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Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

VALUATIONS AND DISTANCE FUNCTIONS ON DIRECTED MULTILATTICES

JUDITA LIHOVÁ

(*Communicated by Tibor Katriňák*)

ABSTRACT. Distance functions corresponding to valuations, isotone and positive valuations on directed multilattices are characterized. As an application, there is proved that congruence relations on a directed modular multilattice of locally finite length form a Boolean algebra.

By a *valuation on a lattice* L a real-valued function v defined on L satisfying

$$v(a) + v(b) = v(a \wedge b) + v(a \vee b)$$

is meant (see, e.g., [2]). A valuation is *isotone* if

$$a \leq b \implies v(a) \leq v(b),$$

and *positive* if

$$a < b \implies v(a) < v(b).$$

The *distance function corresponding to a valuation* v is defined by

$$d(a,b) = v(a \vee b) - v(a \wedge b).$$

The distance function corresponding to a positive (isotone) valuation is a metric (pseudometric). Therefore lattices with positive (isotone) valuations are called metric (pseudometric) lattices.

In [5], the notion of a metric multilattice has been introduced. Applying this definition to the case of lattices, we obtain another definition of a metric lattice:

Metric lattice is a lattice with a metric d satisfying:

L1. $a \leq b \leq c \implies d(a,c) = d(a,b) + d(b,c)$,

L2. $d(a,b) = d(a \wedge b, a \vee b)$.

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Though the latter definition is formally different from the first one, actually they are equivalent, as it is shown in [5]. So we have characteristic properties of distance functions corresponding to positive valuations.

The starting-point of this paper is the question about the relation between the notions of a metric multilattice ([5]) and of a normed multilattice ([1]) (for the definitions see below). In Section 1, there are given characteristic properties of distance functions corresponding to valuations, isotone valuations and positive valuations on directed multilattices. Further, there are described distance functions corresponding to valuations on directed modular multilattices of locally finite length. As an application of the foregoing results, we prove in Section 2 that congruence relations on a directed modular multilattice of locally finite length form a Boolean algebra.

0. Basic notions

Let M be a partially ordered set, $a, b \in M$. Denote by $U(a,b)$ and $L(a,b)$ the set of all upper and lower bounds of the set $\{a,b\}$ in M, respectively. Further, let $a \vee b$ be the set of all minimal elements of the set $U(a,b)$, $a \wedge b$ the set of all maximal elements of $L(a,b)$. For $h \in U(a,b)$ define $(a \vee b)_h = \{t \in a \vee b : \ t \leq h\}$, and for $k \in L(a,b)$ let $(a \wedge b)_k = \{s \in a \wedge b : \ s \geq k\}$.

0.1. Definition. (cf. [1]) A partially ordered set M is said to be a *multilattice* if the sets $(a \vee b)_h$, $(a \wedge b)_k$ are nonempty for all $a, b \in M$, $h \in U(a,b)$, $k \in L(a,b)$. If, moreover, M is a directed set, i.e., the sets $U(a,b)$, $L(a,b)$ are nonempty for all $a, b \in M$, then M is called a *directed multilattice.*

A multilattice M is said to be *modular* if, whenever $a, b, c \in M$, $(a \wedge b) \cap (a \wedge c) \neq \emptyset$, $(a \vee b) \cap (a \vee c) \neq \emptyset$, $b \leq c$, then $b = c$.

The set $\{t \in M : \ a \leq t \leq b\}$ (with $a \leq b$) will be denoted by $\langle a, b \rangle$, and it will be referred to as an *interval.* M is said to have *locally finite length* provided that each its interval contains only finite chains.

1. Valuations and their distance functions

In what follows, M will be a directed multilattice, $\mathbf{G} = (G; +)$ an abelian group.

1.1. Definition. By a *valuation on* M a mapping $v \colon M \to G$ satisfying

V1. $u \in a \wedge b, \ \ w \in a \vee b \implies v(a) + v(b) = v(u) + v(w)$

is meant.

This definition corresponds to the definition of a valuation of third degree, introduced in [1] for multilattices with $\mathbf{G}$ being the additive group of all real numbers.

The simplest example of a valuation is a constant mapping. If M is the multilattice shown in Fig. 1, $\mathbf{G}$ an arbitrary group, then valuations $v\colon M \to G$ are just the mappings satisfying $v(u) = t$, $v(a) = v(b) = t + g$, $v(x) = v(y) = t + 2g$, $v(w) = t + 3g$ for all possible couples of elements $t, g \in G$, as it follows from further considerations.

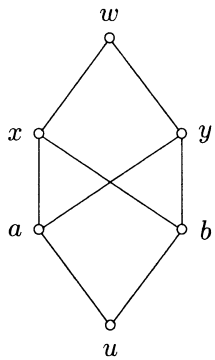

Figure 1.

1.2. LEMMA. *Let $v\colon M \to G$ be a valuation. If $u_1, u_2 \in a \wedge b$, $w_1, w_2 \in a \vee b$ for some $a, b \in M$, then $v(u_1) = v(u_2)$, $v(w_1) = v(w_2)$.*

Proof. Take $u_1, u_2 \in a \wedge b$, and any $w \in a \vee b$. Then $v(u_1) = v(a) + v(b) - v(w) = v(u_2)$ by the definition of a valuation. Analogously $v(w_1) = v(a) + v(b) - v(u_1) = v(w_2)$ for $w_1, w_2 \in a \vee b$. □

1.3. DEFINITION. The *distance function corresponding to a valuation* (*of a valuation*) $v\colon M \to G$ is the mapping $d\colon M \times M \to G$ defined by

$$d(a, b) = v(w) - v(u),$$

where $u \in a \wedge b$, $w \in a \vee b$.

(In view of 1.2, $d(a, b)$ does not depend on the choice of $u \in a \wedge b$, $w \in a \vee b$.)

The following two theorems are evident.

1.4. THEOREM. *If $v\colon M \to G$ is a valuation, then for any $t \in G$ the mapping $v_t\colon M \to G$ defined by $v_t(a) = v(a) + t$ is also a valuation with the same distance function as v.*

1.5. THEOREM. *If k is a positive integer and $v_1, \dots, v_k\colon M \to G$ are valuations with distance functions $d_1, \dots, d_k$, then for any integers $\alpha_1, \dots, \alpha_k$ the* *mapping* $v = \alpha_1 v_1 + \cdots + \alpha_k v_k \colon M \to G$ *defined by* $v(a) = \alpha_1 v_1(a) + \cdots + \alpha_k v_k(a)$ *is a valuation, too. The distance function of* v *is* $d = \alpha_1 d_1 + \cdots + \alpha_k d_k$.

1.6. THEOREM. *The distance function* d *of a valuation* $v \colon M \to G$ *satisfies the following conditions:*

M1. $a \le b \le c \implies d(a,c) = d(a,b) + d(b,c)$,
M2. $u \in a \wedge b, \quad w \in a \vee b \implies d(a,b) = d(u,w)$,
M3. $u \in a \wedge b, \quad w \in a \vee b \implies d(u,a) = d(b,w)$.

Proof. If $a \le b \le c$, then $d(a,c) = v(c) - v(a) = v(c) - v(b) + v(b) - v(a) = d(b,c) + d(a,b)$. Let $u \in a \wedge b$, $w \in a \vee b$. Then $d(a,b) = v(w) - v(u) = d(u,w)$, $d(u,a) = v(a) - v(u) = \big(v(u) + v(w) - v(b)\big) - v(u) = v(w) - v(b) = d(b,w)$, completing the proof. □

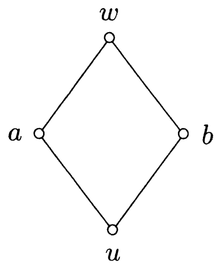

Figure 2.

Obviously, M1 implies that $d(a,a) = 0$ for each $a \in M$, and M2 implies the symmetry of d. We are going to show that the conditions M1, M2, M3 are independent. Let M be as in Fig. 2. Define $d_1, d_2, d_3 \colon M \times M \to \mathbb{R}$ as follows:

$$d_1(x,x) = 0 \qquad \text{for each} \quad x \in M,$$
$$d_1(u,a) = d_1(a,u) = d_1(u,b) = d_1(b,u) = 1,$$
$$d_1(a,w) = d_1(w,a) = d_1(b,w) = d_1(w,b) = 2,$$
$$d_1(u,w) = d_1(w,u) = d_1(a,b) = d_1(b,a) = 3;$$

$$d_2(x,x) = 0 \qquad \text{for each} \quad x \in M,$$
$$d_2(u,a) = d_2(a,u) = d_2(u,b) = d_2(b,u) = d_2(a,w)$$
$$= d_2(w,a) = d_2(b,w) = d_2(w,b) = 1,$$
$$d_2(a,b) = d_2(b,a) = 0,$$
$$d_2(u,w) = d_2(w,u) = 2;$$

$$d_3(x,y) = 1 \qquad \text{for all} \quad x, y \in M.$$

It is easy to see that d_1, d_2 and d_3 satisfy all conditions but M3, M2 and M1, respectively.

Now let us suppose that d is a mapping of $M \times M$ to G satisfying M1, M2, M3. We will prove that d is the distance function of a valuation $v\colon M \to G$. Moreover, we will describe all valuations with the distance function d. Let us pick an element x_0 of M. Using M3 we get that, if $a \in M$, $d(x_0, u)$ is the same for every $u \in a \wedge x_0$, and, analogously, $d(x_0, w)$ does not depend on the choice of $w \in a \vee x_0$. So, if we define $v_0\colon M \to G$ in such a way that

$$v_0(a) = d(x_0, w) - d(u, x_0),$$

where $u \in a \wedge x_0$, $w \in a \vee x_0$, this definition is correct. In view of M3, we have also $v_0(a) = d(x_0, w) - d(a, w) = d(u, a) - d(u, x_0)$.

1.7. Theorem. *The mapping $v_0\colon M \to G$ defined above is a valuation.*

Proof. Let $a, b \in M$, $u \in a \wedge b$, $w \in a \vee b$. We will prove $v_0(a) + v_0(b) = v_0(u) + v_0(w)$. Pick $t \in w \vee x_0$, $w_a \in (a \vee x_0)_t$, $w_b \in (b \vee x_0)_t$, $r \in (u \vee x_0)_{w_a}$. We have $v_0(a) = d(x_0, w_a) - d(a, w_a)$, $v_0(b) = d(x_0, w_b) - d(b, w_b)$, $v_0(u) = d(x_0, r) - d(u, r)$, $v_0(w) = d(x_0, t) - d(w, t)$.

Using M1 we get

$$d(u, a) + d(a, w_a) = d(u, r) + d(r, w_a), \tag{1}$$

$$d(b, w) + d(w, t) = d(b, w_b) + d(w_b, t). \tag{2}$$

By M3, $d(u, a) = d(b, w)$, so subtracting (1), (2) we obtain

$$d(a, w_a) - d(w, t) = d(u, r) + d(r, w_a) - d(b, w_b) - d(w_b, t). \tag{3}$$

Using again M1 we receive $d(w_b, t) = d(x_0, t) - d(x_0, w_b)$, $d(r, w_a) = d(x_0, w_a) - d(x_0, r)$. Substituting into (3) and arranging we get $v_0(a) + v_0(b) = d(x_0, w_a) - d(a, w_a) + d(x_0, w_b) - d(b, w_b) = d(x_0, r) - d(u, r) + d(x_0, t) - d(w, t) = v_0(u) + v_0(w)$. □

1.8. Lemma. *If $a, b \in M$, $a \le b$, then $v_0(b) - v_0(a) = d(a, b)$.*

Proof. Let $a, b \in M$, $a \le b$, $w \in b \vee x_0$, $w' \in (a \vee x_0)_w$. Using M1 we get $v_0(b) - v_0(a) = d(x_0, w) - d(b, w) - d(x_0, w') + d(a, w') = d(x_0, w') + d(w', w) - d(b, w) - d(x_0, w') + d(a, w') = d(a, w) - d(b, w) = d(a, b)$. □

1.9. Theorem. *The distance function of the valuation v_0 constructed above is the starting d.*

Proof. Let $a, b \in M$, $u \in a \wedge b$, $w \in a \vee b$. Using 1.8. and M2 we get $v_0(w) - v_0(u) = d(u, w) = d(a, b)$. □

1.10. THEOREM. *Let $d\colon M\times M\to G$ satisfy the conditions* M1, M2, M3. *Then for each $x_0\in M$ and any $t\in G$ there exists a valuation $v\colon M\to G$ such that $v(x_0)=t$, and d is its distance function. Two valuations having the same distance function differ at most in a constant.*

Proof. The valuation $v_0\colon M\to G$ considered above satisfies $v_0(x_0)=d(x_0,x_0)-d(x_0,x_0)=0$. Then $v_t\colon M\to G$ defined by $v_t(a)=v_0(a)+t$ is such a valuation as we need (see 1.4).

To complete the proof, it is sufficient to show that if v is any valuation with the distance function d, then $v-v_0$ is a constant mapping. Let $a\in M$, $w\in a\vee x_0$. We have $(v-v_0)(a)=v(a)-v_0(a)=v(a)-\big(d(x_0,w)-d(a,w)\big)$. Since d is the distance function of v, there is $d(x_0,w)=v(w)-v(x_0)$, $d(a,w)=v(w)-v(a)$. Consequently, $(v-v_0)(a)=v(a)-\big(v(w)-v(x_0)-v(w)+v(a)\big)=v(x_0)$, which is a constant (independent from the choice of a). □

Now let $\mathbf{G}=(G;+,\le)$ be a partially ordered abelian group. We will define isotone and positive valuations $v\colon M\to G$, and the aim is to characterize their distance functions.

1.11. DEFINITION. Let $v\colon M\to G$ be a valuation. Consider the following conditions:

V2. $a\le b \implies v(a)\le v(b)$,
V3. $a<b \implies v(a)<v(b)$.

If v satisfies V2, it is said to be an *isotone valuation.* We say that v is a *positive valuation* if V3 holds.

1.12. THEOREM. *A valuation $v\colon M\to G$ is isotone if and only if its distance function d satisfies the conditions* M1, M2, M3 *and*

M4. $a,b\in M \implies d(a,b)\ge 0$.

In this case, d is a pseudometric on M.

A valuation $v\colon M\to G$ is positive if and only if its distance function d satisfies M1, M2, M3 *and*

M5. $a,b\in M$, $a\ne b \implies d(a,b)>0$.

In this case, d is a metric on M.

Proof. Let $a,b\in M$. Then $d(a,b)=v(w)-v(u)$, where $u\in a\wedge b$, $w\in a\vee b$. If v is an isotone valuation, then $v(u)\le v(w)$, which implies $d(a,b)\ge 0$. Assuming that v is positive and $a\ne b$, we get $u<w$, $v(u)<v(w)$, so that $d(a,b)>0$.

Now, let us suppose that d fulfils M4. Then $a\le b$ implies $v(b)-v(a)=d(a,b)\ge 0$, hence $v(a)\le v(b)$. If d satisfies M5, then $a<b$ yields $v(b)-v(a)=d(a,b)>0$, which is equivalent to $v(a)<v(b)$.

To complete the proof, it is sufficient to show that the triangle inequality follows from the conditions M1, M2, M3, M4. Let $a, b, c \in M$. Let us pick elements $w_1 \in a \vee c$, $w_2 \in b \vee c$, $w \in w_1 \vee w_2$, $w_3 \in (a \vee b)_w$, $y \in (w_1 \wedge w_2)_c$, and, dually, $u_1 \in a \wedge c$, $u_2 \in b \wedge c$, $u \in u_1 \wedge u_2$, $u_3 \in (a \wedge b)_u$, $x \in (u_1 \vee u_2)_c$. Using M2 and M1 we obtain $d(a,c)+d(b,c) = d(u_1,w_1)+d(u_2,w_2) = d(u_1,a)+d(a,w_1)+d(u_2,b)+d(b,w_2)$. By M3, M1 and M4, $d(a,w_1) = d(u_1,c) = d(u_1,x)+d(x,c) \geq d(u_1,x) = d(u,u_2)$, and, analogously, $d(b,w_2) \geq d(u,u_1)$. Consequently, $d(a,c) + d(b,c) \geq d(u_1,a) + d(u,u_2) + d(u_2,b) + d(u,u_1) = d(u,a) + d(u,b) \geq d(u_3,a) + d(u_3,b) = d(u_3,a) + d(a,w_3) = d(u_3,w_3) = d(a,b)$, again by M1, M4, M3 and M2, as desired. □

1.13. LEMMA. *Let $d\colon M \times M \to G$ be a pseudometric (metric) on a directed multilattice M satisfying* M1 *and* M2. *Then* M3 *holds, too.*

P r o o f. Let $a, b \in M$, $u \in a \wedge b$, $w \in a \vee b$. We want to prove $d(u,a) = d(b,w)$. Using the triangle inequality and M1, M2 we get $d(u,b) + d(b,w) = d(u,w) = d(a,b) \leq d(a,u) + d(u,b)$, $d(u,a) + d(a,w) = d(u,w) = d(a,b) \leq d(a,w) + d(w,b)$, which yields $d(b,w) \leq d(a,u)$ and also $d(u,a) \leq d(w,b)$. Hence $d(u,a) = d(b,w)$. □

In [5], we introduced the notion of a metric multilattice as a multilattice with a metric d fulfilling M1 and M2. In [1], a multilattice with a positive valuation (of third degree) is said to be a normed multilattice. In view of 1.12 and 1.13, in the case of directed multilattices, these two notions are equivalent. Analogously, we can define a *pseudometric directed multilattice* by any of the following three ways:

a) as a directed multilattice with an isotone valuation $v\colon M \to G$;
b) as a directed multilattice with a pseudometric $d\colon M \times M \to G$ satisfying M1 and M2;
c) as a directed multilattice with a mapping $d\colon M \times M \to G^+$ satisfying M1, M2 and M3 ($G^+ = \{x \in G : x \geq 0\}$).

The problem of describing valuations $v\colon M \to G$ is equivalent to that of describing mappings $d\colon M \times M \to G$ satisfying M1, M2, M3. Consider the set $\operatorname{Int} M$ of all (nonempty) intervals of M.

Two intervals $\langle a,b \rangle$, $\langle c,d \rangle$ are said to be *transposed* if either $a \in b \wedge c$, $d \in b \vee c$ or $c \in a \wedge d$, $b \in a \vee d$. The intervals I, J are *projective* if there is a finite sequence of intervals $I = I_0, I_1, \dots, I_n = J$ such that all adjoining intervals I_k, I_{k+1} are transposed. It is easy to see that the relation of projectivity is an equivalence relation on $\operatorname{Int} M$, so it determines a decomposition of $\operatorname{Int} M$.

We will say that a class T of this decomposition is the *sum* of classes T_1, T_2 if T contains an interval $\langle a,b \rangle$ such that there is $\langle a,c \rangle \in T_1$, $\langle c,b \rangle \in T_2$ for some $c \in \langle a,b \rangle$, $c \neq a$, $c \neq b$. The distance function of a valuation coordinates to each interval of a class the same value. Consequently, our problem can be formulated as follows: to describe mappings of the system of all classes of projective intervals to G, assigning to each class T, which is a sum of some classes, the sum of theirs values. All such mappings can be extended to mappings $M \times M \to G$ defining $d(a,b)$ for a, b noncomparable in accordance with M2. Distance functions of isotone valuations assign to every class of projective intervals a nonnegative value, distance functions of positive valuations assign to classes of one-element intervals the value 0, to the others positive values.

We are going to apply this method to directed modular multilattices of locally finite length.

The following theorems (cf. [5]) will be useful:

1.14. THEOREM. *Let M be a modular multilattice of locally finite length, $a, b \in M$, $a \leq b$. Then all maximal chains in the interval $\langle a, b\rangle$ have the same length.*

Under the assumptions of the foregoing theorem, the common length of all maximal chains in $\langle a,b\rangle$, which is also the length of $\langle a,b\rangle$, will be denoted by $l(a,b)$.

1.15. THEOREM. *Let M be a modular multilattice of locally finite length, $a, b, u, w \in M$, $u \in a \wedge b$, $w \in a \vee b$. Then $l(u,a) = l(b,w)$.*

Consider the classes of prime intervals (an interval $\langle a,b\rangle$ is said to be prime if $l(a,b) = 1$) and pick one prime interval from each of them. Let P be the set of all chosen prime intervals. Assign to each $p \in P$ an element $\lambda_p \in G$. Now, if $a, b \in M$, $a \leq b$ put

$$d(a,b) = d(b,a) = \begin{cases} 0 & \text{if } a = b; \\ \lambda_p & \text{if } \langle a,b\rangle \text{ is a prime interval projective} \\ & \text{with } p \in P; \\ \sum\limits_{i=1}^{n} d(x_{i-1}, x_i) & \text{if } a = x_0 < x_1 < \dots < x_n = b, \\ & \langle x_{i-1}, x_i\rangle \text{ are prime intervals.} \end{cases}$$

1.16. THEOREM. *The value $d(a,b)$ does not depend on the choice of a maximal chain in the interval $\langle a,b\rangle$.*

P r o o f. We will prove, by induction on $l(a,b)$, that for any $p \in P$ the number of prime intervals projective with p is the same in all maximal chains of $\langle a,b\rangle$. For intervals of length 0 and 1 the assertion is obviously true. Let $n \geq 1$ and suppose that the theorem is valid for all intervals of the length n. Let $l(a,b) = n+1$, and let $a = a_0 < a_1 < \dots < a_{n+1} = b$, $a = b_0 < b_1 < \dots < b_{n+1} = b$ be two maximal chains. If $a_n = b_n$, the assertion is an immediate consequence of the induction assumption. Let $a_n \neq b_n$. Take any element of the set $(a_n \wedge b_n)_a$ and denote it by c_{n-1}. Evidently, $b \in a_n \vee b_n$, and, by 1.15, $\langle c_{n-1}, a_n\rangle$, $\langle c_{n-1}, b_n\rangle$ are prime intervals. Take any maximal chain $a = c_0 < c_1 < \cdots < c_{n-1}$ of $\langle a, c_{n-1}\rangle$. We can now complete the proof using the induction hypothesis several times together with the fact that either both or none of the intervals $\langle a_n, a_{n+1}\rangle$, $\langle c_{n-1}, b_n\rangle$ are projective with p and analogously for $\langle c_{n-1}, a_n\rangle$, $\langle b_n, b_{n+1}\rangle$. □

1.17. LEMMA. *If the intervals $\langle a,b\rangle$, $\langle r,s\rangle$ are transposed, then $d(a,b) = d(r,s)$.*

P r o o f. We can suppose $a \in b \wedge r$, $s \in b \vee r$. By 1.15, the intervals $\langle a,b\rangle$, $\langle r,s\rangle$ have the same length. Analogously as in the proof of the preceding lemma, we will prove by induction on $l(a,b)$ that for any $p \in P$ the number of prime intervals projective with p is in any maximal chain of $\langle a,b\rangle$ the same as in a maximal chain of $\langle r,s\rangle$. If $l(a,b) \leq 1$, the assertion is evidently true. Let us suppose that $n \geq 1$, and the statement holds if $l(a,b) = n$. Let $l(a,b) = n+1$, $a = a_0 < a_1 < \cdots < a_{n+1} = b$ be a maximal chain. Pick $t \in (a_n \vee r)_s$. If $a_n \in b \wedge t$, then $\langle t,s\rangle$ is a prime interval by 1.15, and either both or none of the intervals $\langle a_n, b\rangle$, $\langle t,s\rangle$ are projective with p. Using the induction assumption to $\langle a, a_n\rangle$, $\langle r,t\rangle$, we complete the proof. Suppose that $a_n \notin b \wedge t$. Then $b \leq t = s$. Since $t \in (a_n \vee r) \cap (b \vee r)$, $a \in (b \wedge r) \cap (a_n \wedge r)$, the modularity of M implies $a_n = b$, a contradiction. □

1.18. COROLLARY. *If $\langle a,b\rangle$, $\langle r,s\rangle$ are projective intervals, then $d(a,b) = d(r,s)$.*

We have assigned to each class of projective intervals an element of G. Evidently, if a class is the sum of classes T_1 and T_2, the value assigned to T is the sum of values assigned to T_1 and T_2. So we have:

1.19. THEOREM. *Let M be a directed modular multilattice of locally finite length, $\mathbf{G} = (G;+)$ a group. Distance functions of valuations $M \to G$ are just the mappings $d\colon M \times M \to G$ defined by $d(a,b) = d(u,w) = \sum\limits_{p\in P} p(u,w)\lambda_p$, where $u \in a \wedge b$, $w \in a \vee b$, and $p(u,w)$ expresses the number of prime intervals projective with p in any maximal chain of $\langle u,w\rangle$, for all possible choices of $\lambda_p \in G$. If $\mathbf{G}$ is a partially ordered group, d is the distance function of an isotone valuation if and only if all λ_p are nonnegative, and d belongs to a positive valuation if and only if all λ_p are positive.*

2. An application

Many authors investigated the lattice of all congruence relations on a lattice, see, e.g., [3], [4]. Using the argument of the preceding section we will prove a generalization of the theorem that the congruence relations on a modular lattice of finite length form a Boolean algebra.

2.1. Definition. (cf. [6]) Let Θ be a binary relation on a directed multilattice M. Then Θ is called a *congruence relation on* M provided that:

(i) Θ is an equivalence relation,

(ii) for all $a, a', b, b' \in M$ the relations $a \,\Theta\, a'$, $b \,\Theta\, b'$ imply $a \vee b \,\Theta\, a' \vee b'$ and $a \wedge b \,\Theta\, a' \wedge b'$.

By $a \vee b \,\Theta\, a' \vee b'$, we mean that

(1) for each $x \in a \vee b$ there exists $y \in a' \vee b'$ such that $x \,\Theta\, y$, and

(2) for each $y \in a' \vee b'$ there exists $x \in a \vee b$ with $x \,\Theta\, y$.

The meaning of $a \wedge b \,\Theta\, a' \wedge b'$ is analogous.

We will use the following theorem proved in [6]:

2.2. Theorem. *Let M be a directed multilattice, Θ a reflexive binary relation on M. Then Θ is a congruence relation if and only if it fulfils the following conditions:*

(i_1) $a \,\Theta\, b \iff u \,\Theta\, w$ *for some* $u \in a \wedge b$, $w \in a \vee b$,

(ii_1) $a \le b \le c$, $a \,\Theta\, b$, $b \,\Theta\, c \implies a \,\Theta\, c$,

(iii_1) $a \le b$, $a \,\Theta\, b \implies a \vee t \,\Theta\, b \vee t$, $a \wedge t \,\Theta\, b \wedge t$ *for any* t.

2.3. Theorem. *Let M be a directed multilattice, $\mathbf{G}$ a partially ordered group. Further, let d be a mapping $M \times M \to G$ such that $(M; d)$ is a pseudometric directed multilattice. Then the relation Θ defined on M by*

$$a \,\Theta\, b \iff d(a, b) = 0$$

is a congruence relation on M.

Proof. By assumption, d is a mapping $M \times M \to G^+$ satisfying the conditions M1, M2, M3 of the preceding section. We will use the foregoing theorem. Obviously, Θ is a reflexive relation satisfying the conditions (i_1), (ii_1). Let $a \le b$, $a \,\Theta\, b$, $t \in M$. We are going to prove $a \vee t \,\Theta\, b \vee t$. The relation $a \wedge t \,\Theta\, b \wedge t$ would be proved analogously. Pick any $w \in b \vee t$. Then there exists $w' \in (a \vee t)_w$. Evidently, $w \in b \vee w'$. Pick $u \in (b \wedge w')_a$. By M3 and M1, $d(w, w') = d(u, b) \le d(a, b) = 0$, so that $d(w, w') = 0$.

Now let $w \in a \vee t$. Choose elements $\bar{w} \in b \vee w$, $w' \in (b \vee t)_{\bar{w}}$, $x \in (b \wedge w)_a$, $y \in (w \wedge w')_x$, $u \in a \wedge t$, $p \in (x \wedge t)_u$, $u' \in (b \wedge t)_p$, $q \in (y \wedge t)_p$, $r \in (p \wedge a)_x$,

$s \in (r \vee u')_b$ (see Fig. 3). We want to prove $d(w, w') = 0$. By M2, M1 and M3, $d(w, w') = d(y, \bar{w}) = d(y, w) + d(w, \bar{w}) = d(y, w) + d(x, b)$. However, $d(x, b) \leq d(a, b) = 0$, hence $d(x, b) = 0$. Consequently, it is sufficient to prove $d(y, w) = 0$. As $\langle b, \bar{w} \rangle$, $\langle x, w \rangle$ are transposed intervals, and so are $\langle x, w \rangle$, $\langle p, t \rangle$, we have $d(b, \bar{w}) = d(p, t)$. Observe that $d(b, \bar{w}) = d(b, w') + d(w', \bar{w})$, $d(p, t) = d(p, q) + d(q, t)$, so that

$$d(b, w') + d(w', \bar{w}) = d(p, q) + d(q, t). \qquad (*)$$

Further, $\langle w', \bar{w} \rangle$, $\langle y, w \rangle$ are transposed intervals, and so are $\langle y, w \rangle$, $\langle q, t \rangle$, which implies $d(w', \bar{w}) = d(y, w) = d(q, t)$. Substituting into $(*)$ we obtain $d(b, w') = d(p, q)$. Besides $d(b, w') = d(u', t)$, as $\langle b, w' \rangle$, $\langle u', t \rangle$ are also transposed intervals. Therefore $d(p, q) = d(u', t)$. Using the above, M1 and M3, we get $d(y, w) = d(q, t) = d(p, t) - d(p, q) = d(p, u') + d(u', t) - d(p, q) = d(u, p')$. Finally, as the intervals $\langle p, u' \rangle$ and $\langle r, s \rangle$ are transposed, there is $d(p, u') = d(r, s) \leq d(a, b) = 0$. We have proved $d(y, w) = 0$, the desired result. □

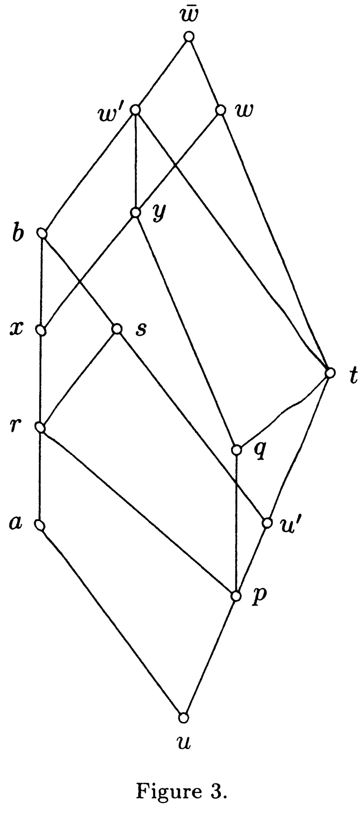

Figure 3.

2.4. THEOREM. *Let M be a directed modular multilattice of locally finite length. Then there exists an order-isomorphism of the system of all congruence relations on M onto the system of all subsets of the set of classes of projective prime intervals of M. Hence, the ordered system of all congruence relations on M is a Boolean algebra.*

P r o o f. Let Θ be any congruence relation on M. If $\langle x,y\rangle$, $\langle x',y'\rangle$ are projective prime intervals, then evidently $x\ \Theta\ y$ is equivalent to $x'\ \Theta\ y'$. Set $\Phi(\Theta) = \{\overline{\langle x,y\rangle} : \langle x,y\rangle \text{ is a prime interval, } x\ \Theta\ y\}$, $\overline{\langle x,y\rangle}$ being the class of all intervals which are projective with $\langle x,y\rangle$.

Evidently, $\Theta_1 \leq \Theta_2$ implies $\Phi(\Theta_1) \subseteq \Phi(\Theta_2)$.

Now let $\Phi(\Theta_1) \subseteq \Phi(\Theta_2)$, $a\ \Theta_1\ b$. Take any $u \in a\wedge b$, $w \in a\vee b$, and a maximal chain $u = a_0 < a_1 < \cdots < a_n = w$. We have $u = a_0\ \Theta_1\ a_1\ \Theta_1 \cdots \Theta_1\ a_n = w$, and since $\langle a_i, a_{i+1}\rangle$ are prime intervals, there is also $u = a_0\ \Theta_2\ a_1\ \Theta_2 \cdots \Theta_2\ a_n = w$, which implies $u\ \Theta_2\ w$, $a\ \Theta_2\ b$.

It remains to show that Φ is onto. Let $\mathcal{S}$ be a set of classes of projective prime intervals. Let $\mathbf{Z}$ be the ordered group of all integers, d be the mapping $M \times M \to \mathbb{Z}$ as in 1.19 for

$$\lambda_p = \begin{cases} 0 & \text{if } \bar{p} \in \mathcal{S}, \\ 1 & \text{if } \bar{p} \notin \mathcal{S}. \end{cases}$$

Then $(M;d)$ is a pseudometric directed multilattice. Denote by Θ the congruence relation on M corresponding to d in the sense of the preceding theorem. We want to show that $\Phi(\Theta) = \mathcal{S}$. There is $\overline{\langle x,y\rangle} \in \Phi(\Theta)$ if and only if $\langle x,y\rangle$ is a prime interval with $x\ \Theta\ y$, and this is evidently equivalent to $\overline{\langle x,y\rangle} \in \mathcal{S}$. The proof is complete. □

2.5. COROLLARY. *A directed modular multilattice of locally finite length is simple (i.e., has only trivial congruence relations) if and only if all its prime intervals are projective.*

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