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CRITERIA OF PROPERTY A FOR THIRD ORDER SUPERLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Some new criteria of property A for superlinear differential equations $y''' + p(t)y' + q(t)|y|^{\alpha} \operatorname{sgn} y = 0$ and $y''' + p(t)y' + q(t)f(y) = 0$ are established. The obtained results extend and improve a sufficient condition for the equation $y''' + p(t)y' + q(t)y^{\alpha} = 0$, where $\alpha > 1$ is a quotient of odd positive integers.

1. Introduction

This paper is concerned with the criteria of property **A** for the third order superlinear differential equations of the form

$$y''' + p(t)y' + q(t)|y|^{\alpha} \operatorname{sgn} y = 0\,, \tag{A}$$

and

$$y''' + p(t)y' + q(t)f(y) = 0\,, \tag{F}$$

where $p\colon I \to (-\infty, 0]$, $q\colon I \to (0,\infty)$, $f\colon \mathbb{R} \to \mathbb{R} = (-\infty,\infty)$, $I = (a,\infty) \subseteq (0,\infty)$ are continuous, $0 < \alpha \in \mathbb{R}$ and $xf(x) > 0$ for $x \neq 0$. In general, we assume that $p(t) \not\equiv 0$ on I.

We restrict our attention to those solutions of equations (A) and (F) which exist on some ray $(t_*,\infty) \subseteq I$ and which are nontrivial in any neighbourhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory.

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DEFINITION 1. *The equation* (F) *is said to have property* **A** *if each solution* u *of that equation is either oscillatory or satisfies conditions:*

There exists a point $T \geq a$ *such that* $(-1)^j u^{(j)}(t) \operatorname{sgn} u(t) > 0$ *for every* $t \geq T$, $j = 0,1,2$, *and*

$$\lim_{t\to\infty} u^{(j)}(t) = 0\,, \qquad j = 0,1,2\,.$$

We shall consider a superlinear case. In the case of equation (A) this means that $\alpha > 1$, that is

$$\int_{\varepsilon}^{\infty} \frac{\mathrm{d}u}{u^{\alpha}} < \infty \qquad \text{for any} \quad \varepsilon > 0\,, \tag{1}$$

and for equation (F) the condition

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{\mathrm{d}u}{f(u)} < \infty \qquad \text{for any} \quad \varepsilon > 0 \tag{2}$$

is required.

In the particular case of equation (A) when $p \equiv 0$ on I the following result holds (see e.g. [6] and [7]):

THEOREM A. *The sufficient and necessary condition for equation* (A) *to have property* **A** *in superlinear case is that*

$$\int^{\infty} t^2 q(t)\, \mathrm{d}t = \infty\,. \tag{3}$$

The authors J. Eliaš [1], L. Erbe [2], J. L. Nelson [8] and P. Šoltés [10] have studied the equation (A) with $\alpha > 1$, where α is a quotient of odd positive integers. Their results may be presented as

THEOREM B. *Suppose that* $p \in C^1(I,\mathbb{R})$. *Let*

$$p \leq 0\,, \quad p' \geq 0 \quad \textit{and} \quad q > 0\,, \quad (p < 0\,, \quad p' \geq 0 \quad \textit{and} \quad q \geq 0) \quad \textit{on} \quad I\,, \tag{4}$$

and

$$\int^{\infty} q(t)\, \mathrm{d}t = \infty\,. \tag{5}$$

Then the equation (A) *has property* **A**.

The purpose of this paper is to establish some new criteria of property **A** for equation (A), which extend and improve Theorem B.

We note that the results here established do not require, on the function p, any monotonicity condition of the form of (4). Furthermore we shall require, on the function q, a weaker condition than the condition (5), sometimes there is needed only the condition (3). So our results can be applied to the cases when the conditions (4) and (5) are not satisfied. In addition we extend these obtained results to equation (F) with additional assumptions of monotonicity on the function f.

In the rest of this paper we suppose that $p(t) \leq 0$ and $q(t) > 0$, $t \in I$ and $xf(x) > 0$ for $x \neq 0$.

2. Lemmas

In this section we present some lemmas requisite to proofs of main results. The proofs of these lemmas may be omitted since they are like or similar to proofs in the references.

Remark 1. If y is a solution of (F), then $z = -y$ is a solution of the equation

$$z''' + p(t)z' + q(t)f_1(z) = 0\,,$$

where $f_1(z) = -f(-z)$ and $zf_1(z) > 0$ for $z \neq 0$. Thus, concerning nonoscillatory solutions of (F) we can restrict our attention only to the positive ones.

LEMMA 1. *There exists a solution y of* (A) ((F)) *with $y \not\equiv 0$ and $y \geq 0$, $y' \leq 0$, $y'' \geq 0$, $t \geq T \geq a$. If $\alpha \geq 1$, then $y > 0$, $y' < 0$, $y'' > 0$, $t \geq T \geq a$.*

Proof. See [1; Theorem 1], [2; Lemma 2.1], [5; Theorem 1.1], [10].

LEMMA 2. *Let y be a nonoscillatory solution of* (A) ((F)), *which exists on I. Then either*

$$y > 0, \quad y' < 0, \quad y'' > 0, \quad y''' < 0 \qquad on \quad I,$$

and (6)

$$\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0\,, \qquad \lim_{t\to\infty} y(t) = L < \infty\,,$$

or there exists $T \in I$ such that

$$y(t) > 0\,, \quad y'(t) \geq 0 \qquad for\ all \quad t \geq T\,. \tag{7}$$

Proof. See [2; Theorem 2.2], [5; Lemma 1.3 and Lemma 2.1].

3. The equation (A)

In this section we shall establish two criteria of property **A** for the equation (A). These criteria extend and improve Theorem B. We recall that we consider a superlinear case, that is, $\alpha > 1$, and so the condition (1) holds.

THEOREM 1. *Let $0 \le \delta \le 2$ be a real number and suppose that*

$$t^{\delta} p(t) \ge -M > -\infty \qquad on \quad I, \tag{8}$$

and

$$\int^{\infty} t^{\delta} q(t)\, \mathrm{d}t = \infty. \tag{9}$$

*Then the equation (A) has property **A**.*

Proof. Let y be a nonoscillatory solution of the equation (A). We first prove that y cannot have property (7). Suppose that there exists $T \in I$, $T \ge a$ such that y satisfies (7). Multiplying (A) by $t^{\delta} y^{-\alpha}(t)$ and integrating by parts from T to $t \ge T$ we obtain

$$\begin{aligned} &\frac{t^{\delta} y''(t)}{y^{\alpha}(t)} - \delta \frac{t^{\delta-1} y'(t)}{y^{\alpha}(t)} + \frac{\alpha}{2} \frac{t^{\delta} y'^2(t)}{y^{\alpha+1}(t)} + \delta(\delta-1) \int_T^t \frac{s^{\delta-2} y'(s)}{y^{\alpha}(s)}\, \mathrm{d}s \\ &\quad + \int_T^t \frac{s^{\delta} p(s) y'(s)}{y^{\alpha}(s)}\, \mathrm{d}s + \frac{1}{2}\alpha(\alpha+1) \int_T^t \frac{s^{\delta} y'^3(s)}{y^{\alpha+2}(s)}\, \mathrm{d}s - \frac{3}{2}\alpha\delta \int_T^t \frac{s^{\delta-1} y'^2(s)}{y^{\alpha+1}(s)}\, \mathrm{d}s \\ &\quad = K - \int_T^t s^{\delta} q(s)\, \mathrm{d}s, \end{aligned} \tag{10}$$

where K is a constant.

From $0 \le \delta \le 2$, (1) and (8) it follows that the first two integrals on the left-hand side of (10) are bounded from below. Using the well-known inequality $Az^2 - Bz \ge -B^2/4A$, $A > 0$ we get

$$\begin{aligned} &\int_T^t \left[\frac{\alpha+1}{2} s^{\delta} \left(\frac{y'(s)}{y(s)} \right)^2 - \frac{3}{2} \delta s^{\delta-1} \frac{y'(s)}{y(s)} \right] \frac{y'(s)}{y^{\alpha}(s)}\, \mathrm{d}s \\ &\qquad \ge -\frac{9\delta^2}{8(\alpha+1)} \int_T^t s^{\delta-2} \frac{y'(s)}{y^{\alpha}(s)}\, \mathrm{d}s > -\infty. \end{aligned}$$

So

$$\frac{t^\delta y''(t)}{y^\alpha(t)} - \delta\frac{t^{\delta-1}y'(t)}{y^\alpha(t)} + \frac{\alpha}{2}\frac{t^\delta y'^2(t)}{y^{\alpha+1}(t)} \le K_1 - \int_T^t s^\delta q(s)\,\mathrm{d}s\,.$$

Integrating the above inequality from T to $t \ge T$ we have

$$\begin{aligned}\frac{t^\delta y'(t)}{y^\alpha(t)} + \frac{3\alpha}{2}\int_T^t \frac{s^\delta y'^2(s)}{y^{\alpha+1}(s)}\,\mathrm{d}s - 2\delta\int_T^t \frac{s^{\delta-1}y'(s)}{y^\alpha(s)}\,\mathrm{d}s& \\ \le K_2 + K_1 t - \int_T^t\int_T^s u^\delta q(u)\,\mathrm{d}u\,\mathrm{d}s\,.&\end{aligned} \tag{11}$$

Since

$$\begin{aligned}&\int_T^t \left[\frac{3\alpha}{2}s^\delta\left(\frac{y'(s)}{y(s)}\right)^2 - 2\delta s^{\delta-1}\frac{y'(s)}{y(s)}\right]y^{1-\alpha}(s)\,\mathrm{d}s \\ &\qquad \ge -\frac{2\delta^2}{3\alpha}\int_T^t s^{\delta-2}y^{1-\alpha}(s)\,\mathrm{d}s \ge -\frac{2\delta^2}{3\alpha}T^{\delta-2}y^{1-\alpha}(T)\int_T^t \mathrm{d}s\,,\end{aligned}$$

from (11) we get

$$\frac{t^\delta y'(t)}{y^\alpha(t)} \le K_4 + K_3 t - \int_T^t\int_T^s u^\delta q(u)\,\mathrm{d}u\,\mathrm{d}s\,.$$

So it follows from (9) that $y' < 0$ for sufficiently large t, hence a contradiction. Therefore the equation (A) cannot have any solution with property (7), hence it follows from Lemma 2 that y has property (6).

Now we prove that $\lim_{t\to\infty} y(t) = 0$. Let y be a (positive) solution with property (6) and $\lim_{t\to\infty} y(t) = L > 0$. From (A) we have

$$t^\delta y'''(t) \le -L^\alpha t^\delta q(t) \qquad \text{for} \quad t > a\,.$$

Integrating the above inequality from a to $t > a$ we get

$$t^\delta y''(t) - \delta t^{\delta-1}y'(t) + \delta(\delta-1)\int_a^t s^{\delta-2}y'(s)\,\mathrm{d}s \le K_5 - L^\alpha\int_a^t s^\delta q(s)\,\mathrm{d}s\,. \tag{12}$$

By the integral condition (9) the right side of (12) tends to $-\infty$ as $t \to \infty$ while all terms on the left side are either positive or bounded. This contradiction proves the theorem.

Remark 2. For $p \equiv 0$ on I it is sufficient to require $\delta = 2$, i.e. the condition (3).

Example 1. Consider a differential equation

$$\begin{gathered} y''' - t^{-2}y' + \beta\big[1 - (\beta - 1)(\beta - 2)\big]t^{\beta(1-\alpha)-3}|y|^{\alpha} \operatorname{sgn} y = 0\,, \\ t \geq a > 0\,, \quad \alpha > 1 \quad \text{and} \quad \beta < 0 \quad \text{or} \quad \beta \in \left(\frac{3-\sqrt{5}}{2}\,,\ \frac{3+\sqrt{5}}{2}\right). \end{gathered} \tag{13}$$

The conditions (4) of Theorem B and (8) of Theorem 1 are satisfied.

Case $\beta < 0$.

The condition (9) of Theorem 1 is satisfied for any $\alpha > 1$ with $\delta = 2$, hence the equation (13) has property $\mathbf{A}$. An example of such solution is $y(t) = t^{\beta}$. Theorem B can be applied only in the case when $\alpha \geq \dfrac{\beta - 2}{\beta}$.

Case $\beta \in \left(\dfrac{3-\sqrt{5}}{2}\,,\ \dfrac{3+\sqrt{5}}{2}\right)$.

The condition (9) is not satisfied. A solution $y(t) = t^{\beta}$ has not property $\mathbf{A}$.

The following theorem concerns the case when condition (8) fails.

LEMMA 3. *Let Q be a polynomial, $Q(z) = Az^3 - Bz + C$, $A > 0$, $B \geq 0$, $z \in \mathbb{R}$. Then*

$$Q(z) \geq C - \frac{2}{3}\left(B^3/3A\right)^{1/2} \qquad \textit{for all} \quad z \geq 0\,. \tag{14}$$

THEOREM 2. *Let $0 \leq \delta \leq 2$. If*

$$\int^{\infty} \Big[q(t) - D\big(-p(t)\big)^{3/2}\Big]t^{\delta}\,\mathrm{d}t = \infty \tag{15}$$

for every positive constant D, then the equation (A) *has property* $\mathbf{A}$.

Proof. Let y be a nonoscillatory solution of (A) and suppose that y

satisfies (7). The identity (10) may be rewritten as

$$\begin{aligned}
&\frac{t^\delta y''(t)}{y^\alpha(t)} - \delta\frac{t^{\delta-1}y'(t)}{y^\alpha(t)} + \frac{\alpha}{2}\frac{t^\delta y'^2(t)}{y^{\alpha+1}(t)} \\
&+ \int_T^t \left[\frac{\alpha^2}{2}\left(y^{\alpha-1}(s)\right)^2\left(\frac{y'(s)}{y^\alpha(s)}\right)^3 + p(s)\frac{y'(s)}{y^\alpha(s)} + q(s)\right] s^\delta \, \mathrm{d}s \\
&+ \delta(\delta-1)\int_T^t \frac{s^{\delta-2}y'(s)}{y^\alpha(s)}\, \mathrm{d}s \\
&+ \frac{\alpha}{2}\int_T^t \left[s^\delta\left(\frac{y'(s)}{y(s)}\right)^2 - 3\delta s^{\delta-1}\frac{y'(s)}{y(s)}\right]\frac{y'(s)}{y^\alpha(s)}\, \mathrm{d}s = K\,.
\end{aligned} \tag{10'}$$

Similarly as in Theorem 1 we can easily prove that the last two integrals on the left-hand side of (10') are bounded from below.

We denote $z(t) = y'(t)/y^\alpha(t)$ for $t \ge T$. By (14) we have

$$\begin{aligned}
&\frac{\alpha^2}{2}\left(y^{\alpha-1}(t)\right)^2 z^3 - \left(-p(t)\right)z + q(t) \ge q(t) - \left(\frac{2}{3}\right)^{3/2}\frac{y^{1-\alpha}(t)}{\alpha}\left(-p(t)\right)^{3/2} \\
&\ge q(t) - \left(\frac{2}{3}\right)^{3/2}\frac{y^{1-\alpha}(T)}{\alpha}\left(-p(t)\right)^{3/2} = q(t) - D_0\left(-p(t)\right)^{3/2} \qquad \text{for all} \quad t \ge T\,.
\end{aligned}$$

After substituting this estimate to (10') we get

$$\begin{aligned}
&\frac{t^\delta y''(t)}{y^\alpha(t)} - \delta\frac{t^{\delta-1}y'(t)}{y^\alpha(t)} + \frac{\alpha}{2}\frac{t^\delta y'^2(t)}{y^{\alpha+1}(t)} \\
&\qquad\qquad \le K_6 - \int_T^t \left[q(s) - D_0\left(-p(s)\right)^{3/2}\right] s^\delta \, \mathrm{d}s\,.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 1, hence it is omitted.

E x a m p l e 2. Consider the differential equation

$$\begin{gathered}
y''' + (3-2t^2)y' + (4t^3-6t)\,\mathrm{e}^{(\alpha-1)t^2}\,|y|^\alpha \operatorname{sgn} y = 0\,, \\
\alpha > 1\,, \quad t > (3/2)^{1/2}\,.
\end{gathered} \tag{16}$$

The conditions of Theorem 2 are satisfied, hence equation (16) has property **A**. An example of such a solution is $y(t) = \mathrm{e}^{-t^2}$. Theorem B cannot be applied to equation (16) since $p'(t) < 0$ on I.

R e m a r k 3. Theorem 2 can be applied to the equation (13), too.

4. The equation (F)

In order to obtain results for superlinear equation (F) similar to those in section 3, we assume that $f \in C^2(\mathbb{R},\mathbb{R})$ and that for all $|u_0| > 0$ there exist constants $K_0 = K_0(u_0) > 0$ and $k_0 = k_0(u_0) > 0$ such that

$$\begin{aligned} &f'(u) \ge k_0\,, \quad 2f'^2(u) - f(u)f''(u) > 0\,, && \text{and} \\ &f'^2(u)/\left[2f'^2(u) - f(u)f''(u)\right] \le K_0\,, && \text{for all} \quad |u| \ge |u_0|\,. \end{aligned} \tag{17}$$

Example 3. The functions f_1, f_2 and $f_3 \colon \mathbb{R} \mapsto \mathbb{R}$ satisfy conditions (2) and (17), where

$$\begin{aligned} f_1(u) &= |u|^\alpha \operatorname{sgn} u\,, && \alpha > 1\,, \\ f_2(u) &= \frac{|u|^{2\alpha} \operatorname{sgn} u}{1 + |u|^\alpha}\,, && \alpha > 1\,, \\ f_3(u) &= \sinh u\,. \end{aligned}$$

THEOREM 3. *Let conditions* (2), (8), (9) *and* (17) *hold. Then the equation* (F) *has property* **A**.

Proof. The proof is similar to the proof of Theorem 1. Let y be a nonoscillatory solution of (F) and suppose that y satisfies (7). Multiplying (F) by $t^\delta / f(y(t))$ and integrating by parts from T to $t \ge T$ we obtain

$$\begin{aligned} &\frac{t^\delta y''(t)}{f(y(t))} - \delta \frac{t^{\delta-1} y'(t)}{f(y(t))} + \frac{t^\delta f'(y(t)) y'^2(t)}{2 f^2(y(t))} \\ &\qquad + \delta(\delta - 1) \int_T^t \frac{s^{\delta-2} y'(s)}{f(y(s))}\, \mathrm{d}s + \int_T^t \frac{s^\delta p(s) y'(s)}{f(y(s))}\, \mathrm{d}s \\ &\qquad + \frac{1}{2} \int_T^t \left[\frac{2f'^2(y) - f(y)f''(y)}{f^3(y(s))} s^\delta y'^3(s) - 3\delta \frac{s^{\delta-1} f'(y)}{f^2(y(s))} y'^2(s) \right] \mathrm{d}s \\ &\qquad\qquad = K - \int_T^t s^\delta q(s)\, \mathrm{d}s\,, \end{aligned} \tag{18}$$

where K is a constant. Since

$$\int_T^t \left[[2f'^2(y) - f(y)f''(y)]\, s^\delta \left(\frac{y'(s)}{f(y)}\right)^2 - 3\delta s^{\delta-1} f'(y)\frac{y'(s)}{f(y)} \right] \frac{y'(s)}{f(y)}\, \mathrm{d}s$$

$$\geq -\frac{9}{4}\delta^2 \int_T^t \frac{f'^2}{2f'^2 - ff''} s^{\delta-2} \frac{y'}{f(y)}\, \mathrm{d}s \geq -\frac{9}{4}\delta^2 K_0 \int_T^t s^{\delta-2} \frac{y'(s)}{f(y(s))}\, \mathrm{d}s > -\infty\,,$$

it follows from (2), (8) and (17) that all integrals on the left-hand side of (18) are bounded from below, hence we have

$$\frac{t^\delta y''(t)}{f(y(t))} - \delta \frac{t^{\delta-1} y'(t)}{f(y(t))} + \frac{t^\delta f'(y(t)) y'^2(t)}{2f^2(y(t))} \leq K_1 - \int_T^t s^\delta q(s)\, \mathrm{d}s\,.$$

Integrating the above inequality from T to $t \geq T$ we obtain

$$\frac{t^\delta y'(t)}{f(y(t))} + \int_T^t \left[\frac{3s^\delta f'(y(s)) y'^2(s)}{2f^2(y(s))} - 2\delta \frac{s^{\delta-1} y'(s)}{f(y(s))} \right] \mathrm{d}s$$
$$\leq K_2 + K_1 t - \int_T^t \int_T^s u^\delta q(u)\, \mathrm{d}u\, \mathrm{d}s\,. \tag{19}$$

Since

$$\int_T^t \left[\frac{3}{2} s^\delta f'(y) \left(\frac{y'(s)}{f(y)}\right)^2 - 2\delta s^{\delta-1} \frac{y'(s)}{f(y)} \right] \mathrm{d}s$$

$$\geq -\frac{2}{3}\delta^2 \int_T^t \frac{s^{\delta-2}}{f'(y)}\, \mathrm{d}s \geq -\frac{2}{3}\delta^2 \frac{T^{\delta-2}}{k_0} \int_T^t \mathrm{d}s\,,$$

from (19) we get

$$\frac{t^\delta y'(t)}{f(y(t))} \leq K_4 + K_3 t - \int_T^t \int_T^s u^\delta q(u)\, \mathrm{d}u\, \mathrm{d}s\,.$$

The rest of the proof is similar to that of Theorem 1 and hence is omitted.

Example 4. Consider the differential equation

$$y''' - \frac{1}{t^2}y' + \frac{5}{t^4 \sinh \frac{1}{t}} \sinh y = 0\,, \qquad t \ge a > 0\,. \tag{20}$$

The conditions of Theorem 3 (with $\delta = 2$) are satisfied, hence equation (20) has property $\mathbf{A}$. An example of such a solution is $y(t) = 1/t$.

Remark 4. In the case when $p \equiv 0$ on I the condition (3) is sufficient for equation (A) to have property $\mathbf{A}$. But in this case the function f need not be twice differentiable. For example, if f is nondecreasing and there exists a constant $C > 0$ such that $|f(uv)| \ge Cf(u)|f(v)|$ for $u \ge 0$, $v \in \mathbb{R}$, then the condition (3) and

$$\int^{\infty} tq(t)f(t)\ \mathrm{d}t = \infty$$

is sufficient for superlinear equation (F) to have property $\mathbf{A}$, see e.g. Theorem 1 in [4] or Corollary 1 in [9].

When the function $f(u)$ does not satisfy either monotonicity conditions or hypotheses for large values of u we refer to [3].

Example 5. The differential equation

$$y''' - \frac{3}{t^2}y' + \frac{2}{t^2(t^2-1)} \sinh y = 0\,, \qquad t \ge a > 1$$

has a nonoscillatory solution $y(t) = \ln t$ which has not property $\mathbf{A}$. All the conditions of Theorem 3 are satisfied except the condition (9).

The last theorem concerns the case when condition (8) fails.

Suppose that $f \in C^2(\mathbb{R},\mathbb{R})$ and that for all $|u_0| > 0$ there exist constants $K_0 = K_0(u_0) > 0$, $k_0 = k_0(u_0) > 0$ and $k_0' = k_0'(u_0) > 0$ such that

$$\begin{aligned} &f'(u) \ge k_0\,, \quad Af'^2(u) - f(u)f''(u) > 0 && \text{and} \\ &f'^2(u)/\big[Af'^2(u) - f(u)f''(u)\big] \le K_0 && \text{or} \\ &Af'^2(u) - f(u)f''(u) \ge k_0' \end{aligned} \tag{21}$$

for all $|u| \ge |u_0|$, where A is some constant, $0 < A < 2$.

Remark 5. The functions f_1, f_2 and f_3 from Example 3 satisfy the condition (21).

THEOREM 4. *Let conditions* (2), (15) *and* (21) *hold. Then the equation* (F) *has property* **A**.

Proof. Let y be a nonoscillatory solution of (F). Suppose that y satisfies (7). The identity (18) may be rewritten as

$$\begin{aligned}
&\frac{t^{\delta}y''(t)}{f(y(t))} - \delta\frac{t^{\delta-1}y'(t)}{f(y(t))} + \frac{t^{\delta}f'(y(t))y'^{2}(t)}{2f^{2}(y(t))} \\
&\quad + \int_{T}^{t}\left[\frac{Af'^{2}-ff''}{2}s^{\delta}\left(\frac{y'(s)}{f(y)}\right)^{2} - \frac{3}{2}\delta s^{\delta-1}f'(y)\frac{y'(s)}{f(y)}\right]\frac{y'(s)}{f(y)}\,\mathrm{d}s \\
&\quad + \delta(\delta-1)\int_{T}^{t}\frac{s^{\delta-2}y'(s)}{f(y(s))}\,\mathrm{d}s \qquad\qquad (18') \\
&\quad + \int_{T}^{t}\left[\frac{B}{2}f'^{2}(y)\left(\frac{y'(s)}{f(y)}\right)^{3} + p(s)\frac{y'(s)}{f(y)} + q(s)\right]s^{\delta}\,\mathrm{d}s = K,
\end{aligned}$$

or

$$\begin{aligned}
&\frac{t^{\delta}y''(t)}{f(y(t))} - \delta\frac{t^{\delta-1}y'(t)}{f(y(t))} + \frac{t^{\delta}f'(y(t))y'^{2}(t)}{2f^{2}(y(t))} \\
&\quad + \int_{T}^{t}\left[\frac{B}{2}f'^{2}(y)s^{\delta}\left(\frac{y'(s)}{f(y)}\right)^{2} - \frac{3}{2}\delta s^{\delta-1}f'(y)\frac{y'(s)}{f(y)}\right]\frac{y'(s)}{f(y)}\,\mathrm{d}s \\
&\quad + \delta(\delta-1)\int_{T}^{t}\frac{s^{\delta-2}y'(s)}{f(y(s))}\,\mathrm{d}s \qquad\qquad (18'') \\
&\quad + \int_{T}^{t}\left[\frac{Af'^{2}-ff''}{2}\left(\frac{y'(s)}{f(y)}\right)^{3} + p(s)\frac{y'(s)}{f(y)} + q(s)\right]s^{\delta}\,\mathrm{d}s = K,
\end{aligned}$$

where B is a constant, $B = 2 - A$.

From $0 \le \delta \le 2$, (2) and (21) it follows that the first two integrals on the left-hand side of (18') or (18") are bounded from below. If we denote $z(t) = y'(t)/f(y(t))$ for all $t \ge T$ and we use (14), then from (18') or (18") we get

$$\frac{t^\delta y''(t)}{f(y(t))} - \delta\frac{t^{\delta-1}y'(t)}{f(y(t))} + \frac{t^\delta f'(y(t))y'^2(t)}{2f^2(y(t))} \le K_6 - \int\limits_T^t \Big[q(s) - D_0\big(-p(s)\big)^{3/2}\Big] s^\delta \, \mathrm{d}s \,.$$

The rest of proof is similar to these of previous theorems and hence is omitted.

Example 6. Consider the differential equation

$$y''' + (3-2t^2)y' + (4t^3-6t)\big(\mathrm{e}^{5t^2} + \mathrm{e}^{2t^2}\big)\frac{y^6 \operatorname{sgn} y}{1+|y|^3} = 0\,, \qquad t > \sqrt{3/2}\,.$$

The conditions of Theorem 4 are satisfied. Hence this equation has property $\mathbf{A}$. An example of such solution is $y(t) = \mathrm{e}^{-t^2}$.

Remark 6. Theorem 4 cannot be applied to equation (20) since the condition (15) is not satisfied.

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