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OSCILLATION OF SOLUTIONS TO NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Our aim in this paper is to give some new sufficient conditions for oscillation of all solutions of first order neutral delay differential equations. Our results extend and improve some well-known results in the literature. We discuss a number of carefully chosen examples which clarify the relevance of our results.

1. Introduction

In recent years the literature on the oscillation theory of neutral delay differential equations is growing very fast. It is relatively a new field with interesting applications in real world life problems. In fact, the neutral delay differential equations appear in modelling of the networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, theory of automatic control and in neuromechanical systems in which inertia plays an important role (see Hale [16], Driver [8], Brayton and Willoughby [5], Popov [23], and Boe and Chang [4] and the references cited therein).

In this paper we shall consider the following first order neutral delay differential equation with variable coefficients,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t) - P(t)x(t-\tau)\big] + Q(t)x(t-\sigma) = 0\,, \qquad t \geqslant t_0\,, \tag{1.1}$$

where

$$P, Q \in C\big([t_0,\infty), \mathbb{R}^+\big)\,, \quad \sigma, \tau \in [0,\infty)\,. \tag{1.2}$$

By a *solution* of equation (1.1) on $[t_1,\infty)$ we mean a function $x \in C\big([t_0-\rho,\infty), \mathbb{R}\big)$ such that $x(t) - P(t)x(t-\tau)$ is continuously differentiable on $[t_0,\infty)$ and such that equation (1.1) is satisfied for $t \geqslant t_0$, where $\rho = \max\{\sigma,\tau\}$.

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Let $t_1 \geqslant t_0$ be a given point and let

$$\phi \in C([t_1-\rho, t_1], \mathbb{R}) \tag{1.3}$$

be a given initial function such that

$$x(t) = \phi(t) \qquad \text{for} \quad t \in [t_1-\rho, t_1]. \tag{1.4}$$

The initial value problem (1.1), (1.4) has a unique positive solution for all $t \geq t_1$. This follows rather easily by the method of steps.

As usual, we say that equation (1.1) is *oscillatory* if every solution of (1.1) is oscillatory, i.e., for every initial point $t_1 \geqslant t_0$ and for every initial function $\phi \in C([t_1-\rho, t_1], \mathbb{R})$ the unique solution of (1.1) and (1.4) has arbitrarily large zeros. Otherwise the solution is called *non-oscillatory.* The oscillation of various functional differential equations has been investigated by several authors. For some contributions we refer to the monographs [1], [2], [3], [11], [15], [20].

The first systematic work about oscillation of neutral delay differential equations is given by Zahariev and Bainov [26]. For the oscillation of equation (1.1) when $P(t)$ and $Q(t)$ are constants, we refer to the articles by Ladas and Sficas [19], Grammatikopoulos et al. [12] and Zhang [27] and the references cited therein. When $P(t)$ is a constant we refer to the articles by Grammatikopoulos, Grove and Ladas [14] and Zhang [27] and the references cited therein. For the oscillation of equation (1.1) we refer to the following results. Grammatikopoulos et al. [13] considered the delay differential equation (1.1) and presented some finite sufficient conditions for oscillation of all solutions when $P(t)$ takes some values in the interval $[0,1]$. Chuanxi and Ladas [7] considered the neutral delay differential equation (1.1) and established new sufficient conditions for oscillation of all solutions under less restrictive hypotheses on $P(t)$. All the above mentioned papers give the oscillation conditions for equation (1.1) when $P(t) \leq 1$ and

$$\int_{t_0}^{\infty} Q(s) \, \mathrm{d}s = \infty. \tag{1.5}$$

Yu, Wang and Chuanxi [25] considered equation (1.1) when $P(t) \equiv 1$ and relaxed condition (1.5) to the condition

$$\int_{t_0}^{\infty} sQ(s) \int_{s}^{\infty} Q(u) \, \mathrm{d}u \, \mathrm{d}s = \infty. \tag{1.6}$$

In fact, Chen, Yu and Huang [6] observed that for (1.1), it is sufficient to have a point t^* so that

$$P(t^* + k\tau) \leq 1, \qquad k = 0, 1, 2, \dots \tag{1.7}$$

without the assumptions (1.5) and (1.6). They compared the oscillation of (1.1) with the absence of positive solutions of the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[y(t) - y(t-\tau)\big] + Q(t)y(t-\sigma) \le 0\,, \qquad t \geqslant t_0\,, \tag{1.8}$$

under the condition

$$P(t-\sigma)Q(t) \le Q(t-\tau)\,. \tag{1.9}$$

However, most of the results in the literature involve conditions depending separately on Q sufficient for oscillation of (1.1) and conditions on P which allow extensions of arguments used in the case where $P(t) \equiv 1$ as in Yu, Wang and Chuanxi [25].

The purpose of this paper is to give some new oscillation criteria of equation (1.1). The paper is organized as follows: In the next section, we present some new sets of sufficient conditions which guarantee oscillation of all proper solutions of the delay differential equation (1.1), and show that the combined growth of P and Q without condition (1.9) can give oscillation criteria for equation (1.1) even when (1.5) and (1.6) fail. In Section 3, we discuss a number of carefully chosen examples which clarify the relevance of our results.

In the sequel, when we write a functional inequality we will assume that it holds for all sufficient large values of t.

Before stating our main results we need the following lemma.

LEMMA 1.1. ([6]) *Assume that* (1.2) *holds,*

$$P(t) > 0 \qquad \text{and} \qquad P(t^* + i\tau) \le 1 \quad \text{for } i = 0,1,2,\dots\,. \tag{1.10}$$

Let $x(t)$ be an eventually positive solution of equation (1.1)*, and set*

$$y(t) = x(t) - P(t)x(t-\tau)\,. \tag{1.11}$$

Then

$$y(t) > 0\,. \tag{1.12}$$

2. Main results

In this section, we will establish several new sharp sufficient conditions for the oscillation of all solutions of equation (1.1).

THEOREM 2.1. *Assume that* (1.2) *and* (1.10) *hold, then either*

$$\liminf_{t\to\infty} \int\limits_{t-\sigma}^{t} Q(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{2.1}$$

or

$$\limsup_{t\to\infty}\int\limits_{t-\sigma}^{t} Q(s)\,\mathrm{d}s > 1 \tag{2.2}$$

implies that every solution of equation (1.1) *oscillates.*

Proof. Without loss of generality, we assume that equation (1.1) has an eventually positive solution $x(t)$. Let $y(t)$ be defined by (1.11), then from (1.1) we have

$$\begin{aligned} y'(t) &= -Q(t)x(t-\sigma) \\ &= -Q(t)\big[y(t-\sigma) + P(t-\sigma)x(t-\tau-\sigma)\big] \\ &= -Q(t)y(t-\sigma) - Q(t)P(t-\sigma)x(t-\tau-\sigma) \\ &= -Q(t)y(t-\sigma) + \frac{Q(t)}{Q(t-\tau)}P(t-\sigma)y'(t-\tau)\,. \end{aligned}$$

Hence,

$$y'(t) - \frac{Q(t)}{Q(t-\tau)}P(t-\sigma)y'(t-\tau) + Q(t)y(t-\sigma) = 0\,. \tag{2.3}$$

By Lemma 1.1, we get

$$y(t) > 0\,, \qquad t \geqslant t_1 \geqslant t_0\,. \tag{2.4}$$

Set

$$\lambda(t) = -\frac{y'(t)}{y(t)}\,. \tag{2.5}$$

Then, (2.3) reduces to

$$\lambda(t) = \lambda(t-\tau)\frac{Q(t)}{Q(t-\tau)}P(t-\sigma)\exp\left(\int\limits_{t-\tau}^{t}\lambda(s)\,\mathrm{d}s\right) + Q(t)\exp\left(\int\limits_{t-\sigma}^{t}\lambda(s)\,\mathrm{d}s\right). \tag{2.6}$$

It is obvious that $\lambda(t) > 0$ for $t \geqslant t_0$. From (2.6) it is clear that

$$\lambda(t) \geqslant Q(t)\exp\left(\int\limits_{t-\sigma}^{t}\lambda(s)\,\mathrm{d}s\right). \tag{2.7}$$

Then, from (2.5) and (2.7), one can see that $y(t)$ is positive solution of the delay differential inequality

$$y'(t) + Q(t)y(t-\sigma) \le 0\,.$$

Then, by [15; Corollary 3.2.2], the delay differential equation

$$y'(t) + Q(t)y(t-\sigma) = 0 \tag{2.8}$$

has an eventually positive solution as well. It is well known that (2.1) or (2.2) implies that (2.8) has no eventually positive solution (see, for example, [15; p. 46, Theorem 2.3.3] and [15; p. 78, Theorem 3.4.3]). This is a contradiction and so the proof is complete. □

Remark 2.1. It is clear that there exist a gap between conditions (2.1) and (2.2) for the oscillation of all solution of equation (1.1). The problem how to fill a gap for equation (2.8) when the limit

$$\lim_{t\to\infty} \int_{t-\sigma}^{t} Q(s)\,\mathrm{d}s \tag{2.9}$$

does not exist has been recently investigated by several authors [10], [18], [17], [9], [21], [24] and [22]. In view of the respective works presented in [10], [18], [17], [9], [21], [24] and [22] and the fact that every solution of equation (1.1) oscillates when (2.8) has no eventually positive solution one can give several sufficient conditions for oscillation. The details are left to the reader.

In the following theorem we present new infinite integral sufficient condition for oscillation of all solutions of equation (1.1) which improve conditions (1.5) and (1.6).

THEOREM 2.2. *Assume that* (1.2) *and* (1.10) *hold, and*

$$0 < d \le \liminf_{t\to\infty} \int_{t}^{t+\sigma} Q(s)\,\mathrm{d}s\,, \tag{2.10}$$

$$\int_{t_0}^{\infty} Q(t)\left[\ln \int_{t}^{t+\sigma} Q(s)\,\mathrm{d}s + 1\right]\mathrm{d}t = \infty\,. \tag{2.11}$$

Then every solution of equation (1.1) *oscillates.*

P r o o f. Without loss of generality, we assume that equation (1.1) has an eventually positive solution $x(t)$. Let $y(t)$ be defined by (2.1), then from Theorem 2.1, $y(t)$ is positive solution of the delay differential equation

$$y'(t) + Q(t)y(t-\sigma) = 0\,. \tag{2.12}$$

As $\lambda(t) = -y'(t)/y(t)$, then $\lambda(t)$ is non-negative and continuous, and there exists $t_1 \ge t_0$ such that $y(t_1) > 0$, and $y(t) = y(t_1)\exp\left(-\int_{t_1}^{t} \lambda(s)\,\mathrm{d}s\right)$. Furthermore $\lambda(t)$ satisfies the generalized characteristic equation

$$\lambda(t) = Q(t)\exp\left(\int_{t-\sigma}^{t} \lambda(s)\,\mathrm{d}s\right),$$

hence,

$$\lambda(t) = Q(t) \exp\left[\frac{1}{A(t)} A(t) \int_{t-\sigma}^{t} \lambda(s)\, \mathrm{d}s\right] \tag{2.13}$$

where $A(t) = \int_{t}^{t+\sigma} Q(s)\, \mathrm{d}s$. By using the inequality (cf. [11; p. 32])

$$\mathrm{e}^{rx} \geq x + \frac{\ln r + 1}{r} \qquad \text{for all} \quad x, r > 0\,, \tag{2.14}$$

we have from (2.13) that

$$A(t)\lambda(t) - Q(t) \int_{t-\sigma}^{t} \lambda(s)\, \mathrm{d}s \geq Q(t)\left[\ln A(t) + 1\right].$$

Then for $N > T$,

$$\int_{T}^{N} \lambda(t)A(t)\, \mathrm{d}t - \int_{T}^{N} Q(t) \int_{t-\sigma}^{t} \lambda(s)\, \mathrm{d}s\, \mathrm{d}t \geqslant \int_{T}^{N} Q(t)\left[\ln A(t) + 1\right]\, \mathrm{d}t\,. \tag{2.15}$$

Interchanging the order of integration, we find that

$$\int_{T}^{N} Q(t)\left(\int_{t-\sigma}^{t} \lambda(s)\, \mathrm{d}s\right) \mathrm{d}t \geq \int_{T}^{N-\sigma} \lambda(t)\left(\int_{t}^{t+\sigma} Q(s)\, \mathrm{d}s\right) \mathrm{d}t\,.$$

Hence

$$\begin{aligned} &\int_{T}^{N} \lambda(t)A(t)\, \mathrm{d}t - \int_{T}^{N-\sigma} \lambda(t)\left(\int_{t}^{t+\sigma} Q(s)\, \mathrm{d}s\right) \mathrm{d}t \\ &\qquad \geqslant \int_{T}^{N} \lambda(t)A(t)\, \mathrm{d}t - \int_{T}^{N} Q(t) \int_{t-\sigma}^{t} \lambda(s)\, \mathrm{d}s\, \mathrm{d}t\,. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), it follows that

$$\int_{T}^{N} \lambda(t)A(t)\, \mathrm{d}t - \int_{T}^{N-\sigma} \lambda(t)\left(\int_{t}^{t+\sigma} Q(s)\, \mathrm{d}s\right) \mathrm{d}t \geq \int_{T}^{N} Q(t)\left[\ln A(t) + 1\right]\, \mathrm{d}t\,. \tag{2.17}$$

Integrating equation (2.12) from t to $t + \sigma$ we have

$$y(t+\sigma) - y(t) + \int_{t}^{t+\sigma} Q(s)y(s-\sigma)\, \mathrm{d}s = 0\,.$$

Then

$$y(t) > \int_t^{t+\sigma} Q(s)y(s-\sigma_1)\,\mathrm{d}s > y(t)\int_t^{t+\sigma} Q(s)\,\mathrm{d}s\,,$$

which implies that

$$A(t) = \int_t^{t+\sigma} Q(s)\,\mathrm{d}s < 1\,. \tag{2.18}$$

Then, by (2.17) and (2.18), we find

$$\int_{N-\sigma}^{N} \lambda(t)\,\mathrm{d}t \ge \int_T^N Q(t)\big[\ln A(t)+1\big]\,\mathrm{d}t\,,$$

hence

$$\ln\frac{y(N-\sigma)}{y(N)} \ge \int_T^N Q(t)\big[\ln A(t)+1\big]\,\mathrm{d}t\,. \tag{2.19}$$

In view of (2.11), we have

$$\lim_{t\to\infty}\frac{y(t-\sigma)}{y(t)} = \infty\,. \tag{2.20}$$

Because of $d \le \liminf\limits_{t\to\infty} \int_t^{t+\sigma} Q(s)\,\mathrm{d}s$ there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k\to\infty$, and there exist $\zeta_k \in (t_k-\sigma, t_k)$ for every k such that

$$\int_{t_k-\sigma}^{\zeta_k} Q(s)\,\mathrm{d}s \geqslant \frac{d}{2} \qquad\text{and}\qquad \int_{\zeta_k}^{t_k} Q(s)\,\mathrm{d}s \geqslant \frac{d}{2}\,. \tag{2.21}$$

Integrating both the sides of equation (2.12) over the intervals $[t_k,\zeta_k]$ and $[\zeta_k, t_k+\sigma]$, we have

$$y(\zeta_k) - y(t_k) + \int_{t_k}^{\zeta_k} Q(s)y(s-\sigma)\,\mathrm{d}s = 0 \tag{2.22}$$

and

$$y(t_k+\sigma) - y(\zeta_k) + \int_{\zeta_k}^{t_k+\sigma} Q(s)y(s-\sigma)\,\mathrm{d}s = 0\,. \tag{2.23}$$

From (2.21), (2.22) and (2.23), we obtain

$$-y(t_k) + \frac{d}{2} y(\zeta_k - \sigma) \le 0 \qquad \text{and} \qquad -y(\zeta_k) + \frac{d}{2} y(t_k) \le 0\,,$$

which implies that

$$\frac{y(\zeta_k - \sigma)}{y(\zeta_k)} \le \left(\frac{2}{d}\right)^2 .$$

This contradicts (2.20) and completes the proof. □

Note that the inequality (2.14) can be rewritten as

$$\mathrm{e}^{rx} \ge x + \frac{\ln(\mathrm{e}\,r)}{r} \qquad \text{for all} \quad x, r > 0$$

and then we have the following result.

COROLLARY 2.1. *Assume that* (1.2) *and* (1.10) *hold, and*

$$\frac{1}{\mathrm{e}} \le \int\limits_t^{t+\sigma} Q(s)\, \mathrm{d}s\,,$$

$$\int\limits_{t_0}^{\infty} Q(t) \ln\left(\mathrm{e} \int\limits_t^{t+\sigma} Q(s)\, \mathrm{d}s \right) \mathrm{d}t = \infty\,.$$

Then every solution of equation (1.1) *oscillates.*

COROLLARY 2.2. *Assume that* (1.2) *and* (1.10) *hold, and*

$$\frac{1}{\mathrm{e}} \le \int\limits_t^{t+\sigma} Q(s)\, \mathrm{d}s\,,$$

$$\int\limits_{t_0}^{\infty} Q(t) \left[\exp\left(\int\limits_t^{t+\sigma} Q(s)\, \mathrm{d}s - \frac{1}{\mathrm{e}} \right) - 1 \right] \mathrm{d}t = \infty\,.$$

Then every solution of equation (1.1) *oscillates.*

P r o o f. The proof is similar to that of Theorem 2.2 by choosing

$$A(t) = \frac{1}{\mathrm{e}} \left\{ \exp\left[\exp\left(\int\limits_t^{t+\sigma} \overline{Q}(s)\, \mathrm{d}s - \frac{1}{\mathrm{e}} \right) - 1 \right] \right\}$$

and will be omitted. □

The following theorems improve Theorem 2.1 and Theorem 2.2, which indicate that the oscillation of all solutions of equation (1.1) depend on P and Q.

THEOREM 2.3. *Assume that* (1.2) *and* (1.10) *hold, then either*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} Q(s)P(s-\sigma)\,\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{2.24}$$

or

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} Q(s)P(s-\sigma)\,\mathrm{d}s > 1 \tag{2.25}$$

implies that every solution of equation (1.1) *oscillates.*

Proof. Without loss of generality, we assume that equation (1.1) has an eventually positive solution $x(t)$. Then from Theorem 2.1 $y(t) > 0$ and its generalized equation is given by (2.6). From (2.6) one can see that $\lambda(t) \geq Q(t)$, then $\lambda(t-\tau) \geqslant Q(t-\tau)$, substituting in (2.6) we have

$$\lambda(t) \geqslant Q(t)P(t-\sigma)\exp\left(\int_{t-\tau}^{t} \lambda(s)\,\mathrm{d}s\right) + Q(t)\exp\left(\int_{t-\sigma}^{t} \lambda(s)\,\mathrm{d}s\right). \tag{2.26}$$

It is obvious that $\lambda(t) > 0$ for $t \geqslant t_0$, and then

$$\lambda(t) \geqslant Q(t)P(t-\sigma)\exp\left(\int_{t-\tau}^{t} \lambda(s)\,\mathrm{d}s\right). \tag{2.27}$$

Then from (2.5) and (2.27) one can see that $y(t)$ is positive solution of the delay differential inequality

$$y'(t) + Q(t)P(t-\sigma)y(t-\tau) \leq 0.$$

Then, by [15; Corollary 3.2.2], the delay differential equation

$$y'(t) + Q(t)P(t-\sigma)y(t-\tau) = 0 \tag{2.28}$$

has an eventually positive solution as well. It is well known that (2.24) or (2.25) implies that (2.28) has no eventually positive solution (see, for example, [15; p. 46, Theorem 2.3.3] and [15; p. 78, Theorem 3.4.3]). This is a contradiction and so the proof is complete. □

THEOREM 2.4. *Assume that* (1.2) *and* (1.10) *hold and* $\tau \geqslant \sigma$, *then either*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \left[Q(s)P(t-\sigma) + Q(s)\right]\,\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{2.29}$$

or

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} \big[Q(s)P(t-\sigma) + Q(s)\big]\, \mathrm{d}s > 1 \tag{2.30}$$

implies that every solution of equation (1.1) *oscillates.*

Proof. Without loss of generality, we assume that equation (1.1) has an eventually positive solution $x(t)$. As in Theorem 2.1 from (2.5) and (2.26) one can see that $y(t)$ is positive solution of the delay differential inequality

$$y'(t) + Q(t)P(t-\sigma)y(t-\tau) + Q(t)y(t-\sigma) \le 0\,.$$

Since $y'(t) \le 0$ and $\tau \geqslant \sigma$,

$$y'(t) + \big[Q(t)P(t-\sigma) + Q(t)\big]y(t-\sigma) \le 0\,.$$

Then by [15; Corollary 3.2.2] the delay differential equation

$$y'(t) + \big[Q(t)P(t-\sigma) + Q(t)\big]y(t-\sigma) = 0 \tag{2.31}$$

has an eventually positive solution as well. It is well known that (2.29) or (2.30) implies that (2.31) has no eventually positive solution (see, for example, [15; p. 46, Theorem 2.3.3] and [15; p. 78, Theorem 3.4.3]). This is a contradiction and completes the proof. □

THEOREM 2.5. *Assume that* (1.2) *and* (1.10) *hold, and* $\tau \geqslant \sigma$. *Then either*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \big[Q(s)P(s-\sigma)P(s-\tau-\sigma) + Q(s)\big]\, \mathrm{d}s > \frac{1}{\mathrm{e}} \tag{2.32}$$

or

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} \big[Q(s)P(s-\sigma)P(s-\tau-\sigma) + Q(s)\big]\, \mathrm{d}s > 1 \tag{2.33}$$

implies that every solution of equation (1.1) *oscillates.*

Proof. Without loss of generality, we assume that equation (1.1) has an eventually positive solution $x(t)$. As in Theorem 2.1, from equation (2.6) it is obvious that $\lambda(t) > 0$ for $t \geqslant t_0$, and $\lambda(t) \geqslant Q(t)$, then $\lambda(t-\tau) \geqslant Q(t-\tau)$, then

$$\lambda(t) \geqslant Q(t)P(t-\sigma)\exp\bigg(\int_{t-\tau}^{t} \lambda(s)\, \mathrm{d}s\bigg) + Q(t)\exp\bigg(\int_{t-\sigma}^{t} \lambda(s)\, \mathrm{d}s\bigg)\,,$$

which guarantees that $\lambda(t) \geqslant Q(t)P(t-\sigma)$, then $\lambda(t-\tau) \geqslant Q(t-\tau)P(t-\tau-\sigma)$. Substitute in (2.26), we get

$$\lambda(t) \geqslant Q(t)P(t-\sigma)P(t-\tau-\sigma)\exp\left(\int_{t-\tau}^{t} \lambda(s)\,\mathrm{d}s\right) + Q(t)\exp\left(\int_{t-\sigma}^{t} \lambda(s)\,\mathrm{d}s\right) \tag{2.34}$$

and then from (2.6), (2.34) and the fact that $\tau \geq \sigma$, $y(t)$ satisfies the inequality

$$y'(t) + \big[Q(t)P(t-\sigma)P(t-\tau-\sigma) + Q(t)\big]y(t-\sigma) \leq 0\,. \tag{2.34}$$

Then by [15; Corollary 3.2.2] the delay differential equation

$$y'(t) + \big[Q(t)P(t-\sigma)P(t-\tau-\sigma) + Q(t)\big]y(t-\sigma) = 0 \tag{2.35}$$

has an eventually positive solution as well. It is well known that (2.32) or (2.33) implies that (2.35) has no eventually positive solution. Then every solution of equation (1.1) oscillates. □

Remark 2.2. In view of the results established in [10], [18], [17], [9], [21], [24] and [22] and the fact that every solution of (1.1) oscillates when each one of equations (2.28), (2.31) and (2.35) has no eventually positive solution, we can present several other criteria for the oscillation of all solutions of equation (1.1). The details are left to the reader.

Further, similar to Theorem 2.2, we can obtain sufficient conditions for the oscillation of (1.1) by using the generalized characteristic equations of (2.28), (2.31) and (2.35). In fact, if we set

$$\begin{aligned} P_1(t) &= Q(t)P(t-\sigma)\,, \\ P_2(t) &= Q(t)P(t-\sigma) + Q(t)\,, \\ P_3(t) &= Q(t)P(t-\sigma)P(t-\tau-\sigma) + Q(t)\,, \end{aligned}$$

then respectively we have the following new sufficient conditions for oscillation of all solutions of equation (1.1):

COROLLARY 2.3. *Assume that* (1.2) *and* (1.10) *hold, and*

$$\frac{1}{\mathrm{e}} \leq \int_{t}^{t+\tau} P_1(s)\,\mathrm{d}s\,,$$

$$\int_{t_0}^{\infty} P_1(t)\ln\left(\mathrm{e}\int_{t}^{t+\tau} P_1(s)\,\mathrm{d}s\right)\mathrm{d}t = \infty\,.$$

Then every solution of equation (1.1) *oscillates.*

COROLLARY 2.4. *Assume that* (1.2) *and* (1.10) *hold, and*

$$\frac{1}{e} \le \int_t^{t+\tau} P_1(s)\,ds\,,$$

$$\int_{t_0}^{\infty} P_1(t)\left\{\exp\left(\int_t^{t+\tau} P_1(s) - \frac{1}{e}\,ds\right) - 1\right\} dt = \infty\,.$$

Then every solution of equation (1.1) *oscillates.*

COROLLARY 2.5. *Assume that* (1.2) *and* (1.10) *hold, and*

$$\frac{1}{e} \le \int_t^{t+\sigma} P_2(s)\,ds\,,$$

$$\int_{t_0}^{\infty} P_2(t) \ln\left(e \int_t^{t+\sigma} P_2(s)\,ds\right) dt = \infty\,.$$

Then every solution of equation (1.1) *oscillates.*

COROLLARY 2.6. *Assume that* (1.2) *and* (1.10) *hold, and*

$$\frac{1}{e} \le \int_t^{t+\sigma} P_2(s)\,ds\,,$$

$$\int_{t_0}^{\infty} P_2(t)\left\{\exp\left(\int_t^{t+\sigma} P_2(s) - \frac{1}{e}\,ds\right) - 1\right\} dt = \infty\,.$$

Then every solution of equation (1.1) *oscillates.*

COROLLARY 2.7. *Assume that* (1.2) *and* (1.10) *hold, and*

$$\frac{1}{e} \le \int_t^{t+\sigma} P_3(s)\,ds\,,$$

$$\int_{t_0}^{\infty} P_3(t) \ln\left(e \int_t^{t+\sigma} P_3(s)\,ds\right) dt = \infty\,.$$

Then every solution of equation (1.1) *oscillates.*

COROLLARY 2.8. *Assume that* (1.2) *and* (1.10) *hold, and*

$$\frac{1}{\mathrm{e}} \le \int\limits_{t}^{t+\sigma} P_3(s)\,\mathrm{d}s\,,$$

$$\int\limits_{t_0}^{\infty} P_3(t)\left\{\exp\left(\int\limits_{t}^{t+\sigma} P_3(s) - \frac{1}{\mathrm{e}}\,\mathrm{d}s\right) - 1\right\}\,\mathrm{d}t = \infty\,.$$

Then every solution of equation (1.1) *oscillates.*

Remark 2.3. Our results can be extended to the more general equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \sum_{j=1}^{m} P_j(t)x(t-\tau_j)\right] + \sum_{i=1}^{n} Q_i(t)x(t-\sigma_i) = 0\,, \qquad t \ge t_0\,.$$

Due to limited space, their statements are omitted here.

3. Examples

In this section we introduce some examples to illustrate our results.

EXAMPLE 3.1. Consider the neutral delay differential equation

$$\left[x(t) - \left(\frac{3}{2} + \sin t\right)x(t-\pi)\right]' + \left(\left(\sqrt{2}+1\right)\frac{2}{\pi} + \cos t\right)x\left(t - \frac{\pi}{2}\right) = 0\,, \qquad t \ge 0\,. \tag{3.1}$$

Here $\sigma = \frac{\pi}{2}$ and

$$Q(t) = \left(\sqrt{2}+1\right)\frac{2}{\pi} + \cos t > 0 \qquad \text{for} \quad t \ge 0$$

and

$$\int\limits_{t-\frac{\pi}{2}}^{t} Q(s)\,\mathrm{d}s = \int\limits_{t-\frac{\pi}{2}}^{t} \left(\sqrt{2}+1\right)\frac{2}{\pi} + \cos s\,\mathrm{d}s = \sqrt{2} + 1 + \sin t + \cos t\,.$$

Hence

$$\liminf_{t\to\infty} \int\limits_{t-\frac{\pi}{2}}^{t} Q(s)\,\mathrm{d}s = 1 > \frac{1}{\mathrm{e}}\,.$$

Then according to Theorem 2.1 and condition (2.1) every solution of equation (3.1) oscillates.

EXAMPLE 3.2. Consider the neutral delay differential equation

$$\Big[x(t)-\Big(\frac{3}{2}+\sin t\Big)x(t-\pi)\Big]'+\Big(\Big(\sqrt{2}+\frac{1}{e}\Big)\frac{2}{\pi}+\cos t\Big)x\Big(t-\frac{\pi}{2}\Big)=0\,,\qquad t\geq 0\,, \tag{3.2}$$

$\sigma=\frac{\pi}{2}$ and

$$Q(t)=\Big(\sqrt{2}+\frac{1}{e}\Big)\frac{2}{\pi}+\cos t>0 \qquad \text{for}\quad t\geq 0$$

and

$$\int_{t-\frac{\pi}{2}}^{t} Q(s)\,ds=\int_{t-\frac{\pi}{2}}^{t}\Big(\sqrt{2}+\frac{1}{e}\Big)\frac{2}{\pi}+\cos s\,ds=\sqrt{2}+\frac{1}{e}+\sin t+\cos t\,.$$

Hence

$$\liminf_{t\to\infty}\int_{t-\frac{\pi}{2}}^{t} Q(s)\,ds=\frac{1}{e}\,.$$

Then condition (2.1) cannot be applied, but one see that

$$\limsup_{t\to\infty}\int_{t-\frac{\pi}{2}}^{t} Q(s)\,ds=2\sqrt{2}+\frac{1}{e}>1\,.$$

Then, by Theorem 2.1 and condition (2.2), every solution of equation (3.2) oscillates.

The following example is adapted from the example in [17].

EXAMPLE 3.3. Consider the neutral delay differential equation

$$\Big[x(t)-\Big(\frac{3}{2}+\sin t\Big)x(t-\pi)\Big]'+\frac{0.6}{\alpha\pi+\sqrt{2}}(2\alpha+\cos t)x\Big(t-\frac{\pi}{2}\Big)=0\,,\qquad t\geq 0\,, \tag{3.3}$$

$\sigma=\frac{\pi}{2}$ and $\alpha=\frac{\sqrt{2}(0.6e+1)}{\pi(0.6e-1)}$,

$$Q(t)=\frac{0.6}{\alpha\pi+\sqrt{2}}(2\alpha+\cos t)>0 \qquad \text{for}\quad t\geq 0$$

and

$$\int_{t-\frac{\pi}{2}}^{t} Q(s)\,ds=\int_{t-\frac{\pi}{2}}^{t}\frac{0.6}{\alpha\pi+\sqrt{2}}(2\alpha+\cos s)\,ds\,.$$

Hence

$$\liminf_{t\to\infty}\int_{t-\frac{\pi}{2}}^{t} Q(s)\,ds=\frac{1}{e} \qquad \text{and} \qquad \limsup_{t\to\infty}\int_{t-\frac{\pi}{2}}^{t} Q(s)\,ds=0.6\,.$$

Then Theorem 2.1 cannot be applied on equation (3.3), but one can see by Corollary 2.2, every solution of equation (3.3) oscillates.

EXAMPLE 3.4. Consider the neutral delay differential equation

$$\Big[x(t) - \Big(\frac{3}{2} + \sin t\Big)x(t-\pi)\Big]' + \Big(\frac{1}{\mathrm{e}} + \frac{1}{t+1}\Big)x(t-1) = 0\,, \qquad t \geq 0\,, \quad (3.4)$$

$$Q(t) = \Big(\frac{1}{\mathrm{e}} + \frac{1}{t+1}\Big)$$

for $t \geq 0$ and

$$\int_{t-1}^{t} Q(s)\,\mathrm{d}s = \int_{t-1}^{t} \Big(\frac{1}{\mathrm{e}} + \frac{1}{s+1}\Big)\,\mathrm{d}s = \log\frac{t+1}{t} + \frac{1}{\mathrm{e}}\,.$$

Hence

$$\liminf_{t\to\infty} \int_{t-1}^{t} Q(s)\,\mathrm{d}s = \frac{1}{\mathrm{e}}\,.$$

Then Theorem 2.1 is failed to apply on equation (3.4), but one can see that for $T > 1$

$$\int_{1}^{T} Q(t)\ln\Bigg(\mathrm{e}\int_{t}^{t+1} Q(s)\,\mathrm{d}s\Bigg)\,\mathrm{d}t = \int_{1}^{T} \Big(\frac{1}{\mathrm{e}} + \frac{1}{t+1}\Big)\ln\Big(\ln\frac{t+2}{t+1} + \frac{1}{\mathrm{e}}\Big)\,\mathrm{d}t \to \infty$$

$$\text{as} \quad T \to \infty\,.$$

Then, by Corollary 2.1 every solution of equation (3.4) oscillates.

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