$ for each $x \in L$. To show that there exist almost principal ideals which are not principal, let A be any lattice without a greatest element and B any lattice with a least element 0. Then $A \times \{0\}$ is an almost principal ideal in the lattice $L = A \times B$ which is not principal; if $(a, b) \in L$, then $(a, 0)$ is the greatest element of the set $\{(a', 0) : \ (a', 0) \leq (a, b)\}$. As an example of an ideal which is not almost principal, we can take the ideal of all finite subsets of an infinite set M in the lattice of all subsets of M.

LEMMA 2.4. *Let A be a weak direct factor of a lattice L. Then A is an almost principal ideal in L.*

Proof. Let $x \in L$, $\varphi(x) = (a_1, b_1)$. Then evidently $\varphi^{-1}\big((a_1, 0)\big)$ is the greatest element of the set $\{a \in A : \ a \leq x\}$. □

Consider the following conditions concerning L and A, where L is a lattice and A is an almost principal ideal in L:

(α) $(x \vee y)(A) = x(A) \vee y(A)$ for all $x, y \in L$;

(β) if $x, t \in L$, $a \in A$, then there exists $a' \in A$ with $(a \vee x) \wedge t = a' \vee (x \wedge t)$.

Evidently the condition (β) can be reformulated in such a way that

$$(a \vee x) \wedge t = \big((a \vee x) \wedge t\big)(A) \vee (x \wedge t)$$

for all $x, t \in L$, $a \in A$.

It is useful to realize that if we want to verity (β), it suffices to consider $x, t \in L - A$, $x \ngeq t$.

LEMMA 2.5. *Let A be a weak direct factor of a lattice L. Then the conditions (α) and (β) are satisfied.*

Proof. To prove (α), let $x, y \in L$, $\varphi(x) = (a_1, b_1)$, $\varphi(y) = (a_2, b_2)$. Then $\varphi(x \vee y) = \varphi(x) \vee \varphi(y) = (a_1 \vee a_2, b_1 \vee b_2)$, so that $(x \vee y)(A) = \varphi^{-1}\big((a_1 \vee a_2, 0)\big) = \varphi^{-1}\big((a_1, 0) \vee (a_2, 0)\big) = \varphi^{-1}\big((a_1, 0)\big) \vee \varphi^{-1}\big((a_2, 0)\big) = x(A) \vee y(A)$.

Now assume that $x, t \in L$, $a \in A$, $\varphi(x) = (a_1, b_1)$, $\varphi(t) = (a_0, b_0)$, $\varphi(a) = (\bar{a}, 0)$. We have $(a \vee x) \wedge t = \Big(\varphi^{-1}\big((\bar{a}, 0)\big) \vee \varphi^{-1}\big((a_1, b_1)\big)\Big) \wedge \varphi^{-1}\big((a_0, b_0)\big) = \varphi^{-1}\Big(\big((\bar{a} \vee a_1) \wedge a_0,\, b_1 \wedge b_0\big)\Big)$. Taking $a' = \varphi^{-1}\Big(\big((\bar{a} \vee a_1) \wedge a_0, 0\big)\Big)$ we obtain $a' \vee (x \wedge t) = \varphi^{-1}\Big(\big((\bar{a} \vee a_1) \wedge a_0, 0\big)\Big) \vee \Big(\varphi^{-1}\big((a_1, b_1)\big) \wedge \varphi^{-1}\big((a_0, b_0)\big)\Big) =$

$\varphi^{-1}\Big(\big(((\bar{a} \vee a_1) \wedge a_0) \vee (a_1 \wedge a_0), b_1 \wedge b_0\big)\Big) = \varphi^{-1}\big(((\bar{a} \vee a_1) \wedge a_0, b_1 \wedge b_0)\big)$. So (β) holds, too. □

Now we will suppose that L is a lattice, A an almost principal ideal in L such that the conditions (α), (β) are satisfied. The aim is to show that under this assumptions, A is a weak direct factor of L.

Let us define binary relations ρ and σ in L as follows:

$$x \,\rho\, y \iff x(A) = y(A);$$
$$x \,\sigma\, y \iff \text{there exists } a \in A \text{ with } (x \wedge y) \vee a = x \vee y.$$

LEMMA 2.6. *The relations ρ and σ are congruence relations in L and $\rho \cap \sigma$ is the identity relation.*

Proof. As to ρ, we want to verify that the mapping $x \mapsto x(A)$ is a homomorphism. Then ρ, as the kernel of this mapping, is a congruence relation in L. In view of (α) it is sufficient to prove $(x \wedge y)(A) = x(A) \wedge y(A)$ for all $x, y \in L$. So let $x, y \in L$. We have $x(A) \wedge y(A) \le x \wedge y$. Now let $a' \in A$, $a' \le x \wedge y$. Then $a' \le x, y$, so that $a' \le x(A), y(A)$, which implies $a' \le x(A) \wedge y(A)$. Therefore $(x \wedge y)(A) = x(A) \wedge y(A)$.

The relation σ is reflexive, since $x \vee x(A) = x$ for each $x \in L$. Now to prove that σ is a congruence relation in L, we will use the statement (ω). The conditions (i), (ii) are trivially satisfied. Now let $x, y, t \in L$, $x \le y$, $x \,\sigma\, y$. Then there exists $a \in A$ with $x \vee a = y$. The relation $x \wedge t \;\sigma\; y \wedge t$ follows immediately from (β) and $x \vee t \;\sigma\; y \vee t$ is evident.

Finally if $x, y \in L$, $x \le y$, $x \,\rho\, y$, $x \,\sigma\, y$, then $y = x \vee y(A) = x \vee x(A) = x$. This implies that $\rho \cap \sigma$ is the identity relation. □

Now consider the quotient lattice $B = L/\sigma$. Then evidently $[a]\sigma = A$ for each $a \in A$ and it is the least element of B. Let us define $\varphi \colon L \to A \times B$ by

$$\varphi(x) = \big(x(A), [x]\sigma\big).$$

LEMMA 2.7. *φ is a one-to-one homomorphism of L into $A \times B$ such that $\varphi(a) = (a, 0)$ for all $a \in A$.*

Proof. Let $\varphi(x) = \varphi(y)$ for some $x, y \in L$. Then $x(A) = y(A)$ and $[x]\sigma = [y]\sigma$, which implies $(x, y) \in \rho \cap \sigma$. Using 2.6 we obtain $x = y$. The fact that φ is a homomorphism can be verified easily. Finally we have $\varphi(a) = \big(a(A), [a]\sigma\big) = (a, A)$ for each $a \in A$ and the proof is complete. □

Hence in view of 2.4, 2.5 and 2.6, 2.7 we have:

THEOREM 2.8. *Let L be a lattice, A its sublattice. Then A is a weak direct factor of L if and only if A is an almost principal ideal of L such that the conditions (α), (β) are satisfied.*

Now look at the conditions (α), (β) in more detail. It is easy to see that if $A = L$, then both (α) and (β) are satisfied. The same holds for $A = \{0\}$ if L has a least element 0. But there exist ideals, even principal ones, which do not satisfy any of (α), (β).

EXAMPLE 2.9. Let L be as in Figure 1, $A = \{0, x\}$. Then neither (α) nor (β) holds. Namely $(y \vee z)(A) = x \neq 0 = y(A) \vee z(A)$ and $(x \vee y) \wedge z = z$ but $x \vee (y \wedge z) = x \neq z$, $0 \vee (y \wedge z) = 0 \neq z$.

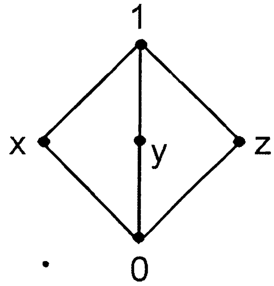

FIGURE 1.

EXAMPLE 2.10. Let L be as in Figure 2. If $A = \{0, x\}$, then (α) is satisfied while (β) does not hold because $(x \vee y) \wedge z = z$ but $x \vee (y \wedge z) = x \neq z$, $0 \vee (y \wedge z) = 0 \neq z$. If $A = \{0, x, z\}$, then (α) does not hold because $(x \vee y)(A) = z \neq x = x(A) \vee y(A)$, while the validity of (β) can be verified easily.

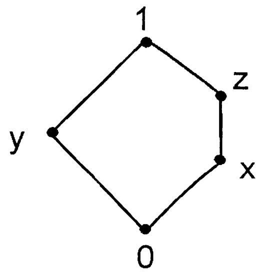

FIGURE 2.

The lattices in the preceding examples are not distributive. A natural question arises what about the validity of (α) and (β) if the lattice L is distributive.

Let us remind (see e.g. [G]) that an element a of a lattice L is said to be

(i) *distributive* if, for all $x, y \in L$,

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y);$$

(ii) *standard* if, for all $x, y \in L$,

$$(a \vee x) \wedge y = (a \wedge y) \vee (x \wedge y);$$

(iii) *neutral* if, for all $x, y \in L$,

$$(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a).$$

LEMMA 2.11. *Let L be a lattice, A an almost principal ideal of L.*

(a) *If all elements of A are dually distributive, then (α) holds.*
(b) *If all elements of A are standard, then (β) holds.*

Proof.

(a) Let $x, y \in L$. Evidently $x(A) \vee y(A) \leq x \vee y$. If $a \in A$, $a \leq x \vee y$, then $a = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$. Since $a \wedge x \leq x(A)$, $a \wedge y \leq y(A)$, we have $a \leq x(A) \vee y(A)$. Therefore $(x \vee y)(A) = x(A) \vee y(A)$.

(b) Let $x, t \in L$, $a \in A$. Then $(a \vee x) \wedge t = (a \wedge t) \vee (x \wedge t)$ and evidently $a \wedge t \in A$. □

As all elements of a distributive lattice are dually distributive and standard, we obtain:

THEOREM 2.12. *If L is a distributive lattice, then weak direct factors of L are just its almost principal ideals.*

The converse statements to (a) and (b) of 2.11 do not hold in general. If we take, e.g., L as in 2.10 and $A = \{0, x, z\}$, then (β) holds, but x is not standard, because $(x \vee y) \wedge z = z \neq x = (x \wedge z) \vee (y \wedge z)$. To see that the converse to (a) does not hold, consider the following example.

EXAMPLE 2.13. Let L be as in Figure 3, $A = (a\rangle$. It is not difficult to verify that (α) is satisfied. But a_1 is not dually distributive, because $a_1 \wedge (x \vee y) > (a_1 \wedge x) \vee (a_1 \wedge y)$.

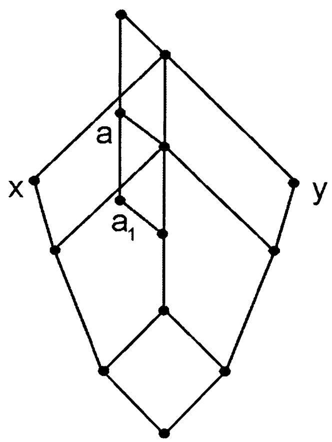

FIGURE 3.

As to principal ideals, we can give a nice description of those satisfying (α) or (β).

THEOREM 2.14. *A principal ideal $A = (a\rangle$ of a lattice L satisfies*

(a) *the condition (α) if and only if a is a dually distributive element of L,*
(b) *the condition (β) if and only if a is a standard element of L.*

Proof. The assertion (a) is obvious because $(x \vee y)(A) = (x \vee y) \wedge a$ and $x(A) \vee y(A) = (x \wedge a) \vee (y \wedge a)$ for any $x, y \in L$.

Let A satisfy (β), $x, y \in L$. Then $(a \vee x) \wedge y = \big((a \vee x) \wedge y\big)(A) \vee (x \wedge y) = \big((a \vee x) \wedge y \wedge a\big) \vee (x \wedge y) = (a \wedge y) \vee (x \wedge y)$. Conversely, let a be standard. Take any $a_1 \leq a$, $x, t \in L$. We have $(a_1 \vee x) \wedge t \leq a_1 \vee x \leq a \vee x$, and so $(a_1 \vee x) \wedge t = (a_1 \vee x) \wedge t \wedge (a \vee x) = \big((a_1 \vee x) \wedge t \wedge a\big) \vee \big((a_1 \vee x) \wedge t \wedge x\big)$ since a is standard. Setting $(a_1 \vee x) \wedge t \wedge a = a_1'$ we obtain $(a_1 \vee x) \wedge t = a_1' \vee \big((a_1 \vee x) \wedge t \wedge x\big) = a_1' \vee (t \wedge x)$, where evidently $a_1' \in A$. □

It is known that an element a of a lattice L is neutral if and only if it is standard and dually distributive (see e.g. [G]). So we have:

COROLLARY 2.15. *A principal ideal $A = (a\rangle$ of a lattice L is a weak direct factor of L if and only if a is a neutral element of A.*

If a lattice L has a greatest element, then almost principal ideals of L are just its principal ideals. So we have:

THEOREM 2.16. *Let L be a lattice with a greatest element. Then weak direct factors of L are just its principal ideals $(a\rangle$ such that a is a neutral element of L.*

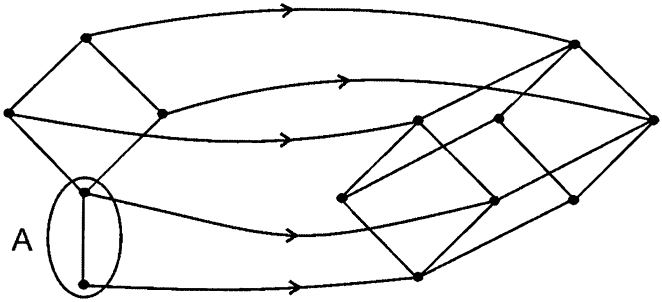

FIGURE 4.

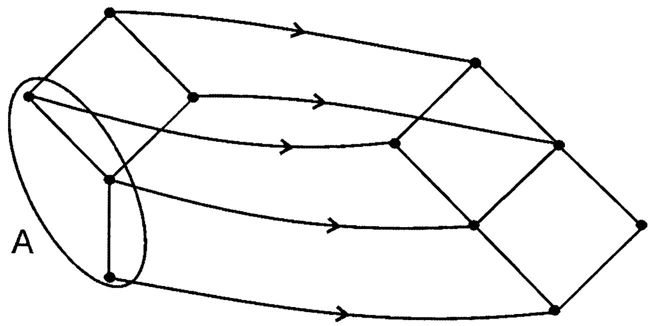

FIGURE 5.

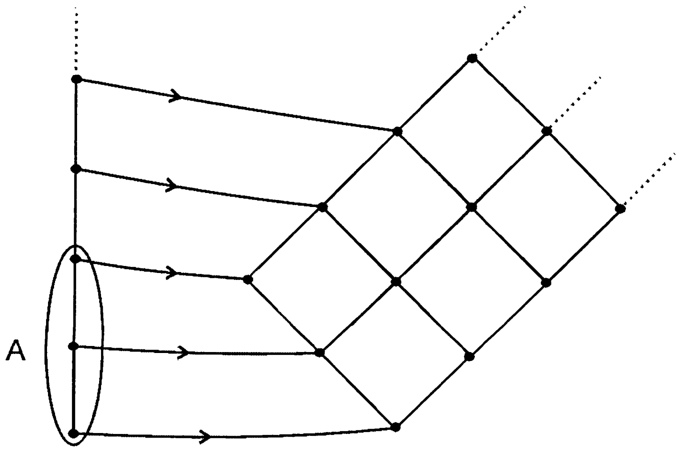

FIGURE 6.

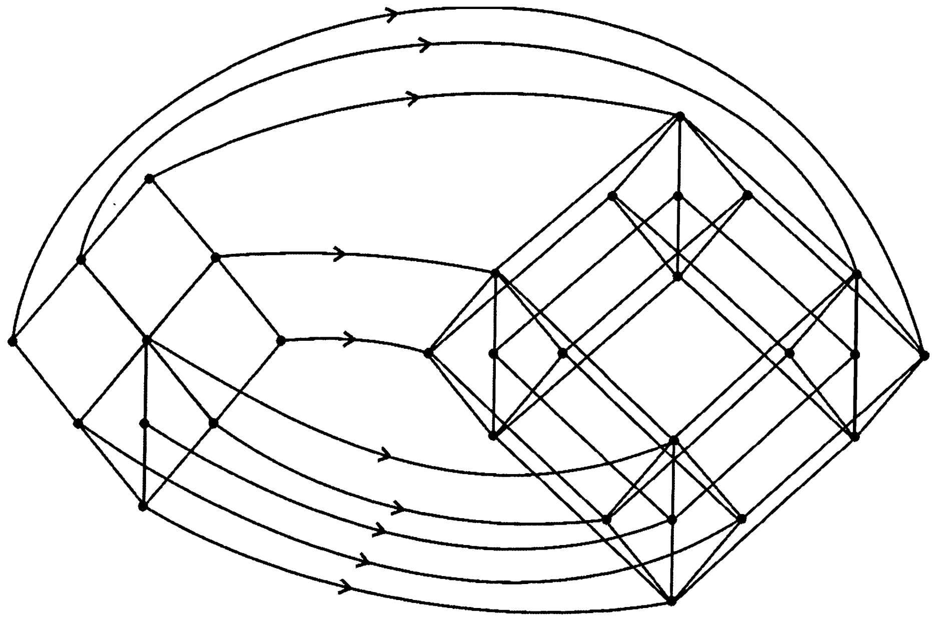

FIGURE 7.

In Figure 4 – Figure 7, there are given some examples of weak direct factors A of L and corresponding embeddings $\varphi\colon L \to A \times B$.

3. Ordered system of weak direct factors

If L is a lattice, let $\operatorname{Id} L$ denote the set of all ideals of L. It is well known that $\operatorname{Id} L$ is a lattice under set inclusion. For $A_1, A_2 \in \operatorname{Id} L$ it is $A_1 \wedge A_2 = A_1 \cap A_2$ and $A_1 \vee A_2 = \{x \in L : \ x \le a_1 \vee a_2 \text{ for some } a_1 \in A_1, \ a_2 \in A_2\}$. This implies that the set of all principal ideals of L is a sublattice of $\operatorname{Id} L$, because $\langle a\rangle \wedge \langle b\rangle = \langle a \wedge b\rangle$, $\langle a\rangle \vee \langle b\rangle = \langle a \vee b\rangle$ for all $a, b \in L$.

Let $\mathcal{W}(L)$ denote the set of all weak direct factors of L. It is $\mathcal{W}(L) \subseteq \operatorname{Id} L$, so we can ask if $\mathcal{W}(L)$ is a sublattice of $\operatorname{Id} L$. First let us look at the system of all almost principal ideals of L.

LEMMA 3.1. *Let A_1, A_2 be almost principal ideals of a lattice L. Then $A = A_1 \cap A_2$ is also an almost principal ideal. Particularly, $x(A) = x(A_1) \wedge x(A_2)$ for each $x \in L$.*

Proof. Let $x \in L$. Then $x(A_1) \wedge x(A_2) \in A_1 \cap A_2 = A$ and evidently $x(A_1) \wedge x(A_2) \leq x$. Now let $a \in A$, $a \leq x$. Then $a \leq x(A_1)$, $a \leq x(A_2)$, so that $a \leq x(A_1) \wedge x(A_2)$. We have proved $x(A_1) \wedge x(A_2) = x(A)$. □

The following example shows that if A_1, A_2 are almost principal ideals, the ideal $A_1 \vee A_2$ need not be almost principal.

EXAMPLE 3.2. Let L, A_1, A_2 be as in Figure 8. Then evidently A_1, A_2 are almost principal ideals, while $A_1 \vee A_2$ fails to be almost principal since the set $\{a \in A_1 \vee A_2 : a \leq x\}$ has not a greatest element.

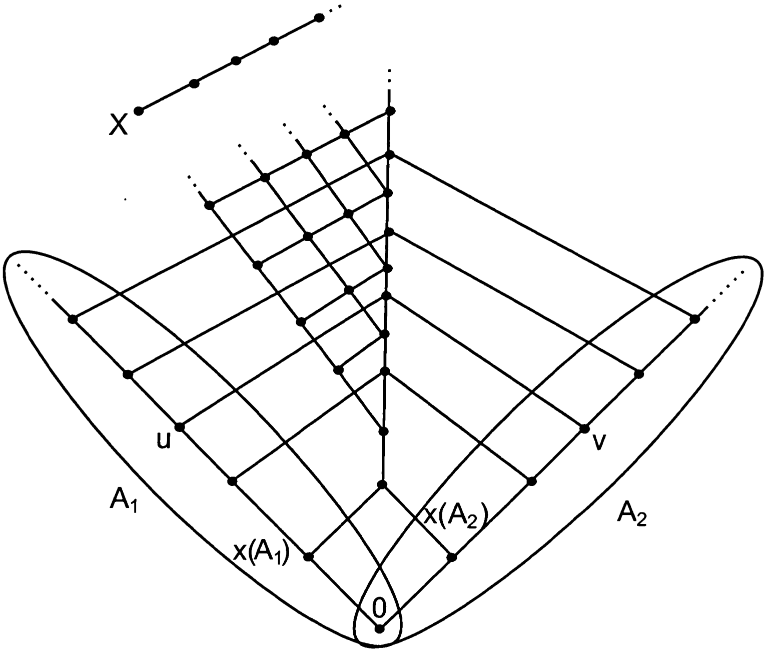

FIGURE 8.

Hence the system of all almost principal ideals of a lattice L need not be a sublattice of $\mathrm{Id}\, L$. But we cannot deduce from this fact that $\mathcal{W}(L)$ is not a sublattice of $\mathrm{Id}\, L$. Namely, if we look at the previous example, we can see that A_1, A_2 are not weak direct factors of L. In fact, A_1, A_2 do not satisfy any of the conditions (α), (β), because, e.g., $(x \vee v)(A_1) = u$, $x(A_1) \vee v(A_1) = x(A_1) \vee 0 = x(A_1) \neq u$ and $(u \vee x) \wedge v = v$, $x \wedge v = x(A_2)$, so there is no $a \in A_1$ with $(u \vee x) \wedge v = a \vee (x \wedge v)$.

We will show that if L is a distributive lattice or a lattice with a greatest element, then $\mathcal{W}(L)$ is a sublattice of the lattice $\mathrm{Id}\, L$.

LEMMA 3.3. *Let L be a distributive lattice, let A_1, A_2 be almost principal ideals of L. Then $A = A_1 \vee A_2$ is also an almost principal ideal. Particularly, $x(A) = x(A_1) \vee x(A_2)$ for each $x \in L$.*

Proof. Let $x \in L$. Then $x(A_1) \vee x(A_2) \in A_1 \vee A_2 = A$ and evidently $x(A_1) \vee x(A_2) \leq x$. Now let $a \in A$, $a \leq x$. Then $a \leq a_1 \vee a_2$ for some $a_1 \in A_1$, $a_2 \in A_2$ and this implies $a = a \wedge (a_1 \vee a_2) = (a \wedge a_1) \vee (a \wedge a_2)$. The element $a \wedge a_1$ belongs to A_1 and $a \wedge a_1 \leq a \leq x$, so that $a \wedge a_1 \leq x(A_1)$. Analogously $a \wedge a_2 \leq x(A_2)$ and hence $a \leq x(A_1) \vee x(A_2)$. We have proved $x(A_1) \vee x(A_2) = x(A)$. □

Using 3.1, 3.3 and 2.12 we obtain immediately:

THEOREM 3.4. *If L is a distributive lattice, then $\mathcal{W}(L)$ is a sublattice of the lattice* $\operatorname{Id} L$.

It is known that the lattice $\operatorname{Id} L$ for any L is closed under infinite joins, too. But $\mathcal{W}(L)$ need not be closed under infinite joins, not even in the case that L is distributive. To see this, let L be the lattice of all subsets of an infinite set M. Then $\bigvee \{ (\{a\}\rangle :\ a \in M \}$ is the ideal of all finite subsets of M, which is not almost principal.

Now we will deal with lattices L having a greatest element. As we have remarked, weak direct factors of L are just principal ideals $(a\rangle$ with a being a neutral element of L (2.16). It is known that the set of all neutral elements of a lattice is closed under joins and meets. So we have:

THEOREM 3.5. *Let L be a lattice with a greatest element. Then $\mathcal{W}(L)$ is a sublattice of the lattice* $\operatorname{Id} L$.

Let us remark that if L is any lattice, the set $\{(a\rangle :\ (a\rangle \text{ satisfies } (\beta)\}$ is a sublattice of the lattice $\operatorname{Id} L$, because the set of all standard elements of L is also closed under joins and meets. But as the set of all dually distributive elements of a lattice L need not be closed under joins, the set $\{(a\rangle :\ (a\rangle \text{ satisfies } (\alpha)\}$ is not a sublattice of $\operatorname{Id} L$ in general. This is the case, e.g., if L is as in Figure 9. It can be verified that dually distributive elements of L are just 0, a, c, 1, while standard elements of L are 0, i, 1. Thus 0, 1 are only neutral elements. Hence $\mathcal{W}(L) = \{\{0\}, L\}$ is a sublattice of $\{(x\rangle :\ x \in L,\ (x\rangle \text{ satisfies } (\beta)\} = \{(x\rangle :\ x \text{ is standard}\} = \{\{0\}, (i\rangle, L\}$. The latter is a sublattice of $\operatorname{Id} L$. On the other hand, $\{(x\rangle :\ x \in L,\ (x\rangle \text{ satisfies } (\alpha)\} = \{(x\rangle :\ x \text{ is dually distributive}\} = \{\{0\}, (a\rangle, (c\rangle, L\}$, which is not a sublattice of $\operatorname{Id} L$.

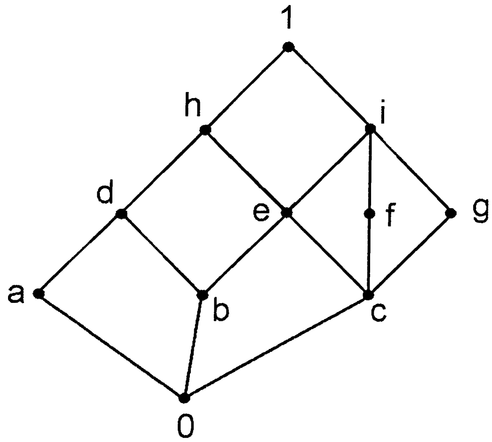

FIGURE 9.

Now let L be a non-distributive lattice without a greatest element. We do not know if the meet and the join of two almost principal ideals of L satisfying the conditions (α), (β) (in the lattice $\operatorname{Id} L$) are also almost principal ideals of L satisfying (α), (β). In other words, the following question is open.

Is $\mathcal{W}(L)$ a sublattice of $\operatorname{Id} L$ for each lattice L?

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