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Svatoslav Staněk

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EXISTENCE RESULTS FOR FUNCTIONAL BOUNDARY VALUE PROBLEMS AT RESONANCE

SVATOSLAV STANĚK

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ABSTRACT. Existence theorems for functional second order differential equations with nonlinear boundary conditions are proved by Leray-Schauder degree theory and the Borsuk theorem. The existence results are formulated simply by sign conditions. Some applications to third and fourth order differential equations with nonlinear boundary conditions are given.

1. Introduction, notation

Let X be the Banach space of continuous functions on $J = [0,1]$ endowed with the sup norm $\|\cdot\|$. Denote by $\mathcal{D}$ the set of all operators $K\colon X \to X$ which are continuous and bounded (i.e., $K(\Omega)$ is bounded for any bounded $\Omega \subset X$). Let $I \subset J$ be an interval, and $\mathcal{A}_I$ be the set of all functionals $\gamma\colon X \to \mathbb{R}$ which are

(i) continuous, $\gamma(0) = 0$,
(ii) increasing (i.e., $x(t) < y(t)$ for $t \in I \implies \gamma(x) < \gamma(y)$),
(iii) $\lim\limits_{n\to\infty} \gamma(\varepsilon x_n) = \varepsilon\infty$ for each $\varepsilon \in \{-1, 1\}$ and any $\{x_n\} \subset X$, $\lim\limits_{n\to\infty} x_n(t) = \infty$ locally uniformly on I.

In this paper, we consider the boundary value problem (BVP for short)

$$x''(t) = f\big(t, x(t), (Fx)(t), x'(t), (Hx')(t)\big)\,, \tag{1}$$

$$\alpha(x) = A\,, \qquad \beta\big(x'(1) - x'\big) = B\,. \tag{2}$$

Here $f\colon J \times \mathbb{R}^4 \to \mathbb{R}$ satisfies the local Carathéodory conditions on $J \times \mathbb{R}^4$ ($f \in \mathrm{Car}(J \times \mathbb{R}^4)$ for short), $F, H \in \mathcal{D}$, $\alpha \in \mathcal{A}_J$, $\beta \in \mathcal{A}_{[0,1)}$, and $A, B \in \mathbb{R}$.

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By a *solution* of (1) we mean a function $x \in AC^1(J)$ (having the absolutely continuous first derivative on J) satisfying (1) a.e. on J. Special cases of (1) are the differential equations

$$x'' = g(t, x, x'), \qquad g \in \mathrm{Car}(J \times \mathbb{R}^2), \tag{3}$$

$$x'' = f_1(t, x, x', x(0), x(1)), \qquad f_1 \in \mathrm{Car}(J \times \mathbb{R}^4), \tag{4}$$

and

$$x'' = f_2(t, x, x', x_+, x_-, x'_+, x'_-), \qquad f_2 \in \mathrm{Car}(J \times \mathbb{R}^6), \tag{5}$$

where $y_+(t) = \max\{y(t), 0\}$, $y_-(t) = \max\{-y(t), 0\}$ for $t \in J$. Equation (4) is obtained by setting $(Fx)(t) = x(0)$, $(Hx)(t) = \int_0^1 x(s)\,\mathrm{d}s$ and $f(t, x, u, v, w) = f_1(t, x, v, u, w+u)$, and then, equation (5) by setting $(Fx)(t) = (Hx)(t) = x_+(t)$ and $f(t, x, u, v, w) = f_2(t, x, v, u, u - x, w, w - v)$.

EXAMPLE 1. Let $\varphi, \varphi_j \in C^0(\mathbb{R})$ be increasing functions mapping $\mathbb{R}$ onto itself, $\varphi(0) = 0 = \varphi_j(0)$ ($j = 1, 2, \dots, n$). Let J_1, $J_1 \subset I$ ($\subset J$), be a compact interval. The functionals belonging to the set $\mathcal{A}_I$ can be given like this:

$$\max\{\varphi(x(t)) : t \in J_1\}, \qquad \min\{\varphi(x(t)) : t \in J_1\},$$

$$\int_{t_0}^{t_1} \varphi(x(s))\,\mathrm{d}s \qquad (t_0, t_1 \in I,\ t_0 < t_1),$$

$$\sum_{j=1}^{n} \varphi_j(x(t_j)) \qquad (t_1 < t_2 < \dots < t_n,\ t_1, t_n \in I).$$

Remark 1. Examples of operators belonging to the set $\mathcal{D}$ are given in [4].

Remark 2. There exists a functional $\gamma \in \mathcal{A}_J$ satisfying the assumptions (i) and (ii) and $\mathrm{Im}\,\gamma = \mathbb{R}$, but the assumption (iii) is not satisfied (see [7; Example 1]).

Observe that the boundary conditions (2) are in general nonlinear, and BVP (1), (2) is at resonance (i.e., the corresponding homogeneous BVP $x'' = 0$, $\alpha(x) = 0$, $\beta(x'(1) - x') = 0$ has nontrivial solutions).

We find sufficient conditions for the existence of solutions of BVP (1), (2). The conditions are formulated only in terms of sign conditions. Our results are proved by the topological degree method and the Borsuk theorem (see, e.g., [1]). Applications to nonlinear BVPs for third and fourth order differential equations are given in the last section.

This paper is a continuation of papers [4] and [5]. In [4], BVPs for equation (1) with boundary conditions $\alpha(x) = A$, $x'(1) = B$ or $\alpha(x) = A$, $x'(0) = B$ or $x(0) = A$, $x(1) = 0$ were considered, where $\alpha\colon X \to \mathbb{R}$ is linear increasing. BVPs for equation (1) with the Neumann or periodic conditions were studied in [5].

For other existence results without growth conditions, see, for example, papers [2], [3], [6] or [8].

NOTATION. For each $L_1 \le 0 \le L_2$, $F, H \in \mathcal{D}$ and any bounded set $\Omega \subset X$ we set

$$\begin{aligned}
\varrho(F;\Omega) &= \sup\{\|Fx\| :\ x \in \Omega\}\,,\\
[L_1, L_2]_X &= \{x :\ x \in X\,,\ \|x\| \le \max\{-L_1, L_2\}\}\,,\\
(L_1, L_2)_X &= \{x :\ x \in X\,,\ L_1 \le x(t) \le L_2 \ \text{for}\ t \in J\}\,,\\
[L_1, L_2; F, H]_{\mathbb{R}} &= \{(x,u,w) :\ (x,u,w) \in \mathbb{R}^3\,,\ |x| \le \max\{-L_1, L_2\}\,,\\
&\qquad |u| \le \varrho(F; [L_1, L_2]_X)\,,\ |w| \le \varrho(H; (L_1, L_2)_X)\}\,,
\end{aligned}$$

and for each $L_1, L_2, S, b \in \mathbb{R}$, $L_1 \le 0 \le L_2$, $S \ge 0$ we set

$$\begin{aligned}
&[L_1, L_2, S, b; F, H]_{\mathbb{R}} =\\
&= \Big\{(x,u,w) :\ (x,u,w) \in \mathbb{R}^3\,,\ |x| \le \max\{-L_1, L_2\} + S\,,\\
&\qquad |u| \le \varrho(F; [0, \max\{-L_1, L_2\} + S]_X)\,,\\
&\qquad |w| \le \varrho\big(H; (L_1 + b(1 - \operatorname{sign}(b)), L_2 + b(1 + \operatorname{sign}(b)))_X\big)\Big\}\,.
\end{aligned}$$

2. Lemmas

LEMMA 1. *Let $\alpha \in \mathcal{A}_J$, $\beta \in \mathcal{A}_{[0,1)}$, $A, B \in \mathbb{R}$. Then the system of nonlinear equations*

$$\alpha(a + bt^2) = A\,, \qquad \beta\big(2b(1-t)\big) = B \tag{6}$$

has a unique solution $(a_0, b_0) \in \mathbb{R}^2$.

Proof. Define the continuous functions $p\colon \mathbb{R}^2 \to \mathbb{R}$ and $q\colon \mathbb{R} \to \mathbb{R}$ by $p(a,b) = \alpha(a + bt^2)$, $q(b) = \beta\big(2b(1-t)\big)$. Since $0 < 1 - t \le 1$ on $[0,1)$, q is increasing on $\mathbb{R}$ and $\lim\limits_{b \to \mp\infty} q(b) = \mp\infty$; hence there exists a unique $b_0 \in \mathbb{R}$ such that $q(b_0) = B$. The function $p(\cdot\,, b_0)$ is increasing on $\mathbb{R}$, $\lim\limits_{a \to \mp\infty} p(a, b_0) = \mp\infty$, and therefore there exists a unique $a_0 \in \mathbb{R}$ with $p(a_0, b_0) = A$. We see that (a_0, b_0) is the unique solution of (6). □

Remark 3. Let $\alpha \in \mathcal{A}_J$, $\beta \in \mathcal{A}_{[0,1)}$. If β (resp. α) is homogeneous (resp. linear) and (cf. Lemma 1) $(a_0, b_0) \in \mathbb{R}^2$ is the unique solution of (6), then $b_0 = \dfrac{B}{2\beta(1-t)}$ (resp. $a_0 = \big(A - b_0\alpha(t^2)\big)/\alpha(1)\big)$. Thus

$$(a_0, b_0) = \left((1/\alpha(1))\left[A - \frac{B\alpha(t^2)}{2\beta(1-t)}\right], \frac{B}{2\beta(1-t)}\right)$$

is the unique solution of (6) provided β is homogeneous and α is linear.

Remark 4. Let $\alpha \in \mathcal{A}_J$, $\beta \in \mathcal{A}_{[0,1)}$, $A, B \in \mathbb{R}$. Let (a_0, b_0) be the unique solution of (6), and set $\varphi(t) = a_0 + b_0 t^2$ for $t \in J$. Then $\alpha(\varphi) = A$, $\beta\big(\varphi'(1) - \varphi'\big) = B$.

Lemma 2. *Let $u, v \in X$, $\alpha \in \mathcal{A}_J$, $\beta \in \mathcal{A}_{[0,1)}$, $c \in [0,1]$. Let*

$$\alpha(x+u) + (c-1)\alpha(-x+u) = c\alpha(u), \tag{7}$$

$$\beta\big(y(1) - y + v\big) + (c-1)\beta\big(-y(1) + y + v\big) = c\beta(v) \tag{8}$$

be satisfied for some $x, y \in X$. Then there exist $\xi \in J$ and $\eta \in [0,1)$ such that

$$x(\xi) = 0, \qquad y(1) = y(\eta).$$

P r o o f. Define $\alpha_1 \in \mathcal{A}_J$, $\beta_1 \in \mathcal{A}_{[0,1)}$ by $\alpha_1(z) = \alpha(z+u) + (c-1)\alpha(-z+u) - c\alpha(u)$, $\beta_1(z) = \beta(z+v) + (c-1)\beta(-z+v) - c\beta(v)$ for $z \in X$. Then (cf. (7), (8))

$$\alpha_1(x) = 0, \qquad \beta_1\big(y(1) - y\big) = 0. \tag{9}$$

Assume that $x(t) \neq 0$ on J, and $y(1) - y(t) \neq 0$ on $[0,1)$. Then $\alpha_1(x) \neq 0$, $\beta_1\big(y(1) - y\big) \neq 0$, which contradicts (9). □

Lemma 3. *Let $u_i, v_i \in X$ $(i = 1,2)$, $\alpha \in \mathcal{A}_J$, $\beta \in \mathcal{A}_{[0,1)}$, $\mu \in [0,\infty)$ and $A, B \in \mathbb{R}$. Then there exist unique $a, b \in \mathbb{R}$ such that the equalities*

$$\alpha(a + b\,\mathrm{e}^{-t} + u_1) - \mu\alpha(-a - b\,\mathrm{e}^{-t} + u_2) = A,$$

$$\beta\big(b(\mathrm{e}^{-t} - 1/\mathrm{e}) + v_1\big) - \mu\beta\big(-b(\mathrm{e}^{-t} - 1/\mathrm{e}) + v_2\big) = B$$

hold.

P r o o f. Define the continuous functions $p\colon \mathbb{R}^2 \to \mathbb{R}$ and $q\colon \mathbb{R} \to \mathbb{R}$ by $p(a,b) = \alpha(a + b\,\mathrm{e}^{-t} + u_1) - \mu\alpha(-a - b\,\mathrm{e}^{-t} + u_2)$, $q(b) = \beta\big(b(\mathrm{e}^{-t} - 1/\mathrm{e}) + v_1\big) - \mu\beta\big(-b(\mathrm{e}^{-t} - 1/\mathrm{e}) + u_2\big)$. Since $0 < \mathrm{e}^{-t} - 1/\mathrm{e} \leq 1 - 1/\mathrm{e}$ for $t \in [0,1)$, q is increasing on $\mathbb{R}$ and $\lim\limits_{b \to \mp\infty} q(b) = \mp\infty$. Hence, $q(b_0) = B$ for a unique $b_0 \in \mathbb{R}$. Since $p(\cdot, b_0)$ is increasing on $\mathbb{R}$, and $\lim\limits_{b \to \mp\infty} p(a, b_0) = \mp\infty$, $p(a_0, b_0) = A$ for a unique $a_0 \in \mathbb{R}$. □

Let $u, v \in X$. We shall consider BVP (1), (10), where

$$\alpha(x+u) = \alpha(u)\,, \qquad \beta\big(x'(1) - x' + v\big) = \beta(v)\,. \tag{10}$$

We assume that f fulfils the assumption

(H_1) There exist constants $L_1, L_2 \in \mathbb{R}$ such that $L_1 \le 0 \le L_2$ and

$$f(t,x,u,L_1,w) \le 0 \le f(t,x,u,L_2,w)$$

for a.e. $t \in J$ and each $(x,u,w) \in [L_1, L_2; F, H]_{\mathbb{R}}$.

To use the topological degree argument and the Borsuk theorem to prove an existence result for BVP (1), (10), we consider an auxiliary BVP defined below.

We define the function $f^* \in \mathrm{Car}(J \times \mathbb{R}^4)$ in the following way

$$f^*(t,x,u,v,w) = \begin{cases} f(t,\tilde{x},\hat{u},L_2,\bar{w}) + v - L_2 & \text{for } v > L_2\,, \\ f(t,\tilde{x},\hat{u},v,\bar{w}) & \text{for } L_1 \le v \le L_2\,, \\ f(t,\tilde{x},\hat{u},L_1,\bar{w}) + v - L_1 & \text{for } v < L_1\,, \end{cases} \tag{11}$$

where $(L = \max\{-L_1, L_2\})$

$$\tilde{x} = \begin{cases} L & \text{for } x > L\,, \\ x & \text{for } |x| \le L\,, \\ -L & \text{for } x < -L\,, \end{cases}$$

$$\tilde{u} = \begin{cases} u & \text{for } |u| \le \varrho\big(F; [L_1, L_2]_X\big)\,, \\ \varrho\big(F; [L_1, L_2]_X\big)\operatorname{sign}(u) & \text{for } |u| > \varrho\big(F; [L_1, L_2]_X\big)\,, \end{cases}$$

$$\bar{w} = \begin{cases} w & \text{for } |w| \le \varrho\big(H; (L_1, L_2)_X\big)\,, \\ \varrho\big(H; (L_1, L_2)_X\big)\operatorname{sign}(w) & \text{for } |w| > \varrho\big(H; (L_1, L_2)_X\big)\,. \end{cases}$$

Consider the auxiliary BVP (12), (10), where

$$x''(t) = f^*\big(t, x(t), (Fx)(t), x'(t), (Hx')(t)\big)\,. \tag{12}$$

Let $L_1(J)$ (resp. $AC^1(J)$) be the Banach space of Lebesgue integrable functions on J (resp. the Banach space of functions with an absolutely continuous derivative on J) with the norm

$$\|x\|_{L_1} = \int_0^1 |x(s)|\,\mathrm{d}s \qquad \big(\text{resp. } \|x\|_{AC^1} = \|x\| + \|x'\| + \|x''\|_{L_1}\,\big)\,.$$

Define the operators

$$U, S, V\colon AC^1(J) \to L_1(J) \times \mathbb{R}^2$$

by

$$U\colon x \mapsto \Big(x''(\cdot)+x'(\cdot), \alpha(x+u)-\alpha(-x+u),$$
$$\beta\big(x'(1)-x'+v\big)-\beta\big(-x'(1)+x'+v\big)\Big),$$
$$S\colon x \mapsto \Big(f^*\big(\cdot, x(\cdot), (Fx)(\cdot), x'(\cdot), (Hx')(\cdot)\big),$$
$$\alpha(u)-\alpha(-x+u), \beta(v)-\beta\big(-x'(1)+x'+v\big)\Big),$$
$$V\colon x \mapsto \big(x'(\cdot), 0, 0\big),$$

and consider the operator equation

$$U(x) = c(S+V)(x) + 2(1-c)V(x), \qquad c \in [0,1]. \tag{13_c}$$

The operator equation (13_c) is equivalent to BVP

$$x''(t) = cf^*\big(t, x(t), (Fx)(t), x'(t), (Hx')(t)\big) + (1-c)x'(t), \tag{14_c}$$

$$\begin{aligned} \alpha(x+u)+(c-1)\alpha(-x+u) &= c\alpha(u), \\ \beta\big(x'(1)-x'+v\big)+(c-1)\beta\big(-x'(1)+x'+v\big) &= c\beta(v), \end{aligned} \tag{15_c}$$

$$c \in [0,1],$$

and we see that x is a solution of BVP (12), (10) if and only if that is a solution of (13_1).

Lemma 4. *The inverse operator $U^{-1}\colon L_1(J)\times\mathbb{R}^2 \to AC^1(J)$ of U exists, and furthermore, U^{-1} is continuous and odd.*

Proof. Let $(z, A, B) \in AC^1(J)\times\mathbb{R}^2$. Consider the operator equation

$$U(x) = (z, A, B), \tag{16}$$

that is, the equations

$$x''+x' = z(t), \tag{17'}$$

$$\begin{aligned} \alpha(x+u)-\alpha(-x+u) &= A, \\ \beta\big(x'(1)-x'+v\big)-\beta\big(-x'(1)+x'+v\big) &= B. \end{aligned} \tag{17''}$$

Since $x(t) = a + b\,\mathrm{e}^{-t} + w(t)$ is the general solution of $(17')$, where a, b are integration constants and $w(t) = \int\limits_0^t\int\limits_0^s \mathrm{e}^{\tau-s} z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$, by Lemma 3 (with $u_1 = u+w$, $u_2 = u-w$, $v_1 = w'(1)-w'+v$, $v_2 = -w'(1)+w'+v$, $\mu = 1$), there exist unique $a_0, b_0 \in \mathbb{R}$ such that $a_0 + b_0 \,\mathrm{e}^{-t} + w(t)$ is the unique solution of (16). Hence, $U^{-1}\colon L_1(J) \times \mathbb{R}^2 \to AC^1(J)$ exists.

If $x \in AC^1(J)$ is a solution of (16), then it follows from (17′) and (17″) that $U(-x) = -(z, A, B) = -U(x)$, which proves that U is odd, and thus U^{-1} is odd as well.

To prove the continuity of U^{-1}, let $\{z_n, A_n, B_n\} \subset L_1(J) \times \mathbb{R}^2$ be a convergent sequence $\{z_n, A_n, B_n\} \to \{z, A, B\}$ as $n \to \infty$. Let us set $x_n = U^{-1}(z_n, A_n, B_n)$ $(n \in \mathbb{N})$ and $x = U^{-1}(z, A, B)$. Then there exist the sequences $\{a_n\}, \{b_n\} \subset \mathbb{R}$ and $a, b \in \mathbb{R}$ such that

$$x_n(t) = a_n + b_n \,\mathrm{e}^{-t} + w_n(t)\,, \qquad x(t) = a + b\,\mathrm{e}^{-t} + w(t)\,,$$

where

$$w_n(t) = \int_0^t \int_0^s \mathrm{e}^{\tau - s} z_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\,, \qquad w(t) = \int_0^t \int_0^s \mathrm{e}^{\tau - s} z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\,,$$

and the equalities

$$\alpha(x_n + u) - \alpha(-x_n + u) = A_n\,, \qquad n \in \mathbb{N}, \tag{18′}$$

$$\beta\big(x_n'(1) - x_n' + v\big) - \beta\big(-x_n'(1) + x_n' + v\big) = B_n\,, \qquad n \in \mathbb{N}, \tag{18″}$$

and (17″) hold. Since $\lim\limits_{n\to\infty} w_n = w$ in $AC^1(J)$, $\{b_n\}$ is bounded by (18″), and $\{a_n\}$ is bounded by (18′). Assume, on the contrary, that $\{b_n\}$ is not convergent. Then there exist convergent subsequences $\{b_{k_n}\}$ and $\{b_{l_n}\}$, $\lim\limits_{n\to\infty} b_{k_n} = b^*$, $\lim\limits_{n\to\infty} b_{l_n} = \tilde{b}$, $b^* \neq \tilde{b}$. Taking the limits in the equalities (cf. (18″))

$$\beta\big(b_n(\mathrm{e}^{-t} - 1/\mathrm{e}) + w_n'(1) - w_n' + v\big) - \beta\big(-b_n(\mathrm{e}^{-t} - 1/\mathrm{e}) - w_n'(1) + w_n' + v\big) = B_n$$

as $k_n \to \infty$ and $l_n \to \infty$, we obtain

$$\beta\big(b^*(\mathrm{e}^{-t} - 1/\mathrm{e}) + w'(1) - w' + v\big) - \beta\big(-b^*(\mathrm{e}^{-t} - 1/\mathrm{e}) - w'(1) + w' + v\big) = B$$

and

$$\beta\big(\tilde{b}(\mathrm{e}^{-t} - 1/\mathrm{e}) + w'(1) - w' + v\big) - \beta\big(-\tilde{b}(\mathrm{e}^{-t} - 1/\mathrm{e}) - w'(1) + w' + v\big) = B$$

respectively. Since $\beta \in \mathcal{A}_{[0,1)}$, $b^* = \tilde{b}$, hence $\{b_n\}$ is convergent, and then (cf. the second equality in (17″)) $\lim\limits_{n\to\infty} b_n = b$. Similarly, $\{a_n\}$ is convergent and $\lim\limits_{n\to\infty} a_n = a$. Hence, $\lim\limits_{n\to\infty} x_n = x$ in $AC^1(J)$, and consequently, U^{-1} is continuous. □

LEMMA 5. *Let* $\Omega = \{x: \ x \in AC^1(J), \ \|x\| \le K, \ \|x'\| \le K, \ |x''(t)| \le q(t)$ *for a.e.* $t \in J\}$, *where* $q \in L_1(J)$, *and* K *is a positive constant. Then the operator*

$$U^{-1}\big(cS + (2-c)V\big) \colon \Omega \to AC^1(J)$$

is compact for each $c \in [0,1]$.

Proof. Fix $c \in [0,1]$. We first show that the operator $cS + (2-c)V\colon \Omega \to L_1(J) \times \mathbb{R}^2$ is compact. Let $\{x_n\} \subset \Omega$. Then the sequences $\{\|x_n\|\}$ and $\{\|x'_n\|\}$ are bounded, and, moreover, $\{x'_n(t)\}$ is equicontinuous on J since $|x''_n(t)| \le q(t)$ for a.e. $t \in J$ and each $n \in \mathbb{N}$. Applying the Arzelà-Ascoli theorem there exist an $x \in C^1(J)$ and a subsequence of $\{x_n\}$, which we denote by $\{x_n\}$ again, such that $\lim\limits_{n\to\infty} x_n^{(i)} = x^{(i)}$ in X for $i = 0, 1$. Since $F, H \in \mathcal{D}$, α and β are continuous, and $f \in \mathrm{Car}(J \times \mathbb{R}^4)$, $\lim\limits_{n\to\infty} Fx_n = Fx$, $\lim\limits_{n\to\infty} Hx'_n = Hx'$ in X, $\lim\limits_{n\to\infty}\big(\alpha(u) - \alpha(-x_n + u)\big) = \alpha(u) - \alpha(-x+u)$, $\lim\limits_{n\to\infty}\big(\beta(v) - \beta(-x'_n(1) + x'_n + v)\big) = \beta(v) - \beta\big(-x'(1) + x' + v\big)$ in $\mathbb{R}$, and there exists an $h \in L_1(J)$ such that $\big|f^*\big(t, x_n(t), (Fx_n)(t), x'_n(t), (Hx'_n)(t)\big)\big| \le h(t)$ for a.e. $t \in J$ and each $n \in \mathbb{N}$. Consequently, an application of the Lebesgue dominated convergence theorem shows that

$$\lim_{n\to\infty} f^*\big(t, x_n(t), (Fx_n)(t), x'_n(t), (Hx'_n)(t)\big) = f^*\big(t, x(t), (Fx)(t), x'(t), (Hx')(t)\big)$$

in $L_1(J)$. Hence, $\lim\limits_{n\to\infty} S(x_n) = S(x)$, $\lim\limits_{n\to\infty} V(x_n) = V(x)$ in $L_1(J) \times \mathbb{R}^2$, and since S and V are continuous in Ω, the operator $cS + (2-c)V\colon \Omega \to L_1(J) \times \mathbb{R}^2$ is compact. Now, by Lemma 4, $U^{-1}\colon L_1(J) \times \mathbb{R}^2 \to AC^1(J)$ is continuous, and consequently, $U^{-1}\big(cS + (2-c)V\big)\colon \Omega \to AC^1(J)$ is compact. □

3. Existence theorems

LEMMA 6. *Let* f *satisfy* (H_1) *and*

$$|f(t, x, u, v, w)| \le p(t)$$

for a.e. $t \in J$ *and each* $(x, u, w) \in [L_1, L_2; F, H]_{\mathbb{R}}$, $|v| \le L$ *with a* $p \in L_1(J)$ *and* $L = \max\{-L_1, L_2\}$. *Let* x *be a solution of BVP* (14_c), (15_c) *for a* $c \in [0,1]$. *Then*

$$\begin{gathered} \|x\| \le L, \qquad \|x'\| \le L, \\ |x''(t)| \le p(t) + L + \max\{L - L_2, L + L_1\} \qquad \text{for a.e. } t \in J. \end{gathered} \tag{19}$$

Proof. Fix $n \in \mathbb{N}$. Assume $\max\{x'(t);\ t \in J\} = x'(t_0) \geq L + 1/n$ with a $t_0 \in J$. By Lemma 2, we can assume $t_0 \in [0,1)$. Then there exists an $\varepsilon > 0$ such that $x'(t) > L$ on the interval $[t_0, t_0 + \varepsilon] \subset J$, thus (cf. (H_1) and (11))

$$x'(t_0+\varepsilon)-x'(t_0) = \int_{t_0}^{t_0+\varepsilon} x''(t)\ \mathrm{d}t \geq c \int_{t_0}^{t_0+\varepsilon} \bigl(x'(t)-L_2\bigr)\ \mathrm{d}t+(1-c) \int_{t_0}^{t_0+\varepsilon} x'(t)\ \mathrm{d}t > 0\,,$$

which is a contradiction. Similarly for $\min\{x'(t);\ t \in J\} < -L - 1/n$. Hence, $\|x'\| \leq L+1/n$ for each $n \in \mathbb{N}$, and consequently, $\|x'\| \leq L$. Since (cf. Lemma 2) $x(\xi) = 0$ for a $\xi \in J$, we see $\|x\| \leq L$. Then (cf. (11))

$$\begin{aligned}|x''(t)| &\leq c\bigl|f^*\bigl(t, x(t), (Fx)(t), x'(t), (Hx')(t)\bigr)\bigr| + (1-c)|x'(t)| \\ &\leq |p(t)| + \max\{L - L_2, L + L_1\} + L \qquad \text{for a.e.} \quad t \in J\,.\end{aligned}$$

□

Lemma 7. *Let f satisfy* (H_1). *Then BVP* (12), (10) *has a solution x satisfying*

$$\|x\| \leq L\,, \qquad \|x'\| \leq L\,, \tag{20}$$

where $L = \max\{-L_1, L_2\}$.

Proof. Let $p \in L_1(J)$ be as in Lemma 6, and let ε be a positive constant. Set

$$\begin{aligned}\Omega_\varepsilon = \bigl\{x :\ & x \in AC^1(J)\,,\ \|x\| < L + \varepsilon\,,\ \|x'\| < L + \varepsilon\,, \\ & |x''(t)| < p(t) + L + \max\{L - L_2, L + L_1\} + \varepsilon \ \text{ for } \ a.e.\ t \in J\bigr\}\,.\end{aligned}$$

Then $\Omega_\varepsilon \subset AC^1(J)$ is an open bounded subset of $AC^1(J)$ and symmetric with respect to $0 \in \Omega_\varepsilon$. Set $W(c,x) = \bigl(U^{-1}(cS+(2-c)V)\bigr)(x)$ for $(c,x) \in [0,1]\times\bar{\Omega}_\varepsilon$. By Lemma 5 (with $\Omega = \bar{\Omega}_\varepsilon$, $K = L+\varepsilon$, $q(t) = p(t)+L+\max\{L-L_2, L+L_1\}+\varepsilon$) and the Bolzano-Weierstrass theorem, $W(c,x)$ is compact and $W(c,x) \neq x$ for each $(c,x) \in [0,1] \times \partial\Omega_\varepsilon$ by Lemma 6. Hence, (see, e.g., [1]) $D\bigl(I - U^{-1}(S+V), \Omega_\varepsilon, 0\bigr) = D\bigl(I - U^{-1}(2V), \Omega_\varepsilon, 0\bigr)$, where "$D$" denotes the Leray-Schauder degree. Since U^{-1} is odd and V is linear, $D\bigl(I - U^{-1}(2V), \Omega_\varepsilon, 0\bigr) \neq 0$ by the Borsuk theorem. Hence the operator equation $x = \bigl(U^{-1}(S+V)\bigr)(x)$ has a solution x in $\bar{\Omega}_\varepsilon$, and, by Lemma 6, x satisfies (20), which proves our lemma. □

Theorem 1. *Let f satisfy* (H_1). *Then BVP* (1), (10) *has a solution x satisfying*

$$\|x\| \leq L\,, \qquad L_1 \leq x'(t) \leq L_2 \quad \textit{for} \ \ t \in J\,, \tag{21}$$

where $L = \max\{-L_1, L_2\}$.

P r o o f . By Lemma 7, there exists a solution of BVP (12), (10) satisfying (20). To prove our theorem, it is enough to show that (cf. (11))

$$L_1 \le x'(t) \le L_2 \qquad \text{for} \quad t \in J\,.$$

Assume $L_2 < L$ and $L_2 < \max\{x'(t);\ t \in J\} = x'(t_0)$ for a $t_0 \in [0,1)$ (see Lemma 2). Then there exists $\tau > 0$ such that $x'(t) > L_2$ for $t \in [t_0, t_0+\tau] \subset J$; hence, (cf. (H_1))

$$\begin{aligned} x'(t_0+\tau) - x'(t_0) &= \int\limits_{t_0}^{t_0+\tau} x''(t)\,\mathrm{d}t \\ &= \int\limits_{t_0}^{t_0+\tau} \Big(f^*\big(t, x(t), (Fx)(t), x'(t), (Hx')(t)\big) + x'(t) - L_2\Big)\,\mathrm{d}t \\ &\ge \int\limits_{t_0}^{t_0+\tau} \big(x'(t) - L_2\big)\,\mathrm{d}t > 0\,, \end{aligned}$$

which is a contradiction. Similarly, for $\min\{x'(t);\ t \in J\} < L_1\ (> -L)$. $\square$

COROLLARY 1. *Let L_1, L_2 be constants such that $L_1 \le 0 \le L_2$, and*

$$g(t, x, L_1) \le 0 \le g(t, x, L_2)$$

for a.e. $t \in J$ and each $x \in [-L, L]$, where $L = \max\{-L_1, L_2\}$. Then BVP (3), (10) *has a solution x satisfying* (21).

THEOREM 2. *Let $A, B \in \mathbb{R}$ and $(a_0, b_0) \in \mathbb{R}^2$ be the unique solution of* (6). *Let there exist constants $L_1, L_2 \in \mathbb{R}$ such that $L_1 \le 0 \le L_2$ and*

$$f(t, x, u, L_1 + 2b_0 t, w) \le 2b_0 \le f(t, x, u, L_2 + 2b_0 t, w) \tag{22}$$

for a.e. $t \in J$ and each $(x, u, w) \in [L_1, L_2, S, b_0; F, H]_{\mathbb{R}}$ with $S = \max\{|a_0|, |a_0 + b_0|\}$.

Then BVP (1), (2) *has a solution x satisfying*

$$\|x\| \le S + \max\{-L_1, L_2\}\,, \qquad L_1 + 2b_0 t \le x'(t) \le L_2 + 2b_0 t \quad \textit{for} \ \ t \in J\,. \tag{23}$$

P r o o f . Set $\varphi(t) = a_0 + b_0 t^2$ for $t \in J$. Then $\|\varphi\| = S$, and, by Remark 4, $\alpha(\varphi) = A$, $\beta(\varphi'(1) - \varphi') = B$. Define the operators $F^*, H^* : X \to X$ by $F^*(x) = F(x + \varphi)$, $H^*(x) = H(x + \varphi')$. Then $F^*, H^* \in \mathcal{D}$ and

$$\begin{aligned} &\varrho\big(F^*; [L_1, L_2]_X\big) \le \varrho\big(F; [L_1 - S, L_2 + S]_X\big)\,, \\ &\varrho\big(H^*; (L_1, L_2)_X\big) \le \varrho\Big(H; \big(L_1 + b_0\big(1 - \operatorname{sign}(b_0)\big), L_2 + b_0\big(1 + \operatorname{sign}(b_0)\big)\big)_X\Big)\,. \end{aligned} \tag{24}$$

Define the function $h \in \mathrm{Car}(J \times \mathbb{R}^4)$ by

$$h(t,x,u,v,w) = f\big(t, x+\varphi(t), u, v+\varphi'(t), w\big) - 2b_0\,,$$

and consider BVP

$$x''(t) = h\big(t, x(t), (F^*x)(t), x'(t), (H^*x')(t)\big)\,, \tag{25}$$

$$\alpha(x+\varphi) = \alpha(\varphi)\,, \qquad \beta\big(x'(1) - x' + \varphi'(1) - \varphi'\big) = \beta(\varphi'(1) - \varphi')\,. \tag{26}$$

We see that $x = x_1 + \varphi$ is a solution of BVP (1), (2) satisfying (23) if and only if x_1 is a solution of BVP (25), (26) satisfying (21) (with $x = x_1$). Applying (22) and (24) we obtain

$$h(t,x,u,L_1,w) \le 0 \le h(t,x,u,L_2,w)$$

for a.e. $t \in J$ and each $(x,u,w) \in [L_1, L_2; F^*, H^*]_{\mathbb{R}}$. So, by Theorem 1 (with $f = h$, $F = F^*$, $H = H^*$, $u = \varphi$, $v = \varphi'(1) - \varphi'$), there is a solution x of BVP (25), (26) satisfying (21). This completes the proof. □

COROLLARY 2. *Let $A, B \in \mathbb{R}$ and $(a_0, b_0) \in \mathbb{R}^2$ be the unique solution of* (6). *Let there exist constants $L_1, L_2 \in \mathbb{R}$ such that $L_1 \le 0 \le L_2$ and*

$$g(t, x, L_1 + 2b_0 t) \le 2b_0 \le g(t, x, L_2 + 2b_0 t)$$

for a.e. $t \in J$ and each $|x| \le S + \max\{-L_1, L_2\}$ with $S = \max\big\{|a_0|, |a_0 + b_0|\big\}$. Then BVP (3), (2) *has a solution x satisfying* (23).

EXAMPLE 1. Let $\xi \in J$ and $\tau \ge 0$, $\varepsilon \ge 0$ be constants. Consider BVP

$$x''(t) = t + x(\xi) + \big(2 + \tau\big(x'(t^2)\big)^2 + \varepsilon|x(t)|\big)x'(t)\,, \tag{27}$$

$$\max\big\{x(t);\ t \in J\big\} = A\,, \quad x'(1) = B + x(1) - x(0)\,. \tag{28}$$

If we set $(Fx)(t) = x(\xi)$, $(Hx)(t) = x(t^2)$, $\alpha(x) = \max\big\{x(t);\ t \in J\big\}$ and $\beta(x) = \int_0^1 x(t)\,\mathrm{d}t$ for $x \in X$, we see that BVP (27), (28) is the special case of BVP (1), (2) with $f(t,x,u,v,w) = t + u + \big(2 + \tau w^2 + \varepsilon|x|\big)v$. One can verify that system (6) has the unique solution $(a_0, b_0) = \big(A - \frac{1}{2}B(1 + \operatorname{sign} B), B\big)$. Assume $0 \le A \le B$. Then (27) satisfies the assumptions of Theorem 2 with $-L_1 = L_2 = 3B + 1$, $S \le B$, $b_0 = B$. Hence for each $0 \le A \le B$, $\tau \ge 0$ and $\varepsilon \ge 0$, BVP (27), (28) has a solution x satisfying

$$\|x\| \le 4B + 1\,, \qquad -3B - 1 + 2Bt \le x'(t) \le 3B + 1 + 2Bt \quad \text{for } t \in J\,.$$

4. Applications

In this section, we give some applications of the above results to BVPs for third and fourth order differential equations.

Consider BVP

$$x^{(4)} = p(t, x, x', x'', x''') \,, \tag{29}$$

$$\alpha(x) = 0 \,, \qquad \beta(x') = 0 \,, \qquad \gamma(x'') = 0 \,, \qquad \delta\big(x'''(1) - x'''\big) = 0 \,, \tag{30}$$

where $p \in \mathrm{Car}(J \times \mathbb{R}^4)$ and $\alpha, \beta, \gamma \in \mathcal{A}_J$, $\delta \in \mathcal{A}_{[0,1)}$.

THEOREM 3. *Let $L_1 \le 0 \le L_2$ be constants such that the inequalities*

$$p(t, x, u, v, L_1) \le 0 \le p(t, x, u, v, L_2)$$

are satisfied for a.e. $t \in J$ and each $(x, u, v) \in [-L/2, L/2] \times [-L/2, L/2] \times [-L, L]$, $L = \max\{-L_1, L_2\}$. Then BVP (29), (30) has at least one solution x satisfying

$$\|x\| \le L/2 \,, \qquad \|x'\| \le L/2 \,, \qquad \|x''\| \le L \,, \qquad L_1 \le x'''(t) \le L_2 \quad \text{for } t \in J \,. \tag{31}$$

P r o o f. Let $u \in X$. By [4; Lemma 6], BVPs

$$x'' = u(t) \,, \qquad \alpha(x) = 0 \,, \qquad \beta(x') = 0 \tag{32}$$

and

$$x'' = u(t) \,, \qquad \beta(x) = 0 \,, \qquad \gamma(x') = 0 \tag{33}$$

have a unique solution x_1 and x_2, respectively. Moreover,

$$\|x_j\| \le \|u\|/2 \,, \quad \|x_j'\| \le \|u\| \,, \qquad j = 1, 2 \,. \tag{34}$$

Define the operators $F, H \colon X \to X$ by

$$F(u) = x_1 \,, \qquad H(u) = x_2 \,,$$

where x_1 and x_2 is the unique solution of BVP (32) and (33), respectively. Then $F, H \in \mathcal{D}$ (see the proof of Theorem 7 in [4]) and $\varrho\big(F; [L_1, L_2]_X\big) \le L/2$, $\varrho\big(H; [L_1, L_2]_X\big) \le L/2$. By the substitution $u = x''$, BVP (29), (30) can be written as

$$u''(t) = p\big(t, (Fu)(t), (Hu')(t), u(t), u'(t)\big) \,, \qquad \gamma(u) = 0 \,, \qquad \delta\big(u'(1) - u'\big) = 0 \,. \tag{35}$$

Set $f(t, x, u, v, w) = p(t, u, w, x, v)$ for $(t, x, u, v, w) \in J \times \mathbb{R}^4$. Then f satisfies the assumptions of Theorem 1 (with $u = v = 0$ in (10)), and consequently, there exists a solution u of BVP (35) such that $\|u\| \le L$, $L_1 \le u'(t) \le L_2$ for $t \in J$.

Obviously, there exists a unique $x \in AC^3(J)$ satisfying $\alpha(x) = 0$, $\beta(x') = 0$ and $x''(t) = u(t)$ on J. This function x is a solution of BVP (29), (30) for which (31) holds. $\square$

Similarly, for BVP

$$x''' = q(t, x, x', x''), \tag{36}$$

$$\alpha(x) = 0, \qquad \beta(x') = 0, \qquad \gamma\big(x''(1) - x''\big) = 0, \tag{37}$$

where $q \in \mathrm{Car}(J \times \mathbb{R}^3)$, $\alpha, \beta \in \mathcal{A}_J$ and $\gamma \in \mathcal{A}_{[0,1)}$, we can prove the following theorem.

THEOREM 4. *Let $L_1 \le 0 \le L_2$ be constants such that the inequalities*

$$q(t, x, u, L_1) \le 0 \le q(t, x, u, L_2)$$

are satisfied for a.e. $t \in J$ and each $(x, u) \in [-L, L] \times [-L, L]$, $L = \max\{-L_1, L_2\}$. Then BVP (36), (37) has a solution x satisfying

$$\|x\| \le L, \qquad \|x'\| \le L, \qquad L_1 \le x''(t) \le L_2 \quad \textit{for } t \in J.$$

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Department of Mathematical Analysis
Faculty of Science
Palacký University
Tomkova 40
CZ–779 06 Olomouc
Czech Republic