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SACHS TRIANGULATIONS, GENERATED BY DESSINS D'ENFANT, AND REGULAR MAPS

HEINZ-JUERGEN VOSS

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ABSTRACT. In 1963, H. Sachs developed the concept of describing finite groups by triangulations of closed oriented surfaces. This paper presents a new method for constructing Sachs triangulations. The method starts with an arbitrary map on a closed orientable surface, called dessin d'enfant. With the help of it, a Sachs triangulation is generated. The method is used to construct reflexible regular maps on closed oriented surfaces.

In 1963, H. Sachs [4] developed the concept of describing finite groups by triangulations of closed oriented surfaces. Later, this description was studied by W. Voss and H.-J. Voss; for a survey, see H.-J. Voss [7]. J. Širáň, M. Škoviera and H.-J. Voss [6] used these Sachs triangulations in constructing regular maps.

This paper presents a new method for constructing Sachs triangulations. The method starts with an arbitrary map D on a closed orientable surface, called dessin d'enfant. With the help of D, a Sachs triangulation $T(D)$ is generated. The main result of this paper is formulated in Theorem 10 (Section 5). The method is used to construct reflexible regular maps M on closed oriented surfaces of type $\{p, q\}$, for all integers p and q with $q \geq 3$ and $p \geq 3q$, where p denotes the vertex valence, and q the face length. If $q = 3$, and $p \geq 3$ is an odd number, then M is even a Sachs triangulation. In this way, a reflexible regular Sachs triangulation of each odd valence p $(p \geq 3)$ and a reflexible regular triangulation of each even valence p $(p \geq 4)$ are obtained.

This new method is described in Section 2, Section 1 gives the basic concepts. The valences of the Sachs triangulations constructed by this method are considered in Section 3. In Sections 4, 5 and 6, regular maps, regular maps of given type, and reflexible regular maps of given type are constructed, respectively. The concluding remarks of Section 7 present a construction similar to that of D. S. Archdeacon, P. Gvozdjak, and J. Širáň [1], and some open problems.

AMS Subject Classification (1991): Primary 05C10, 05C25.
Key words: regular map, Sachs triangulation.

1. Concepts

Let G be a finite group. A *triple* (x,y,z) of elements $x,y,z \in G$ is said to be *regular* if $x \neq y \neq z \neq x$, and $xyz = e$. With $xyz = e$ also $yzx = e$ and $zxy = e$. If (x,y,z) is a regular triple, then (y,x,z') with $z' = (yx)^{-1}$ is regular, too. To each regular triple (x,y,z) an *oriented triangle* $D(x,y,z)$ with arcs (x,y), (y,z), (z,x), and vertices x, y, z is assigned in such a way that $D(x,y,z) = D(x',y',z')$ if and only if $(x',y',z') \in \{(x,y,z),(y,z,x),(z,x,y)\}$; otherwise $D(x,y,z)$ and $D(x',y',z')$ are disjoint (Figure 1.1). In the set Σ of all triangles thus obtained, an arc occurs at most once. If (x,y) is in some triangle of Σ, then (y,x) is in a triangle of Σ, too. In the sense of combinatorial topology, oppositely directed arcs (x,y), (y,x) are identified to obtain an edge $[x,y]$ (Figure 1.2). Identifying step by step all oppositely directed arcs, a set Θ of triangulations of closed oriented surfaces is obtained.

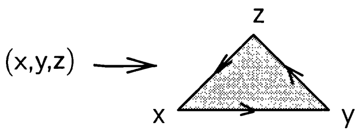


FIGURE 1.1.

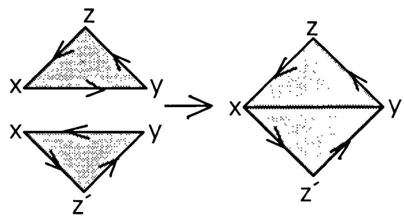


FIGURE 1.2.

If the elements x, y of the regular triple (x,y,z) commute, then $xyz = yxz = e$, and the triangulation T is formed by the two triangles $D(x,y,z)$ and $D(y,x,z)$ only. Hence, T is the sphere with a cycle of length 3 on it which triangulates that sphere. It is called the *trivial triangulation.*

Consequently, in the following, we shall consider *non-commuting* elements of G. In fact, two non-commuting elements x, y form with $z = (xy)^{-1}$ a regular triple (x,y,z). Thus, to each pair of non-commuting elements $x,y \in G$ a triangulation $T(x,y)$ of a closed oriented surface $S(x,y)$ is obtained. Such a triangulation is called a *Sachs triangulation.* It is a component of the set Θ defined above. The octahedron $T\big((12)(34),(1234)\big)$ is presented as an example in Figure 1.3. It belongs to the symmetric group S_4 on the four symbols $1,2,3,4$.

The *subgroup* of G generated by x and y is denoted by $G(x,y)$. Any two different triangles in $T(x,y)$ are either disjoint, or have precisely one vertex in common, or have precisely one edge and its end vertices in common. Such triangulations are said to be *simple.* The *basic disc* $K(x,y)$ of $T(x,y)$ *through* y with *center* x consists of the edge $[x,y]$ and all triangles of $T(x,y)$ incident with the vertex x (see Figure 1.4). An *automorphism* a of a triangulation is a triple of one-to-one mappings of the vertex-set, the edge-set, and the triangle-set onto the vertex-set, the edge-set and the triangle-set, respectively, preserving incidences. The *inner automorphism* f_x of G with $f_x(g) = x^{-1}gx$ for each $g \in G$ induces an orientation preserving automorphism of $T(x, y)$ with the property that $K(x, y)$ is *rotated clockwise over two sectors around* x. That is to say, if $y = y_1, y_2, \dots, y_s$, $y_1 = y$ are the neighbours of x, which appear in this order on the bounding cycle of $K(x, y)$ clockwise, then $f_x(x) = x$ and $f_x(y_i) = y_{i+2}$ for all $s \geq i \geq 1$, where the indices are to be taken modulo s.

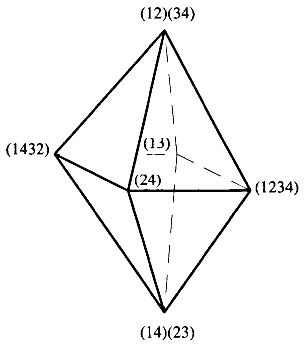


FIGURE 1.3.

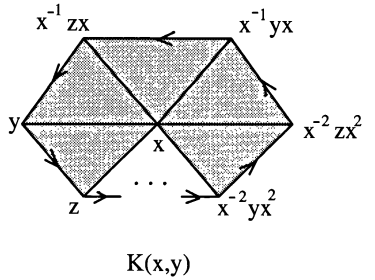


FIGURE 1.4.

In general, an automorphism of an arbitrary triangulation T of a surface S that rotates the star neighbourhood $K(x, y)$ as above is called a 2-*rotation* (*based at* x). If for every vertex x of the triangulation T there is a 2-rotation based at x, then T is called a 2-*rotary triangulation.* Thus each Sachs triangulation is a 2-rotary simple triangulation. These 2-rotations have the following property: if $D(a, b, c)$ is a face of $T(x, y)$, then $f_a f_b f_c$ is the identical automorphism e because $f_a f_b f_c(g) = f_c(f_b(f_a(g))) = (abc)^{-1}gabc = g$ for all $g \in G$.

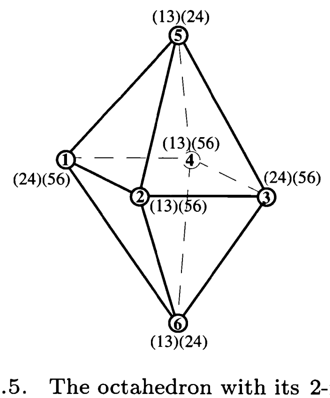


FIGURE 1.5. The octahedron with its 2-rotations.

Now we consider an arbitrary 2-rotary simple triangulation T. It can easily be proved that the product of the 2-rotations of the vertices of a triangle face taken in the anticlockwise order is the identity automorphism e. A triangulation T is said to have an *obstacle* if T has two different arcs with the same 2-rotations. An example is the octahedron with the vertices $1,2,3,4,5,6$ depicted in Figure 1.5. Each 2-rotation $(13)(24)$, $(24)(56)$, and $(13)(56)$ appears twice on the octahedron. Obviously, the octahedron has obstacles.

THEOREM 1. *Let T be a 2-rotary simple triangulation without an obstacle. Then a Sachs triangulation is obtained whenever to each vertex its 2-rotation is assigned.*

The original question posed by H. Sachs [4] was the following: How are properties of groups related to properties of their Sachs triangulations and vice versa? For further reading concerning this question, the reader is referred to [7]. In the present paper, the method is used in constructing regular maps of given type (Theorem 10).

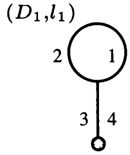
2. Construction of Sachs triangulations from dessins d'enfant

In the Conference on Topological Graph Theory in Donovaly, Slovakia, in 1994, R. Nedela and M. Škoviera presented the concept of a dessin d'enfant. This concept was first introduced by A. Grothendieck [2]. In their talk, R. Nedela and M. Škoviera (see [3]) used the notation "dessin d'enfant" for very elementary maps, from which they constructed large and complicated maps of high symmetries. In this paper, very elementary dessins d'enfant, namely, plane trees and plane trees with precisely one additional loop are used in constructing large Sachs triangulations. From these triangulations regular maps with desired properties are derived. The main result is presented in Theorem 10.

We define a *dessin d'enfant* D to be a map on an oriented closed surface; half-edges are allowed, i.e., edges starting in a vertex and ending in no vertex (see Figure 2.2). Since D is a map, the underlying graph is connected.

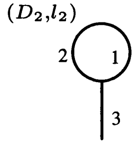
To each such map D, three permutations on the set of the arcs of D are assigned, namely, the *edge inversion* L, the *edge rotation* R, and the *face boundary permutation* B. In order to define these permutations, each edge is replaced by two opposite directed arcs; each half-edge at vertex P is replaced by an arc leaving P and ending in an auxiliary pendant point which does not count as a vertex. Let these arcs be denoted by $1,2,\dots,n$. In the figures, the label of an arc a is put into the face bounding a from the left. Even, if the dessins d'enfant are drawn without arrows, the direction of each arc can easily be determined.

The *edge rotation* R of D is obtained by the anticlockwise 1-rotations of the outward directed arcs around all vertices of D. The *edge inversion* L of D consists of all transpositions (i,j) such that the arcs i, j belong to the same edge. The *boundary of a face* F consists of all arcs bounding F in the anti-clockwise sense. The *boundary permutation* B of D is obtained by the anti-clockwise 1-rotations of the bounding arcs around all faces of D (see Figures 2.1 and 2.2). It can easily be seen that $RLB = e$.



R = (124)
L = (12)(34)
B = (243)

FIGURE 2.1.



R = (123)
L = (12)
B = (23)

FIGURE 2.2.

In the following, this situation will be interpreted as follows. Let a dessin d'enfant D be endowed with a labelling l of its n arcs by $1, 2, \dots, n$, where l is a bijection of the arc set of D onto $\{1, 2, \dots, n\}$. In D, let the arcs of D be fixed, and only the labels of the arcs be shifted under the actions of R, L, B. So R is now obtained by the anticlockwise 1-rotations of the labels of the outward directed arcs around all vertices of D, the permutation L consists of all transpositions (λ, λ') of labels λ, λ' of arcs forming the same edge, and B is obtained by the anticlockwise 1-rotations of the labels of the bounding arcs of all faces of D.

Let (D, l) be a dessin d'enfant D with a labelling l of its arcs, and let R, L, B be the edge rotation, the edge inversion L and the boundary permutation B of (D, l). In what follows, we assume that the triple (R, L, B) is regular. The Sachs triangulation $T(R, L) = T(L, B) = T(B, R)$ is also denoted by $T(D, l)$.

Given a labelled dessin d'enfant (D, l), we denote the triangle with vertices R, L, B by $\Delta = \Delta(D, l)$. If Δ can be mapped by a sequence of 2-rotations onto a triangle Δ', then there exists a labelling l' of D such that the permutations assigned to the vertices of Δ' are the edge rotation R', the edge inversion L', and the boundary B' of (D, l').

In Figures 2.3 and 2.4, we have presented the triangulations $T(D_1, l_1)$ and $T(D_2, l_2)$, where (D_1, l_1) and (D_2, l_2) are the dessins d'enfant of Figures 2.1 and 2.2, respectively. Each labelled (D, l) is depicted in the triangle, to which it is related. In order to get the triangulation $T(D_1, l_1) = T\big((123), (12)(34)\big)$ of the sphere in Figure 2.3, the left hand part has to be put above the right hand part, and the outer cycles have to be identified (permutations are written without parentheses, different cycles of the permutations are separated by a line). The Sachs triangulation $T(D_1, l_1)$ belongs to the symmetric group S_3 on three symbols. In order to obtain the triangulation $T(D_2, l_2) = T\big((123), (12)\big)$ of the sphere in Figure 2.4, oppositely directed arcs have to be identified. The Sachs triangulation $T(D_2, l_2)$ belongs to the alternating group A_4 on four symbols.

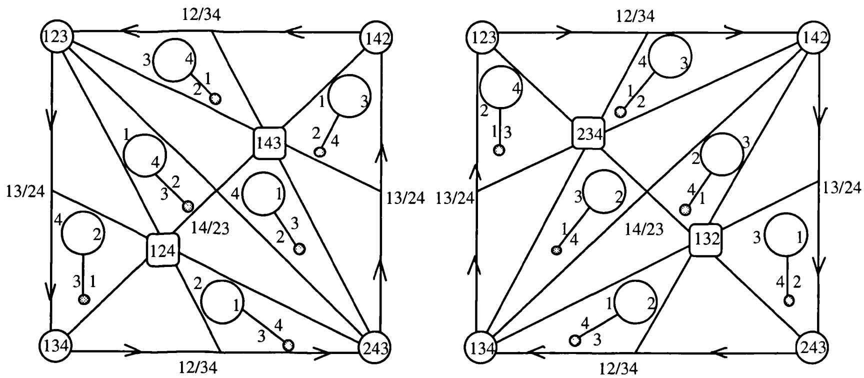


FIGURE 2.3. Opposite directed arcs have to be identified.

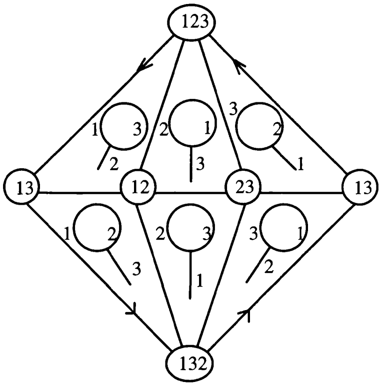


FIGURE 2.4. Opposite directed arcs have to be identified.

Obviously, the labellings of D which are assigned to triangles of $T(D, l)$ are obtained by taking the orbit of the original labelling l under the group $\langle R, L, B \rangle$, where $\langle R, L, B \rangle$ denotes the subgroup of the symmetric group S_n of n symbols, generated by R, L, B.

How to find all triangles Δ' together with the labellings l' of D so that (D, l') is assigned to Δ'? These triangles Δ' and these labellings l' of D can recursively be determined according to the following rules.

- **Rule I.**
  Edge rotation at the vertex R: 2-rotate (D,l) anticlockwise around R and apply R to (D,l), i.e., shift the labels of (D,l) around each vertex of D according to R (in Figure 2.3, these vertices are depicted by small rectangles).

- **Rule II.**
  Edge inversion at the vertex L: 2-rotate (D,l) anticlockwise around L and apply L to (D,l), i.e., exchange the labels of each pair of arcs of (D,l) belonging to the same edge (in Figure 2.3, these vertices have valency 4 and appear as simple crossings).

- **Rule III.**
  Boundary shifting at the vertex B: 2-rotate (D,l) anticlockwise around B and apply B to (D,l), i.e., shift the labels around each face of (D,l) according B (in Figure 2.3, these vertices are depicted by small circles).

The easy proofs of the validity of the Rules I – III are omitted here.

By this method, D can be spread over the whole triangulation, having different labellings at different triangles. Moreover, by these rules, we have a new method for constructing the Sachs triangulation $T(R,L) = T(D,l)$. If there is an orientation preserving, label preserving homeomorphism of (D,l) onto (D,l') (i.e., image and preimage of each arc have the same labels), then R, L, B of (D,l) and R', L', B' of (D,l') are equal. Since in every two distinct triangles of $T(D,l)$ the sets of assigned group elements are different, there is no nontrivial orientation preserving, label preserving homeomorphism between two labelled dessins d'enfant belonging to different triangles of $T(D,l)$.

3. Properties of the constructed Sachs triangulations

3.1. The distribution of labelled D's on $T(D,l)$.

Let (D,l), $T(D,l)$ and R, L, B be defined as before. The triangle with the vertices R, L, B is again denoted by Δ. If the rules I, II, III are successively applied to the dessin d'enfant (D,l) of Δ, then either to each triangle or to every second triangle of $T(D,l)$ the dessin d'enfant D with a certain labelling is assigned. The first case is shown by the Sachs triangulation $T\big((123),(12)\big)$ of the symmetric group S_3 on three symbols, depicted in Figure 2.4. The second case is shown by the Sachs triangulation $T\big((123),(12)(34)\big)$ of the alternating group A_4 on four letters, depicted in Figure 2.3. In the first case, let us consider the dessins d'enfant (D,l_1), (D,l_2) of two neighbouring triangles (which have an edge in common). It is obvious that $|\{R_1,L_1,B_1\}\cap\{R_2,L_2,B_2\}|=2$, and an edge rotation coincides with a boundary permutation or an edge inversion, or a boundary permutation coincides with an edge inversion. Hence, at least two permutations of $\{R,L,B\}$ have the same cycle structure and are conjugate in $\langle R,L,B\rangle$.

3.2. The orbits of $T(D,l)$.

The *cycle structure of a permutation* p can be described by the type(p). The type(p) is the n-tuple $(b_1,b_2,\dots b_n)$, where b_i means that p has in its cycle representation b_i cycles of length i, $n\geq i\geq 1$.

It is well known that two permutations of the same conjugacy class have the same type. On the Sachs triangulation $T(x,y)$, there are group elements of at most three conjugacy classes because on $T(x,y)$, there are at most three vertex orbits under the actions of the 2-rotations of $T(x,y)$ (for more details, see [7]). Hence, only permutations of at most three types appear on each Sachs triangulation.

In the following, only dessins d'enfant D will be considered in which the edge rotation R, edge inversion L and boundary B do not commute, and type$(R)\neq$ type$(L)\neq$ type$(B)\neq$ type(R). Then (R,L,B) is a regular triple, and our construction results in a nontrivial Sachs triangulation $T(D,l)=T(R,L)$ having three different orbits $O(R)$, $O(L)$, and $O(B)$. By Section 3.1, a labelled dessin d'enfant is assigned only to every second triangle of $T(D,l)$. If the type condition type$(R)\neq$ type$(L)\neq$ type$(B)\neq$ type(R) is not fulfilled, then both possibilities described in 3.1 can occur. This is shown by the triangulations $T\big((123),(12)(34)\big)$ and $T\big((123),(12)\big)$. However, the related triples $\big((123),(12)(34),(234)\big)$ of the group A_4, and $\big((123),(12),(23)\big)$ of the group S_3, respectively, do not satisfy the type condition type$(R)\neq$ type$(L)\neq$ type$(B)\neq$ type(R).

3.3. The valences of $T(D,l)$.

If X is a fixed vertex of $T(D,l)$, then all labelled dessins d'enfant appearing in the basic disc $K(X)$ around X are obtained by 2-rotating one of them around X and applying precisely one of the Rules I, II, III. Let this one be denoted by (D,l), being related to the triangle D of $K(X)$. These 2-rotations end when a labelling l' of D describing the same permutations as (D,l) is obtained for the first time. Hence Δ is reached, and (D,l') can be obtained from (D,l) by an orientation preserving and label preserving homeomorphism of (D,l) onto (D,l') (see Figure 2.3).

These observations imply that the valences of all vertices are even.

If each labelled dessin d'enfant (D,l') assigned to a triangle of $K(X)$ can be mapped onto (D,l) by an orientation preserving and label preserving homeomorphism, then the valency $v(T:X)$ of X in $T(D,l)$ is two, and $T(D,l)$ is a trivial triangulation. This case is excluded if D has at least five arcs, at least two of which form an edge.

In the following, let D satisfy these requirements. Then each triangulation $T(D,l)$ is nontrivial and $v(T:L)=4$, $v(T:R)\geq 4$, and $v(T:B)\geq 4$.

4. Constructions of regular maps with group labellings

4.1. Construction of regular maps.

Throughout this paper, only maps on closed oriented surfaces will be considered.

A 2-rotary map M is called *regular of type* $\{p,q\}$ if the automorphism group of the map M acts regularly (i.e., simply transitively) on the arcs of M, where p is the vertex valence, and q is the length of the face boundaries. In the proof of Theorem 2, we need the following reformulation: a 2-rotary map M is regular if and only if the 1-rotation of each face and the rotation around the middle of each edge are automorphisms of M.

Let (D,l), $T(D,l)$, R, L and B be defined as before. Next, all vertices of $O(R)$ of T are deleted as well as all edges incident with them. In the remaining map, each vertex of $O(L)$ has valence 2. If each of these vertices together with its two incident edges is replaced by an edge, then a map T_R is obtained having valence $v(T:B)/2$. A map T_B is obtained in a similar way.

Since the Sachs triangulation $T(D,l)$ is a 2-rotary map, we have:

THEOREM 2. *T_R and T_B are dual regular maps of types $\{v(T:B)/2, v(T:R)/2\}$ and $\{v(T:R)/2, v(T:B)/2\}$, respectively.*

Proof. A 1-rotation of T_R around a vertex corresponds to a 2-rotation of T around this vertex of $O(B)$, a rotation around the middle of an edge of T_R corresponds to a 2-rotation of T around a vertex of $O(L)$, and a 1-rotation around the center of a face of T_R corresponds to the 2-rotation of T around a vertex of $O(R)$.

The proof for T_B is similar. □

Next, all $O(R),O(L)$-edges of T are deleted. In the remaining map, each vertex of $O(L)$ has valence 2. The triangulation cT_R is obtained by replacing each of these vertices together with its two incident edges by an edge. It is a 2-rotary triangulation having 1-rotations around the vertices of $O(B)$; the valences of cT_R are $v(cT_R:B)=v(T:B)$ and $v(cT_R:R)=v(T:R)/2$. The triangulation cT_B is defined similarly.

In order to relate the structures of cT_R and T with the structure of T_R, some well known concepts are used.

Let M again denote a map on a closed oriented surface. A cellular subdivision of M can be derived from M by introducing a new vertex in each face and joining it with all vertices of the boundary by an edge. A barycentric subdivision of M is obtained from M by subdividing each edge by a vertex, introducing a new vertex in each face and joining it with all vertices of the boundary by an edge.

THEOREM 3.

(1) *T is a barycentric subdivision of both regular maps T_R and T_B.*
(2) *T is a cellular subdivision of $T_L := T \setminus O(L)$.*
(3) *cT_R and cT_B are cellular subdivisions of T_R and T_B, respectively.*

4.2. Construction of group labellings.

The concept of a Sachs labelling of a triangulation is generalized to arbitrary maps. Let f be a mapping of the vertex-set of a map M into a finite group G. The mapping f is said to be a *group labelling* if for each face F of M the following conditions hold:

(i) the group elements assigned to the vertices of F are pairwise distinct;
(ii) the anticlockwise product of these elements around the boundary of F is the identity element e of G.

If there are no two distinct edges having the same labels at their end vertices, then the group labelling f is called a *Sachs labelling.*

By construction, T has a Sachs labelling.

What kind of labellings of T_R, T_L and T_B can be derived from the labelling of T?

THEOREM 4. *Let the orders of R, L, and B coincide with $v(T:R)/2$, $v(T:L)/2$, and $v(T:B)/2$, respectively. If the boundaries of the faces of T_R, T_L, or T_B have even length, then the anticlockwise product of the group elements of the boundary of each face is the identity.*

If the boundaries of the faces of T_R or T_B have odd length, then replace each group element by its square. Then the anticlockwise product of the squares around the boundary of each face is the identity.

P r o o f . We prove the theorem only in the case where the boundaries of the faces of T_R have an odd length $s = v(T:R)/2$. The other cases can be proved similarly.

Let $B_1, B_2, \dots, B_s$ be the group elements on the boundary δF of a face F of T_R in anticlockwise order. Then in T, the face F of T_R corresponds to a basic disc $K(R)$. On its boundary δF, a vertex L_i lies between B_i and B_{i+1}, $s \ge i \ge 1$ (indices modulo s).

Since T is a Sachs triangulation, $B_i L_i = R^{-1} = L_i B_{i+1}$. With $(L_i)^2 = e$, $s \ge i \ge 1$, and $R^s = e$ this implies:

$$B_1 L_1 L_1 B_2 B_2 L_2 L_2 B_3 \dots B_s L_s L_s B_1 = R^{-2s} = e.$$

Hence,

$$B_1^2 B_2^2 \dots B_s^2 = e\,.$$

□

Note that cT_R and cT_B have two classes $O(R)$ and $O(B)$ of vertices.

THEOREM 5. *If in cT_R (or cT_B) each label of $O(R)$ (or $O(B)$, respectively) is replaced by its square, then the anticlockwise product of the labels around each triangle face is the identity.*

Proof. Let (B_1, B_2, R') be a triangle face of cT_R.

In T, the edge (B_1, B_2) of cT_R is subdivided by a vertex L, and (B_1, L, R') and (R', L, B_2) are two neighbouring triangles of T. Hence $(R'B_1L)(LB_2R') = R'B_1B_2R' = e$ and $B_1B_2R'^2 = e$.

The proof for cT_B is similar. □

5. Construction of regular maps

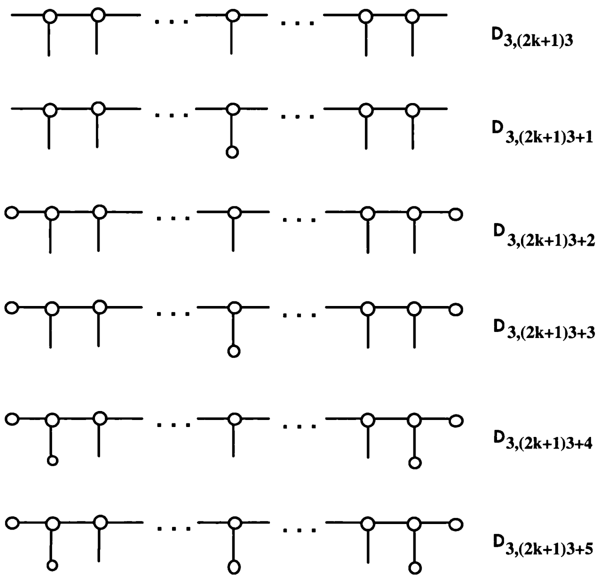


FIGURE 5.1.

The path with j vertices will be denoted by p_j. Let k and q be integers with $k \geq 1$ and $q \geq 3$. Let $D_{q,(2k+1)q}$ denote the map obtained from an embedding of p_{2k+1} into the sphere by adding $q-2$ half-edges to each inner vertex of p_{2k+1} and $q-1$ half-edges to each end vertex of p_{2k+1} in such a way that all half-edges lie on the same side of p_{2k+1} (see Figure 5.1 and Figure 5.2).

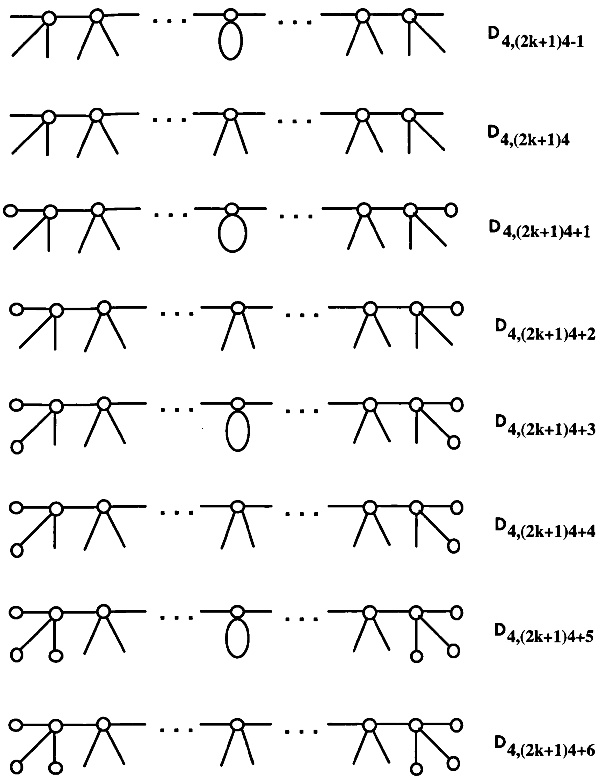


FIGURE 5.2.

Obviously, the permutations R, L, B are of different type, because R has precisely $2k+1$ cycles of length q, L has precisely $2k$ cycles of length 2 and $(2k+1)(q-2)+2$ fixpoints, and B consists of one cycle of length $(2k+1)q$.

The map $D = D_{q,(2k+1)q}$ is asymmetric in the sense that there is no nontrivial orientation preserving homeomorphism of D onto itself. Nevertheless, there is an orientation reversing homeomorphism ρ of D onto itself, a special reflection of D.

If q is odd, then ρ has precisely one fixed half-edge h. The map $D_{q,(2k+1)q+1}$ is obtained from $D_{q,(2k+1)q}$ by adding a new vertex to h in such a way that h becomes an edge.

If q is even, then ρ has no fixed half-edges, but there is precisely one pair of half-edges h', h'' with $\rho(h') = h''$ (and $\rho(h'') = h'$), being neighbours on the boundary of the only face of $D_{q,(2k+1)q}$. Then $D_{q,(2k+1)q-1}$ is obtained from $D_{q,(2k+1)q}$ by identifying the two endpoints of h', h'' (which are not vertices) in such a way that a loop is obtained bounding a face.

Let $e = 1$ if q is odd, and $e = -1$ if q is even. The maps $D_{q,(2k+1)q+2i}$ and $D_{q,(2k+1)q+e+2i}$, $q - 1 \geq i \geq 1$, are obtained from $D_{q,(2k+1)q}$ and $D_{q,(2k+1)q+e}$, respectively, by adding an end vertex to each of $2i$ half-edges $h_1, \rho h_1, h_2, \rho h_2, \dots, h_i, \rho h_i$ in such a way that these half-edges become edges. Obviously, $h \notin \{h_1, \rho h_1, \dots, h_i, \rho h_i\}$, because $\rho(h) = h$. Thus $D_{q,p}$ is defined for all $q \geq 3$ and $p \geq 3q$ if q is odd, and $p \geq 3q - 1$ if q is even.

The construction implies:

Proposition 6. *The dessin d'enfant $D_{q,p}$ has no nontrivial orientation preserving homeomorphism onto itself. Moreover, there is one orientation reversing homeomorphism ρ of $D_{q,p}$ onto itself.*

By construction of $D_{q,p}$, the permutations R, L, B have pairwise different types; besides fixpoints, R has only cycles of length q, L has only transpositions, and B has only one cycle of length p. The orders of R, L, and B are $o(R) = q$, $o(L) = 2$, and $o(B) = p$, respectively. By Proposition 6, there is no nontrivial orientation preserving homeomorphism of $D_{q,p}$ onto itself; hence there is also no label preserving one. Applying Rules I, II, and III to the basic discs around the vertices of $O(R)$, $O(L)$, and $O(B)$, respectively, the sequence of 2-shifts only ends if all labels are again in their starting position. Consequently, $v(T : L) = 4$, $v(T : R) = 2q$, and $v(T : B) = 2p$. Theorem 3 of Section 4.1 implies:

Theorem 7. *The Sachs triangulation $T(D_{q,p})$, where $q \geq 3$ and $p \geq 3q$, is a barycentric subdivision of the regular map $T_R(D_{q,p})$ of type $\{p, q\}$ and also of the regular map $T_B(D_{q,p})$ of type $\{q, p\}$.*

Now, we assign to each vertex of T_R the square of the former label. Since $v(T : L) = 2o(L)$, $v(T : R) = 2o(R)$, and $v(T : B) = 2o(B)$, Theorem 4 implies: the anticlockwise product of these labels around each face is the identity.

When is this labelling a Sachs labelling of T_R?

We start with two lemmas.

Lemma 8. *Any two vertices of $O(B)$ have different labels in T.*

P r o o f. Assume there are two vertices $X, Y \in O(B)$ with the same label B^*. Then, in both of the two disjoint basic discs $K(X)$ and $K(Y)$ of T, all 1-shifts of the boundary B^* around the outer face of D occur. Since D is a tree or a tree with one loop, respectively, the set S_X (and S_Y) of the labellings of D in $K(X)$ (and $K(Y)$) are uniquely determined by B^*. Hence $S_X = S_Y$, and

T has distinct triangles with the same permutations assigned to their vertices. This contradiction proves the lemma. □

The type of a permutation x is denoted by $\text{type}(x) = i^\alpha j^\beta$ if x has α cycles of length i, β cycles of length j, and no other cycles. The validity of the following lemma is obvious.

LEMMA 9. *For any two permutations $x, y \in S_n$ of the same type $1^b(2k+1)^1$ ($k \geq 1$ an integer),*

$$x \neq y \implies x^2 \neq y^2 .$$

THEOREM 10. *The map $T_R(D_{q,p})$, $q \geq 3$, $p \geq 3q$, is a regular map on a closed oriented surface of type $\{p, q\}$. Moreover, for each integer $q \geq 3$ and each odd integer $p \geq 3q$ a Sachs labelling of $T_R(D_{q,p})$ is obtained by replacing each label B by its square B^2. Further, for each even integer $q \geq 4$ and each integer $p \geq 3q$, the original labels form a Sachs labelling.*

The regularity of the map $T_R(D_{q,p})$ follows immediately from Theorem 2.

We remark that for odd $q \geq 3$ and even $p \geq 3q$ the existence of a Sachs labelling of $T_R(D_{q,p})$ is an open problem.

P r o o f. The first part of this assertion follows from Theorem 2. By Lemma 8, any two vertices of $T_R(D_{q,p})$ have different “original” labels. By Lemma 9, any two vertices have different “squared” labels. Thus Theorem 4 implies that for odd p the squared labels and for even q the original labels form a Sachs labelling. □

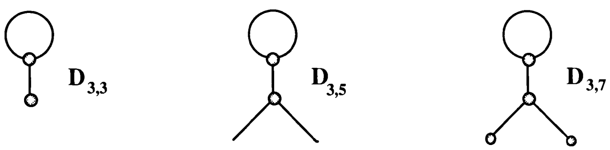


FIGURE 5.3.

If the method is applied to the dessins d'enfant of Figure 5.3, then also regular Sachs triangulations of vertex valences 3 (tetrahedron), 5 (icosahedron), and 7 are obtained. With case $q = 3$ of Theorem 10 this implies:

THEOREM 11. *The triangulations $T_R(D_{3,p})$ are regular maps for each odd vertex valence $p \geq 3$. The squared labels form a Sachs labelling.*

If the method is applied to the dessins d'enfant of Figure 5.4, then also regular triangulations of vertex valences 4 (octahedron), 6, and 8 are obtained.

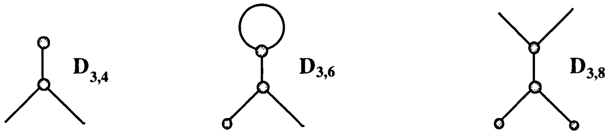


FIGURE 5.4.

THEOREM 12. *The triangulations $T_R(D_{3,p})$ are regular maps for each even vertex valence $p \geq 4$. The squared labels form a group labelling.*

Notice that in general this group labelling is not a Sachs labelling.

Proof. The first part of this assertion follows from Theorem 2.
Next let B_1, B_2, B_3 be the labels of a triangle Δ of $T_R(D_{3,p})$, $p \geq 10$.

$$B_1^2 \neq B_2^2 \neq B_3^2 \neq B_1^2 . \tag{1}$$

Proof of (1). If, without loss of generality, $B_1^2 = B_2^2$, then the boundary of $K(B_1)$ in $T(D_{3,p})$ has only the vertices L, R, $B_1^{-1}LB_1$, $B_1^{-1}RB_1$, $B_1^{-2}LB_1^2 = L$. Hence, $v(T : B_1) = 4$. This contradicts the consequence of Theorem 7 that $v(T : B_1) = 2p \geq 20$. Thus (1) is proved for $p \geq 10$.

(1) implies with Theorem 4 the validity of the second part of Theorem 12 for $p \geq 10$. The case $p = 4$, 6, or 8 is proved by a direct construction. □

For $T_B(D_{q,p})$ the following result can be proved in the same way.

THEOREM 13. *The map $T_B(D_{q,p})$, $q \geq 3$, $p \geq 3q$, is a regular map on a closed oriented surface of type $\{q,p\}$.*

Similarly, the following result can be proved:

THEOREM 14. *The cellular subdivision $cT_R(D_{q,p})$ of $T_R(D_{q,p})$ is a Sachs triangulation. Its Sachs labelling is obtained by replacing each group element of $O(R)$ by its square.*

6. Reflexible maps

6.1. Inverse Sachs triangulations.

Let T be a triangulation of a closed oriented surface with a group labelling. Replace each group element by its inverse. If (x, y, z) is a regular triple of group elements assigned to some triangle of T, then (z^{-1}, y^{-1}, x^{-1}) is a regular triple, too. Hence, replacing each group element by its inverse results in a group labelling of T^i, where T^i is obtained from T by reversing the orientation of the underlying surface.

Next, let (D,l), R, L, B and $T(D,l)$ be defined as before. The dessin d'enfant (D,l) is assigned to the triangle Δ labelled (R,L,B). The triangle Δ^* labelled (R^*,B,L) has with Δ the common edge (L,B), where $R^* = L^{-1}BL$.

Let (D^i,l) denote the labelled dessin d'enfant obtained by reflecting (D,l) (changing the orientation of the underlying surface) and reversing the direction of all arcs, where the labels are fixed at the arcs, i.e., image and preimage have the same label. Then ${R^*}^{-1}$, L^{-1}, and B^{-1} are the rotation, edge inversion and the boundary permutation of (D^i,l) respectively, and (D^i,l) is assigned to Δ^* of T^i. Thus $T^i \cong T^i(D,l) \cong T(D^i,l)$.

6.2. Reflexible maps.

Let D and D^* be two homeomorphic dessins d'enfant. Then the Sachs triangulations $T(D)$ and $T(D^*)$ are homeomorphic.

If D is a reflexible map, i.e., D and D^i are homeomorphic, the Sachs triangulation $T(D)$ and $T^i(D) \cong T(D^i)$ are also homeomorphic. Hence, $T(D)$ is a reflexible map.

Theorem 15. *With D also $T(D)$ is a reflexible map.*

Remark. The nonreflexible dessin d'enfant $D_{3,6}$ of Figure 5.4 generates a nonreflexible Sachs triangulation $T(D)$.

Corollary 16. *Besides $D_{3,6}$, the dessins d'enfant used in Section 5 are all reflexible. Consequently, besides $T_R(D_{3,6})$, all regular maps constructed in Section 5 – see Theorems 10–14 – are all reflexible.*

7. Concluding remarks

The method presented here constructs certain Sachs triangulations. If a regular triple (x,y,z) of permutations x, y, z is given, one of which has type 1^a2^b, $b>0$, say y, then a dessin d'enfant with $R=x$, $L=y$, and $B=z$ can easily be found.

If all three permutations are not of type 1^a2^b, $b>0$, then a similar construction can be developed with generalized dessins d'enfant. This method will be presented in a further paper.

In this volume, D. S. Archdeacon, P. Gvozdjak, and J. Širáň [1] present a method in constructing regular maps, which is similar to the method presented here. The idea of this construction can be described in our notation as follows: In our construction of $T(x,y)$, two triangles with the same group elements are identified. This occurs if there is an orientation preserving, label preserving homeomorphism of one of the assigned dessins d'enfant onto the other one. In the construction of D. S. Archdeacon, P. Gvozdjak and J. Širáň in [1], only two triangles with the same dessin d'enfant are identified, i.e., in both triangles, the two dessins d'enfant have the labels at the same places. If (D,l) has no orientation preserving, label preserving homeomorphism, then both methods results in the same map. For example, this is the case for all $D_{q,p}$.

We conclude the paper with three open problems.

PROBLEM 1. Let D be a regular map. Is $T_R(D)$ isomorphic with D?

PROBLEM 2. Has each 2-rotary simple triangulation a Sachs labelling?

In all investigated cases such a labelling could be found. For Problem 2, see also Section 1, especially Theorem 1. Problem 2 is also open for the regular triangulation $D_{3,p}$, where p is an even integer.

PROBLEM 3. Is there a Sachs triangulation of genus 2?

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