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QUASICONTINUOUS SELECTIONS FOR CLOSED-VALUED MULTIFUNCTIONS

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ABSTRACT. We present a new result in the selection theory. A technique which worked only for the spaces $\mathbb{R}^n$ (equipped with a linear structure) is adapted and used in the topological context. The main result is: Let X be a regular topological space which is a union of pairwise disjoint regularly semiopen precompact sets. Let Y be a topological space, metrizable by a complete metric. Let $F\colon X \to Y$ be an l.s.c. multifunction with closed values. Then F has a quasicontinuous selection. Moreover, if X is a locally compact T_2 space, then for any finite subset A of X there exists a quasicontinuous selection of F which is continuous at any point of A.

1. Introduction

The research in the selection theory was started by Michael in 1956 (see for example [7], [8]) by proving several continuous selection theorems. Then, the problem of the existence of selections of various types (measurable, Carathéodory, Darboux etc.) was studied in many papers.

The first work dealing with the problem of existence of quasicontinuous selections for multifunctions was a paper of Matejdes [6]. The paper gives some conditions for the existence of quasicontinuous selections for multifunctions $F\colon X \to Y$ with compact values, where X is a Baire space and Y is a compact metric space.

A reason for proving quasicontinuous selection theorems when we cannot prove continuous ones is the relatively good connection between the continuity and quasicontinuity in spite of the generality of the latter. In general, if a continuous multifunction $F\colon X \to Y$ has nonconvex compact values, or even finite ones, it need not have a continuous selection ([3]).

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Even if the spaces X and Y are extremely nice (let for example $X = [a, b]$ be a compact interval in $\mathbb{R}$ and $Y = \mathbb{R}$), an l.s.c., u.s.c. and Hausdorff-continuous multifunction $F: [a, b] \to \mathbb{R}$ need not have a continuous selection (see [5]).

In this paper we prove some quasicontinuous selection theorems for multifunctions with closed values.

In 1988 Bressan [1] applied a useful technique which enabled him to obtain some new results in selection theory. The idea behind the technique was to consider the space $\mathbb{R}^n$ as a union of a system of pairwise disjoint n-dimensional "intervals" which were "sufficiently small".

In this paper we develop a similar technique. However, we introduce a topological concept of regularly semiopen set, which enables us to work with far more general spaces than $\mathbb{R}^n$ is. On these spaces only a topological stucture is considered.

2. Notation and terminology

In what follows we denote by $\mathbb{N}$ the set of all positive integers.

In this paper by a "*regular space*" we mean a topological space in which every point x and every nonempty closed set A not containing the point x can be separated by two disjoint open sets. So "regular" does not imply "T_1".

If $F: X \to Y$ is a multifunction from a given topological space X into the space of all nonempty subsets of a space Y, then for any set $A \subset Y$ we denote

$$F^-(A) = \{x \in X : \ F(x) \cap A \neq \emptyset\},$$
$$F^+(A) = \{x \in X : \ F(x) \subset A\}.$$

A *selection* for F is any function $f: X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$. By $\operatorname{int}(A)$ and $\overline{A}$ we denote the interior and the closure of A, respectively.

Let X be a topological space. A set $A \subset X$ is called *regularly open* if and only if $A = \operatorname{int}(\overline{A})$ (see [10]). In what follows by r.o. we mean "regularly open".

DEFINITION 1. Let X be a topological space. A set $A \subset X$ is said to be *semiopen* if there exists an open set $B \subset X$ such that $B \subset A \subset \overline{B}$. A set $A \subset X$ is said to be *regularly semiopen* (in what follows also *r.s.o.*) if there exists an r.o. set $B \subset X$ such that $B \subset A \subset \overline{B}$.

Let $P = \{A_\alpha : \ \alpha \in \Gamma\}$ (where Γ is an indexing set) be a collection of pairwise disjoint regularly semiopen subsets of X such that $\overline{A_\alpha}$ is compact for every $\alpha \in \Gamma$. Let K be a subset of X such that $K = \bigcup_{\alpha \in \Gamma} A_\alpha$. Then P is called an *r.s.o.* (*regularly semiopen*) *partition* of K. A topological space X with at least one r.s.o. partition of X will be called a *pt-space*.

For a later use, let us observe that a set A is r.s.o. if and only if A is semiopen and $\operatorname{int}(A)$ is r.o.. Of course, the space $\mathbb{R}^n$ equipped with the usual topology is a pt-space for every positive integer n.

DEFINITION 2. Let X, Y be two arbitrary topological spaces. A function $f\colon X \to Y$ is said to be *quasicontinuous* at $x \in X$ if for any open set V such that $f(x) \in V$ and any open set U such that $x \in U$ there exists a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at each point $x \in X$ it is said to be quasicontinuous (on X).

Remark 1. We shall use the following useful result of Neubrunnová [9]: a function $f\colon X \to Y$ is quasicontinuous if and only if $f^{-1}(G)$ is semiopen for any open subset G of Y.

3. Technical results

Many results presented in this section may be known and proven elsewhere. Nevertheless, we list them and prove them in order to make this paper self-contained. Namely the four propositions of this section contain many known assertions. The first three lemmas prepare the proof of Lemma 4. This lemma is important for proving our main results. It shows how to "cut" regularly semiopen sets which enables to construct their r.s.o. partitions.

PROPOSITION 1. *The following assertions are true.*

(i) *An open set O is r.o. if and only if* $\operatorname{int}(\overline{O}) \subset O$ *holds.*
(ii) *The intersection of two r.o. sets is an r.o. set.*
(iii) *For every closed set A the set* $\operatorname{int}(A)$ *is r.o..*

Proof.

(i) It suffices to prove that every open set O satisfying $\operatorname{int}(\overline{O}) \subset O$ is r.o.. Since $O \subset \overline{O}$ holds, then $O = \operatorname{int}(O) \subset \operatorname{int}(\overline{O})$ is true. So we have $O = \operatorname{int}(\overline{O})$.

(ii) Let A, B be two r.o. sets. Since $A = \operatorname{int}(\overline{A})$ and $B = \operatorname{int}(\overline{B})$, we obtain $\operatorname{int}(\overline{A \cap B}) \subset \operatorname{int}(\overline{A} \cap \overline{B}) = \operatorname{int}(\overline{A}) \cap \operatorname{int}(\overline{B}) = A \cap B$. According to (i) the set $A \cap B$ is r.o.

(iii) If A is closed, then the following holds: $\operatorname{int}(A) \subset \overline{\operatorname{int}(A)} \subset \overline{A} = A$. Hence $\operatorname{int}(A) \subset \operatorname{int}(\overline{\operatorname{int}(A)}) \subset \operatorname{int}(A)$ so $\operatorname{int}(A) = \operatorname{int}(\overline{\operatorname{int}(A)})$ holds, i.e. $\operatorname{int}(A)$ is r.o. □

DEFINITION 3. Let X be a topological space. A set $A \subset X$ is said to be a *regularly closed set* if and only if there exists an r.o. set B such that $A = \overline{B}$.

PROPOSITION 2. *The closure of any semiopen set is a regularly closed set.*

P r o o f . First we show that a closure of any open set is a regularly closed set. Let A be an open set. Let us denote $O = \operatorname{int}(\overline{A})$. Then $O \subset \overline{A}$, so $\overline{O} \subset \overline{A}$ and $\operatorname{int}(\overline{O}) \subset \operatorname{int}(\overline{A}) = O$. So we have $\operatorname{int}(\overline{O}) \subset O$, and by Proposition 1(i), O is an r.o. set. Moreover $A \subset O \subset \overline{A}$ holds so we obtain $\overline{A} = \overline{O}$, i.e. $\overline{A}$ is regularly closed. Now, let C be a semiopen set. Then $\operatorname{int}(C) \subset C \subset \overline{\operatorname{int}(C)}$ holds, therefore $\overline{C} = \overline{\operatorname{int}(C)}$. So the closure of C can be represented as a closure of an open set. Hence it is a regularly closed set. □

PROPOSITION 3. *The complement of a regularly closed set is regularly open. The complement of a regularly open set is regularly closed.*

P r o o f . Let A be a regularly closed set, let $A = \overline{O}$ where O is an r.o. set. We need to prove $X - A = \operatorname{int}(\overline{X - A})$. Since $O = \operatorname{int}(\overline{O})$ the following holds:

$$\begin{aligned} X - A &= X - \overline{A} = X - \overline{O} \\ &= X - \overline{\operatorname{int}\overline{O}} = \operatorname{int}(X - \operatorname{int}(\overline{O})) \\ &= \operatorname{int}(X - \operatorname{int}(A)) = \operatorname{int}(\overline{X - A}). \end{aligned}$$

So $X - A$ is an r.o. set.

Let $B = \operatorname{int}(\overline{B})$ be an r.o. set. Then $X - B = X - \operatorname{int}(\overline{B}) = \overline{X - \overline{B}}$ holds. The set $X - \overline{B}$ is open, hence, by Proposition 2 the set $X - B$ is regularly closed. □

PROPOSITION 4. *The following assertions hold.*

(i) *Every regularly closed set is r.s.o..*
(ii) *A set Z is regularly closed if and only if $Z = \overline{\operatorname{int}(Z)}$ holds.*
(iii) *A union of two regularly closed sets is a regularly closed set.*

P r o o f .

(i) This is obvious. It suffices to examine Definition 3 and Definition 1.

(ii) First let $Z = \overline{\operatorname{int}(Z)}$ holds. Then by Proposition 2 the set Z is regularly closed. Now let us suppose that Z is a regularly closed set. Then there exists an open set B such that: $B = \operatorname{int}(\overline{B})$ and $Z = \overline{B}$. Hence $Z = \overline{\operatorname{int}(\overline{B})} = \overline{\operatorname{int}(Z)}$.

(iii) Since A and B are regularly closed, their complements are regularly open sets and the intersection of these complements is a regularly open set (Proposition 1(ii)). So according to Proposition 3 the set $A \cup B = X - \big((X - A) \cap (X - B)\big)$ is a regularly closed set. □

LEMMA 1. *Let A be an r.s.o. set and B be r.o.. Then the set $A \cap B$ is r.s.o..*

P r o o f . Let A be an r.s.o. set and B be r.o.. Then $B = \operatorname{int}(\overline{B})$, $A \subset \overline{\operatorname{int}(A)}$ and $\operatorname{int}(A) = \operatorname{int}(\overline{\operatorname{int}(A)})$ hold. Let $C = \operatorname{int}(A) \cap B$. Since C is an intersection of two r.o. sets, C is r.o.. Let x be an element of $B \cap A$. Let O be an open neighbourhood of x. Since x is an element of B, the set $O \cap B$ is an open nonempty neighbourhood of x. The point x is also an element of $\overline{\operatorname{int}(A)}$, therefore $(O \cap B) \cap \operatorname{int}(A) \neq \emptyset$ so $O \cap (B \cap \operatorname{int}(A)) = O \cap C \neq \emptyset$ holds. Therefore x is an element of $\overline{C}$. Hence $C \subset A \cap B \subset \overline{C}$ holds and by Definition 1, $A \cap B$ is an r.s.o. set. □

Lemma 2. *Let K, F be two r.s.o. sets. Let $F \subset \overline{K}$ holds. Then the set $F \cap K$ is an r.s.o. set.*

P r o o f. Let $H = F \cap K$, $D = \operatorname{int}(F) \cap \operatorname{int}(K)$. The set D is r.o. since it is an intersection of two r.o. sets. Let c be an element of H. Let O be an arbitrary open neighbourhood of c. Then the set $U = O \cap \operatorname{int}(F)$ is nonempty, since $c \in F \subset \overline{\operatorname{int}(F)}$ holds. The set U is an open subset of F. Since $F \subset \overline{K} = \overline{\operatorname{int}(K)}$, then $U \cap \operatorname{int}(K) \neq \emptyset$. Hence $(O \cap \operatorname{int}(F)) \cap \operatorname{int}(K) \neq \emptyset$ is true or, when we rewrite it differently, $O \cap (\operatorname{int}(F) \cap \operatorname{int}(K)) = O \cap D \neq \emptyset$ holds. Therefore the point c is an element of $\overline{D}$ and this implies $D \subset H \subset \overline{D}$. That is, the set H is r.s.o.. □

Lemma 3. *Let X be a topological space. Let $K \subset X$ be an r.s.o. set and let $K = \bigcup_{i=1}^{l} A_i$ where $\{A_i : i = 1, 2, \dots, l\}$ is a finite collection of r.s.o. sets. Let a be an element of $\operatorname{int}(A_1) \cap \operatorname{int}(K) \subset X$. Then there exists a finite r.s.o. partition $P = \{H_i : i = 1, 2, \dots, l\}$ of K such that $H_i \subset \overline{A_i}$ for $i = 1, 2, \dots, l$ holds and a is an element of $\operatorname{int}(H_1) \cap \operatorname{int}(K)$.*

P r o o f. First observe that the equality $\overline{K} = \bigcup_{i=1}^{l} \overline{A_i}$ holds. Let $C_i = \overline{A_i}$ for $i = 1, 2, \dots, l$. The sets C_i are regularly closed. Set $D_1 = C_1$ and for $j = 2, 3, \dots, l$, $D_i = \overline{C_i - \left(\bigcup_{j=1}^{i-1} D_j\right)}$. Then for $k = 1, 2, \dots, l$, $D_k \subset \overline{A_k} \subset \overline{K}$. From the definition of the sets D_i it is easy to see that $\overline{K} = \bigcup_{i=1}^{l} D_i$ and $a \in \operatorname{int}(D_1)$.

Next we prove by induction that the sets $D_1, D_2, \dots, D_l$ are regularly closed.

1. The set D_1 is regularly closed since $D_1 = \overline{A_1}$ holds.
2. Let for $j = 1, 2, \dots, s < l$, the set D_j be regularly closed.

Since $D_{s+1} = \overline{C_{s+1} - (D_1 \cup \dots \cup D_s)} = \overline{C_{s+1} \cap (X - (D_1 \cup \dots \cup D_s))}$, we can see, that D_{s+1} is a closure of the intersection of a regularly closed set (C_{s+1}) and a regularly open one. (The set $X - (D_1 \cup \dots \cup D_s)$ is r.o. by Proposition 3(iii) and by Proposition 4.) According to Lemma 1 this intersection is an r.s.o. set. So our set D_{s+1} is a closure of an r.s.o. set and according to Proposition 2 it is regularly closed.

This finishes our proof by induction.

Now, let us denote $F_1 = D_1$ and for $1 < i \le l$, $F_i = D_i - \bigcup_{j<i} D_j$. The sets F_i are r.s.o. since each of them is an intersection of a regularly closed set with an r.o. one. It can be seen that for $i = 1, 2, \dots, l$, $F_i \subset D_i \subset \overline{A_i}$ holds and that $\overline{K} = \bigcup_{i=1}^{l} F_i$ and $a \in \operatorname{int}(F_1) \cap \operatorname{int}(K)$ are true.

Set $H_i = F_i \cap K$ for $i = 1, 2, \dots, l$. According to Lemma 2 the sets H_i are r.s.o.. We have $K = \bigcup_{i=1}^{l} H_i$ and $a \in \operatorname{int}(H_1) \cap \operatorname{int}(K)$. Since the sets F_i were pairwise disjoint, the sets H_i are pairwise disjoint too. So the collection of sets $P = \{H_i : i = 1, 2, \dots, l\}$ is an r.s.o. partition of K. The proof of the lemma is now complete. $\square$

LEMMA 4. *Let X be a topological space. Let $K \subset X$ be an r.s.o. set and let $\{B_i : i = 1, 2, \dots, l\}$ be a finite open cover of K consisting of r.o. sets B_i. Let a be an element of $\operatorname{int}(B_1) \cap \operatorname{int}(K) \subset X$. Then there exists a finite r.s.o. partition $P = \{H_i : i = 1, 2, \dots, l\}$ of K such that $H_i \subset \overline{B_i}$ for $i = 1, 2, \dots, l$ and a is an element of $\operatorname{int}(H_1) \cap \operatorname{int}(K)$.*

Proof. Set $V_i = B_i \cap \operatorname{int}(K)$ for $i = 1, 2, \dots, l$. The sets V_i are r.s.o. and $a \in \operatorname{int}(V_1) \cap \operatorname{int}(K)$. Since each of the sets V_i is a subset of K, the following holds:

$$\overline{V_i} \subset \overline{K} \qquad \text{for} \quad i = 1, 2, \dots, l, \tag{$*$}$$

and since $\operatorname{int}(K) = \bigcup_{i=1}^{l} V_i$, we obtain $\overline{K} = \overline{\operatorname{int}(K)} = \bigcup_{i=1}^{l} \overline{V_i}$. Using the inclusion $(*)$ we obtain by Lemma 2 that for $i = 1, 2, \dots, l$ the set $A_i = \overline{V_i} \cap K$ is r.s.o.. For each $i \in \{1, 2, \dots, l\}$, $A_i \subset \overline{B_i}$ holds, and $a \in \operatorname{int}(A_1) \cap \operatorname{int}(K)$ and $K = \bigcup_{i=1}^{l} A_i$ hold too. According to Lemma 3 there exists a finite r.s.o. partit on $P = \{H_i : i = 1, 2, \dots, l\}$ of K such that $H_i \subset \overline{A_i}$ for $i = 1, 2, \dots, l$ and a is an element of $\operatorname{int}(H_1) \cap \operatorname{int}(K)$. We see, that for $i = 1, 2, \dots, l$, $H_i \subset \overline{A_i} \subset \overline{B_i}$. The proof is complete. $\square$

4. The main result

THEOREM 1. *Let X be a compact regular topological space (which need not be T_1), Y be a topological space, metrizable by a complete metric. Let $F\colon X \to Y$ be an l.s.c. multifunction with closed values. Then for every point (a, b) of the graph of F there exists a quasicontinuous selection f of F such that $f(a) = b$ and f is continuous at a.*

P r o o f . Let d be such a metric, that (Y, d) is a complete metric space with the original topology of Y.

We shall construct now by induction a sequence of functions $\{f_n\}_{n=1}^{\infty}$, $f_n\colon X \to Y$, a sequence $\{D_n\}_{n=1}^{\infty}$ of r.s.o. partitions of X and a sequence $\{V_n\}_{n=1}^{\infty}$ of finite collections of subsets of Y such that for every positive integer n the following will hold:

(1) D_n is a finite r.s.o. partition of X, $D_n = \{D_1^n, \dots, D_{l(n)}^n\}$ and $X = \bigcup_{j=1}^{l(n)} D_j^n$. The collection V_n is a finite collection of open subsets of Y, $V_n = \{V_1^n, \dots, V_{l(n)}^n\}$ and the diameter of every element of V_n is smaller then 2^{-n}. For $i = 1, 2, \dots, l(n)$, $\overline{D_i^n} \subset F^-(V_i^n)$ holds.

(2) If $n > 1$, then for every element j of $\{1, 2, \dots, l(n)\}$ there exists an element k of $\{1, 2, \dots, l(n-1)\}$ such that $D_j^n \subset \overline{D_k^{n-1}}$ and $V_j^n \subset V_k^{n-1}$. Moreover, if $j = 1$, then $k = 1$, so $D_1^n \subset \overline{D_1^{n-1}}$.

(3) The functions f_n are constant on every element of D_n and $f_n(D_i^n) \in V_i^n$ for $i \in \{1, 2, \dots, l(n)\}$.

(4) The point a is an element of $\operatorname{int}(D_1^n)$, and b is an element of V_1^n, where $f_n(D_1^n) = b$ holds.

(5) If $n > 1$, then for every element x of X, $d(f_n(x), f_{n-1}(x)) < 2^{-n}$.

(6) For every element x of X, $d(f_n(x), F(x)) = \inf\limits_{y \in F(x)} \{d(f_n(x), y)\} < 2^{-n}$.

We note also, that (1) and (3) imply the quasicontinuity of the functions f_n.

Let us start by constructing f_1. Let us consider such a metric on Y, that $\operatorname{diam}(Y) < 2^{-1}$. Then we can define $D_1 = \{X\}$; $V_1 = \{Y\} = \{B(b, 2^{-1})\} = \{\{y \in Y : d(b, y) < 2^{-1}\}\}$ and $f_1(x) = b$ for all x from X.

Now, let $k \in \mathbb{N}$. Let us suppose, that for $j = 1, 2, \dots, k$ the systems V_j, D_j and the functions f_j are defined such that (1), (2), … , (6) hold for them.

Let us define D_{k+1}, V_{k+1} and f_{k+1} as follows: When $D_k = \{D_1^k, \dots, D_{l(k)}^k\}$ and $V_k = \{V_1^k, \dots, V_{l(k)}^k\}$, we define for $m \in \{1, 2, \dots, l(k)\}$:

$$I_m = \{B(y, r) : y \in Y,\ r < 2^{-k-2} \text{ and } B(y, r) \subset V_m^k\},$$

$$J_m = \{F^-(V) : V \in I_m\},$$

$$H_m = \{H : H \text{ is r.o. and there exists } S \text{ from } J_m \text{ such that } \overline{H} \subset S\}.$$

Since $\overline{D_m^k}$ is a compact subset of $F^-(V_m^k)$, H_m is an open cover of $\overline{D_m^k}$ from which we choose a finite subcover $R_m = \{C_1^m, \dots, C_{j(m)}^m\}$. (This is the step where the regularity of X is needed. In nonregular spaces nothing guarantees that H_m is an open cover — or a nonempty set.)

Moreover if $m = 1$, we choose $R_1 = \{C_1^1, \dots, C_{j(1)}^1\}$ in such a way, that $a \in C_1^1 \in H_1$ holds. This is possible since for D_k and V_k, (1) and (4) are true, so $a \in \operatorname{int}(D_1^k) \subset F^-(V_1^k)$ and $b \in V_1^k$ hold. From the definition of H_1 and I_1 we see that there exist sets $B_1 \in H_1$ and $W \in I_1$ such that $a \in B_1$, $\overline{B_1} \subset F^-(W)$ and $b \in W$. Let us put $C_1^1 = B_1$. Then $a \in C_1^1 \cap \operatorname{int}(D_1^k)$ holds. Let us extract a finite cover $\{C_1^1, \dots, C_{j(1)}^1\} \subset H_1$ of the set $\overline{D_1^k}$ in such a way that C_1^1 is the set mentioned above.

From the definition of H_m and I_m it follows that for every m from $\{1, 2, \dots$ $\dots, l(k)\}$ there exists a finite collection of open balls $\{W_1^m, \dots, W_{j(m)}^m\}$ such that for every index o from $\{1, 2, \dots, j(m)\}$ the following holds: $\overline{C_o^m} \subset F^-(W_o^m)$, $W_o^m \subset V_m^k$, $\operatorname{diam}(W_o^m) < 2^{-k-1}$.

In particular, we put $W_1^1 = W$.

For $m = 1, 2, \dots, l(k)$ the collection of regularly open sets $R_m = \{C_1^m, \dots$ $\dots, C_{j(m)}^m\}$ is a finite cover of the r.s.o. set D_m^k. For $m = 1$, moreover, C_1^1 is r.o. and $a \in \operatorname{int}(C_1^1) \cap \operatorname{int}(D_1^k)$ holds. According to Lemma 4 there exists a finite r.s.o. partition $P_m = \{F_1^m, \dots, F_{j(m)}^m\}$ of the set D_m^k such that $F_i^m \subset \overline{C_i^m} \subset F^-(W_i^m)$ for $i = 1, 2, \dots, j(m)$ and, moreover, for $m = 1$, $a \in \operatorname{int}(F_1^1) \cap \operatorname{int}(D_1^k)$, $b \in W_1^1$ and $F_1^1 \subset \overline{D_1^k}$ hold.

Set $D'_{k+1} = \bigcup_{m=1}^{l(k)} P_m$. Let $l(k+1)$ be the number of nonempty elements of D'_{k+1}. Set $D_1^{k+1} = F_1^1$.

Let us denote all other nonempty elements of D'_{k+1} by $D_2^{k+1}, D_3^{k+1}, \dots$ $\dots, D_{l(k+1)}^{k+1}$. We put $D_{k+1} = \{D_1^{k+1}, D_2^{k+1}, \dots, D_{l(k+1)}^{k+1}\}$.

We define a collection V_{k+1} as follows: We put $V_1^{k+1} = W_1^1$. If $o \in \{2, 3, \dots, l(k+1)\}$, then there exist two positive integers s and t such that $D_o^{k+1} = F_s^t$. For such s and t let us denote $V_o^{k+1} = W_s^t$.

It is easy to verify now that for D_{k+1} and $V_{k+1} = \{V_1^{k+1}, \dots, V_{l(k+1)}^{k+1)}\}$, (1) and (2) hold. For $o = 2, 3, \dots, l(k+1)$ the sets V_o^{k+1} are open balls and $\operatorname{diam}(V_o^{k+1}) < 2^{-k-1}$ holds. Let us denote the center of each of these balls V_o^{k+1} by y_o^{k+1}. For $o = 1$ we put $y_1^{k+1} = b$. We define a function $f_{k+1} \colon X \to Y$ as follows:

For $o = 1, 2, \dots, l(k+1)$, $f_{k+1}(x) = y_o^{k+1}$ if and only if $x \in D_o^{k+1}$.

From the way we constructed the collections of sets D_{k+1}, V_{k+1} and the function f_{k+1}; it follows that for $n = k+1$ the conditions (1), (2), ... , (6) are fulfilled. Our construction by induction is finished.

We have constructed a sequence of quasicontinuous functions $\{f_n\}_{n=1}^{\infty}$, $f_n \colon X \to Y$, a sequence $\{D_n\}_{n=1}^{\infty}$ of r.s.o. partitions of X and a sequence $\{V_n\}_{n=1}^{\infty}$ of finite collections of subsets of Y such that (1), (2), ... , (6) hold for them. By (5) the sequence $\{f_n\}_{n=1}^{\infty}$ is Cauchy, and since (Y, d) is a complete metric space, there exists a function $f \colon X \to Y$ which is a uniform limit of the functions f_n. According to [11] the function f is quasicontinuous. Since each f_n was continuous at the point a, the function f is continuous at a too.

Next we show that for every x from X, $f(x) \in F(x)$. Let $\varepsilon > 0$ be an arbitrary positive real number. Let us choose a positive integer n such that $2^{-n} < \frac{\varepsilon}{2}$. Let $l \in \mathbb{N}$ be such that $\forall(k \geq l)\forall(x \in X)\big(d(f(x), f_k(x)) < \frac{\varepsilon}{2}\big)$. Let $m = \max\{n, l\}$. Since, by (6), $d\big(f_m(x), F(x)\big) \leq +2^{-m}$, for every x from X, $d\big(f(x), F(x)\big) \leq d\big(f(x), f_m(x)\big) + d\big(f_m(x), F(x)\big) < \frac{\varepsilon}{2} + 2^{-m} < \varepsilon$. The values of F are closed, therefore $\forall(x \in X)\big(f(x) \in F(x)\big)$. This completes the proof. $\square$

We are now ready to present our main result.

THEOREM 2. *Let X be a regular topological space which is a pt-space. Let Y be a topological space, metrizable by a complete metric. Let $F\colon X \to Y$ be an l.s.c. multifunction with closed values. Then F has a quasicontinuous selection. Moreover, if*

(M) *X is a locally compact T_2 space and $\{a_1, a_2, \dots, a_n\}$ is a finite subset of X ($a_i \neq a_j$ for $i \neq j$), and $\{b_1, b_2, \dots, b_n\} \subset Y$ is such that $\forall\big(i \in \{1, 2, \dots, n\}\big)\big(b_i \in F(a_i)\big)$,*

then F has a quasicontinuous selection $f\colon X \to Y$ such that for all $i \in \{1, 2, \dots, n\}$, f is continuous at the point a_i and $f(a_i) = b_i$.

Proof.

I. We show that if X is locally compact T_2, then for any finite subset $\{a_1, a_2, \dots, a_n\}$ of X with exactly n points there exists an r.s.o. partition P of X such that for all i, $1 \leqq i \leqq n$, there exists $A \in P$ such that $a_i \in \operatorname{int}(A)$.

Since X is a pt-space, there exists an r.s.o. partition $Q = \{A_\gamma : \gamma \in \Gamma\}$. Since X is locally compact and T_2, there exists a finite system $\{V_1, V_2, \dots, V_n\}$ of pairwise disjoint regularly open subsets of X such that for all k, $1 \leqq k \leqq n$, $a_k \in V_k$, $\overline{V_k}$ is compact and $\overline{V_i} \cap \overline{V_j} = \emptyset$ for $i \neq j$.

The set $K = \bigcup\limits_{i=1}^{n} \overline{V_i} = \overline{\bigcup\limits_{i=1}^{n} V_i}$ is compact and regularly closed (see Propositon 2). Let us define P as follows: $P = \{\overline{V_1}, \overline{V_2}, \dots, \overline{V_n}\} \cup \{A - K : A \in Q\}$. For each A from Q the set $A - K = A \cap (X - K)$ is an intersection of an r.s.o. set and an r.o. set (Proposition 4). So by Lemma 1 it is an r.s.o. set. Since Q was an r.s.o. partition of X, P is also such one.

II. Let $X = \bigcup\limits_{\alpha \in \Gamma} A_\alpha$ where $P = \{A_\alpha : \alpha \in \Gamma\}$ is an r.s.o. partition of X. Let us consider an arbitrary index $\alpha \in \Gamma$. By Theorem 1 the multifunction F, restricted to $\overline{A_\alpha}$ has a quasicontinuous selection $f_\alpha\colon \overline{A_\alpha} \to Y$, where we consider $\overline{A_\alpha}$ endowed with the inherited topology. (Moreover, if (M) holds we can suppose, that the partition P is the one constructed in I. Then, according

Theorem 1, f_α can be constructed on every A_α in such a way that if $a_i \in A_\alpha$, then f_α is continuous in a_i and $f_\alpha(a_i) = b_i$.)

Let us define a function $f\colon X \to Y$ as follows:

$$f(x) = f_\alpha(x) \iff x \in A_\alpha .$$

It is obvious that f is a selection of F. We will show that f is quasicontinuous. It suffices to verify that f satisfies the conditions of Definition 2. So let x be an arbitrary point of X. Let U be an open neighbourhood of x, let W be an open neighbourhood of its image $f(x)$. There exists an index $\alpha \in \Gamma$ such that x is an element of A_α. Let us denote $V = U \cap A_\alpha$. The set V is a relatively open neighbourhood of x in the space A_α. Since the function $f_\alpha\colon A_\alpha \to Y$ is quasicontinuous, there exists a nonempty relatively open subset O' of the space A_α such that $O' \subset V$ and $f(O') \subset W$. Since O' is relatively open, there exists an open subset P of X such that $O' = P \cap A_\alpha$. So the set O' is an intersection of an open set with a semiopen one. Therefore it is semiopen. So the set $O = \operatorname{int}(O'$ is a nonempty open subset of U. For the set O, $f(O) = f_\alpha(O) \subset f_\alpha(O') \subset W$. So f is quasicontinuous at x, and since x was an arbitrary point of X, f is proved to be quasicontinuous.

Moreover, if (M) holds, f is continuous at each a_i. Indeed, there exists an index α such that $A_\alpha = \overline{V_i}$ ($\overline{V_i}$ as constructed in I) and f coincides on $\overline{V_i}$ with f_α, which is continuous at $a_i \in \operatorname{int}(\overline{V_i})$. Of course, $f(a_i) = f_\alpha(a_i) = b_i$ holds for each of our finitely many indices i. □

The following example shows that the assumption of regularity of X in Theorem 2 is essential. If X is not regular, there exist multifunctions $F\colon X \to \mathbb{R}$ which have no somewhat continuous selection. (A function $f\colon X \to Y$ is somewhat continuous if and only if for every open subset V of Y the following holds: if $f^{-1}(V) \neq \emptyset$, then $\operatorname{int}(f^{-1}(V)) \neq \emptyset$. So somewhat continuity is far weaker than quasicontinuity.)

EXAMPLE 1. Let $X = (N, T)$ be the set of natural numbers endowed with the cofinite topology $T = \{G \subset N : N - G \text{ is finite}\}$. Let $Y = \mathbb{R}$. Let us define a multifunction $F\colon X \to Y$ as follows:

$$\forall (k \in \mathbb{N})\big(F(k) = N - \{k\}\big) .$$

X is a compact topological space, Y is is a complete metric one. F is an l.s.c. multifunction with closed values. But F has no constant selection and it is easy to see that every somewhat continuous function $f\colon X \to Y$ has to be constant.

OPEN QUESTION. Is every regular, locally compact topological space a pt-space?

Remark 2. If X is $\mathbb{R}^n$, then the selection f can be constructed to be measurable if we choose the "right" partitions of X consisting of measurable sets. More precisely these partitions can be constructed with aid of cubes in $\mathbb{R}^n$. (I.e. in the proof of Theorem 1 we could choose the sets H from H_m to be open cubes in $\mathbb{R}^n$.)

Remark 3. As the referee communicated to the author, in [2], Bressan and Colombo continued the work started in [1]. In [2], they do not need to work with linear structure. Their results are valid for paracompact spaces, so for T_4 spaces. Theorem 1 in our article is valid also for non T_1 spaces. The continuity property of a selection in our case is still the same: the selection is quasicontinuous and continuous in a chosen point. In [2], the type of generalized continuity of the selection depends on the topological properties of the space X.

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