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A PROPERTY OF CONNECTIONS OF MECHANICAL SYSTEMS OF HIGHER ORDER

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ABSTRACT. In the case of regular Lagrangian L of order 1 on the tangent bundle TM of a smooth manifold M it is known that its mechanical system S_L satisfying the fundamental equation of the Lagrangian formalism $i_S \, \mathrm{d}\omega_L = -\, \mathrm{d}E$ determines a connection Γ which is Lagrangian, i.e. $\mathrm{d}\omega_L(X,Y) = 0$ for all Γ-horizontal vectors X,Y on TM. In this paper the form of this property is studied in the case of higher order tangent bundles.

Let T^rM be the space of all r-jets from $\mathbb{R}$ into a smooth manifold M with source $0 \in \mathbb{R}$. Let L be a real smooth function on T^rM. Roughly speaking, the main idea of the Lagrangian formalism in the classical mechanics of order r consists in a construction of a symplectic structure $(T^{2r-1}M, \mathrm{d}\omega_L)$ and a mechanical system S_L of L on $T^{2r-1}M$, the integral curves of which satisfy the Euler differential equation of order $2r$, see [6], if $r = 1$ and [8] in the case of higher order.

In [3] we constructed connections which are determined by a semispray S on T^rM, see also [1]. One of them given by the Lie derivation $L_S J_1$ of the canonical morphism J_1 on T^rM with respect to S was studied by a number of authors first of all in the case when S is a mechanical system of a regular Lagrangian L on TM, [2, 4, 7]. For example, [4] showed that the connection $\Gamma = L_S J_1$ is Lagrangian, i.e. $\Gamma = \mathrm{Orth}_{\mathrm{d}\omega_L} \Gamma$, i.e. $\mathrm{d}\omega_L(X,Y) = 0$ for all Γ-horizontal vectors X, Y on TM. In this paper we specify this property in the case of higher order. All manifolds and mappings will be smooth.

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Introduction

Throughout this paper $Tf\colon TM \to TN$ denotes the differential of a map $f\colon M \to N$. We recall some canonical properties of the manifold T^rM. Let (x^i) be a chart on M. Then

$$h = j_0^r(\gamma\colon \mathbb{R} \to M) = \left(x_0^i = \gamma^i(0),\, x_1^i = \frac{\mathrm{d}\gamma^i(0)}{\mathrm{d}t}, \dots, x_r^i = \frac{\mathrm{d}^r\gamma^i(0)}{\mathrm{d}t^r}\right)$$

determines the induced chart on T^rM. Let $\pi_k^r\colon T^rM \to T^kM$,

$$(x_0^i, \dots, x_k^i, \dots, x_r^i) \mapsto (x_0^i, \dots, x_k^i)$$

denote the canonical projection of r-jets onto their k-subjets. Then (π_k^r) or $V\pi_k^r$ is the abbreviated notation for the fibre manifold $\pi_k^r\colon T^rM \to T^kM$ or for the vector bundle of all π_k^r-vertical vectors on (π_k^r), respectively.

There are the canonical vector fibre morphisms

$$J_1 = \sum_{p=1}^{r} p\,\mathrm{d}x_{p-1}^i \otimes \partial/\partial x_p^i\,, \quad J_k = \frac{1}{k!}J_1^k = \sum_{u=k}^{r}\binom{u}{k}\mathrm{d}x_{u-k}^i \otimes \partial/\partial x_u^i$$

and the canonical vector fields

$$C_1 = \sum_{p=1}^{r} p x_p^i \partial/\partial x_p^i\,, \quad C_k = \frac{1}{k!}J_1^{k-1}C_1 = \sum_{u=k}^{r}\binom{u}{k} x_{u-k+1}^i \partial/\partial x_u^i$$

on T^rM, where $k = 2, \dots, r$. Readers are referred for constructions of these objects to [8] or [3].

Recall the following embedding $i_r\colon T^{r+1}M \to TT^rM$,

$$h = j_0^{r+1}\gamma(t) \mapsto j_0^1\big(t \mapsto j_{s=0}^r\gamma(t+s)\big)\,,$$
$$(x_0^i, \dots, x_{r+1}^i) \mapsto (x_0^i, \dots, x_r^i, x_1^i, \dots, x_{r+1}^i)\,.$$

By [8], a semispray on T^rM (a differential equation of order $r+1$), is a vector field S on T^rM such that $J_1S = C_1$, i.e. S is of the expression

$$S = \sum_{j=0}^{r-1} x_{j+1}^i \partial/\partial x_j^i + b^i(x_0, \dots, x_r^i)\,\partial/\partial x_r^i\,.$$

It is clear that $S\colon T^rM \to TT^rM$ determines a unique section $\bar{S}\colon T^rM \to T^{r+1}M$ such that $S = i_r \cdot \bar{S}$, $\bar{S}(x_0^i, \dots, x_r^i) = (x_0^i, \dots, x_r^i, b^i)$. Then $S \mapsto \bar{S}$ is a bijection between the set of all semisprays on $T^r M$ and the set of all sections of the fibred manifold (π_r^{r+1}).

To construct connections from a semispray S we introduced in [3] a map $\tau_S\colon V\pi_0^r \to TT^r M$ as follows. Let J_1 be the canonical vector bundle morphism on $T^{r+1}M$ and $h \in T^{r+1}M$, $u = \pi_r^{r+1}h$. Let $J_1^h\colon T_u T^r M \to (V\pi_0^{r+1})_h$ denote a vector morphism such that $J_1^h(Y) = J_1(Z)$, where $Z \in T_h T^{r+1}M$ and $T\pi_r^{r+1}(Z) = Y$. Then we define a vector bundle morphism $\bar{J}_1\colon V\pi_0^{r+1} \to TT^r M$ such that for $W \in \left(V\pi_0^{r+1}\right)_h$ there holds $\bar{J}_1(W) = Y \in T_u T^r M$, where $J_1^h(Y) = W$. It is clear that

$$\bar{J}_1(x_0^i, \dots, x_{r+1}^i, 0, c_1^i, \dots, c_{r+1}^i) = (x_0^i, \dots, x_r^i, c_1^i, \frac{1}{2}c_2^i, \dots, \frac{1}{r+1}c_{r+1}^i).$$

Let $\bar{S}\colon T^r M \to T^{r+1}M$ be a semispray. Using the restriction of $T\bar{S}$ on $V\pi_0^r$ we set

$$\tau_S = \bar{J}_1 \cdot T\bar{S}|_{V\pi_0^r} = \sum_{s=1}^{r} \frac{1}{s}\,\mathrm{d}x_s^i \otimes \partial/\partial x_{s-1}^i + \frac{1}{r+1}\sum_{p=1}^{r} \frac{\partial b^i}{\partial x_p^j}\,\mathrm{d}x_p^j \otimes \partial/\partial x_r^i .$$

Recall some needed facts about connections on a fibred manifold $\pi\colon Y \to M$. A 1-form ω on Y is said to be π-semibasic if $\omega(X) = 0$ for any $X \in V\pi$. A connection Γ on Y is determined by its horizontal projection h_Γ that is a π-semibasic tangent value 1-form on Y such that $T\pi\cdot h(X) = T\pi(X)$. In a chart (x^i, y^α) on Y $h_\Gamma = \mathrm{d}x^i \otimes \partial/\partial x^i + \Gamma_i^\alpha\,\mathrm{d}x^i \otimes \partial/\partial y^\alpha$. Then $v_\Gamma = \mathrm{id}_{TY} - h_\Gamma = (\mathrm{d}y^\alpha - \Gamma_i^\alpha\,\mathrm{d}x^i) \otimes \partial/\partial y^\alpha$ is the vertical projection of Γ and $H\Gamma$ denotes the vector fibred manifold of all Γ-horizontal vectors $X \in \operatorname{Im} h = \operatorname{Ker} v_\Gamma$. It is obvious that if h_Γ is the horizontal projection of a given connection Γ on Y and $\varphi\colon Y \to T^*M \otimes VY$ is a π-semibasic $V\pi$-value 1-form on Y, then $h_\Gamma + \varphi$ is the horizontal projection of the other connection on Y denoted by $\Gamma + \varphi$.

Let J_k or S be the canonical vector bundle morphism or a semispray on $T^r M$, respectively. In [3] we showed that $k!\,\tau_S^k \cdot J_k$ is the horizontal projection of a connection ${}^{r-k}\Gamma_S$ on the fibre manifold (π_{r-k}^r). We are interested in the connections ${}^{r-1}\Gamma_S$ and ${}^0\Gamma_S$. Their coordinate expressions are as follows

$${}^{r-1}\Gamma_S\colon \tau_S J_1 = \mathrm{d}x_0^i \otimes \partial/\partial x_0^i + \dots + \mathrm{d}x_{r-1}^i \otimes \partial/\partial x_{r-1}^i + \frac{1}{r+1}\sum_{p=1}^{r} \frac{\partial b^i}{\partial x_p^j}\, p\,\mathrm{d}x_{p-1}^j \otimes \partial/\partial x_r^i , \tag{1}$$

$${}^0\Gamma_S\colon r!\,\tau_S^r J_r = \mathrm{d}x_0^i \otimes \partial/\partial x_0^i + r!\sum_{s=1}^{r} B_{r,s}^i \partial/\partial x_s^i , \tag{2}$$

where

$$B^i_{n,s} = \frac{1}{r-n+s+1} \cdots \frac{1}{r+1} A^i_s \,, \qquad A^i_1 = \frac{\partial b^i}{\partial x^j_r} \, \mathrm{d}x^j_0$$

and for $s = 2, \dots, r$

$$A^i_s = \frac{\partial b^i}{\partial x^j_{r-s+1}} \frac{1}{r-s+2} \cdots \frac{1}{r} \, \mathrm{d}x^j_0 + \sum_{j=1}^{s-1} \frac{\partial b^i}{\partial x^u_{r-s+1+j}} B^u_{s-1,j} \,.$$

Remark. The connection ${}^{r-1}\Gamma_S$ coincides with the one determined by $L_S J_1$, see [8]. Other connections on (π^r_{r-k}) can be constructed from S by π^r_{r-k}-semibasic $V\pi^r_{r-k}$-value 1-forms on $T^r M$. For example such forms are $J_k + L_S J_{k+1}, \dots, J_{r-1} + L_S J_r$. We refer to [3] for details.

Connections induced by mechanical systems of higher order

First we will give a brief survey of the main classical mechanics notions on a smooth manifold M. We refer to [8] for detailed information on higher order mechanics.

Every canonical morphism J_s on $T^r M$ determines a derivative of first order $\mathrm{d}_{J_s} = [i_{J_s}, \mathrm{d}] = i_{J_s}\mathrm{d} - \mathrm{d}\, i_{J_s}$, where d denotes the standard exterior derivative and

$$i_{J_s}\omega(X_1, \dots, X_p) = \sum_{j=1}^{p} \omega\big(X_1, \dots, J_s(X_j), \dots, X_p\big) \,.$$

Let $f \colon T^k_1 M \to \mathbb{R}$ be a smooth function. Regard $\mathrm{d}f$ as a real function on $TT^k M$. Let $i_k \colon T^{k+1} M \to TT^k M$ be the above recalled canonical embedding. Then $\mathrm{d}_T f = i^*_k(\mathrm{d}f) = \sum_{p=0}^{k} f_{x^i_p} x^i_{p+1}$ is a function on $T^{k+1}M$, where we introduced the notation $f_{x^i_p} := \dfrac{f}{\partial x^i_p}$ for further use. It is clear that d_T is a derivation operator which can be extended to a derivative (due to Tulczyjew, [9]) of order 0 commutative with d in the algebra λ that is the quotient set of $\bigcup_k \lambda(T^r M)$ by the equivalence relation according to which two forms $\alpha \in \lambda(T^k M)$, $\beta \in \lambda(T^j M)$, $k \geq j$, are equivalent if $\alpha = (\pi^k_j)^* \beta$, see [8].

Let $L \colon T^r M \to \mathbb{R}$ be a real function (Lagrangian of order r). Then the mechanical system of L is a vector field S_L on $T^{2r-1}M$ such that

$$i_{S_L} \, \mathrm{d}\omega_L = -\,\mathrm{d}E \,, \tag{3}$$

where

$$\omega_L = \sum_{i=1}^{r} (-1)^i \, \mathrm{d}_T^{i-1} \, \mathrm{d}_{J_i} L \,, \qquad E = \sum_{i=1}^{r} (-1)^i \, \mathrm{d}_T^{i-1} \, C_i(L) + L \,.$$

By induction it can be checked that

$$\omega_L = \sum_{p=1}^{r} R_i^p \, \mathrm{d}x_{p-1}^i \,, \qquad E = \sum_{p=1}^{r} R_i^p x_p^i + L \,, \tag{4}$$

where

$$R_i^p = \sum_{q=0}^{r-p} (-1)^{q+1} \, \mathrm{d}_T^q \, L_{x_{p+q}^i} \,, \qquad p = 1, \dots, r \,, \tag{5}$$

$$\mathrm{d}_T^q \, L_{x_s^i} = \sum_{n=1}^{q} L^i_{x_s} x_{p_1}^{i_1} \dots x_{p_n}^{i_n} \sum_{t_1 + \dots + t_u = q} A_{t_1 \dots t_u} x_{p_1 + t_1}^{i_1} \dots x_{p_u + t_u}^{i_n} \,,$$

where $t_1 \le t_2 \le \dots \le t_u$ are positive integers and

$$A_{(t_{11} \dots t_{1s_1}) \dots (t_{k1} \dots t_{ks_k})} = \frac{q!}{(t_{11}!)^{s_1} s_1! \dots (t_{k1}!)^{s_k} s_k!} \,,$$
$$t_{j1} = t_{j2} = \dots = t_{js_j} \,.$$

Let us emphasize that R_i^p is a local function of variables $x_0^i, \dots, x_{2r-p}^i$.

LEMMA 1. *A semispray* $S = \sum_{s=0}^{2r-2} x_{s+1}^i \partial / \partial x_s^i + b^i \partial / \partial x_{2r-1}^i$ *is a solution of the equation* $i_S \, \mathrm{d}\omega_L = - \mathrm{d}E$ *if and only if* $\mathrm{d}R_i^1(S) = -L_{x_0^i}$.

Proof.
$\mathrm{d}\omega_L = \sum_{p=1}^{r} \mathrm{d}R_i^p \wedge \mathrm{d}x_{p-1}^i \,, \quad \mathrm{d}E = \sum_{p=1}^{r} \left(x_p^i \, \mathrm{d}R_i^p + R_i^p \, \mathrm{d}x_p^i \right) + \sum_{p=0}^{r} L_{x_p^i} \, \mathrm{d}x_p^i \,.$
Then the equation $i_S \, \mathrm{d}\omega_L = - \mathrm{d}E$ is of the form

$$\sum_{p=0}^{r-1} \mathrm{d}R_i^{p+1}(S) \, \mathrm{d}x_p^i + \sum_{p=1}^{r} R_i^p \, \mathrm{d}x_p^i + \sum_{p=0}^{r} L_{x_p^i} \, \mathrm{d}x_p^i = 0$$

that is satisfied if and only if

$$\mathrm{d}R_i^1(S) + L_{x_0^i} = 0 \,, \tag{6}$$

$$\mathrm{d}R_i^{p+1}(S) + R_i^p + L_{x_p^i} = 0 \,, \qquad p = 1, \dots, r-1 \,, \tag{7}$$

$$R_i^r + L_{x_r^i} = 0 \,. \tag{8}$$

According to (5) the relation (8) is correct. It is easy to see that $\mathrm{d}_T R_i^{p+1} = -R_i^p - L_{x_p^i}$. Then (7) is satisfied if and only if $\mathrm{d}R_i^{p+1}(S) = \mathrm{d}_T R_i^{p+1}$. This is right since

$$\mathrm{d}R_i^{p+1} = \sum_{q=0}^{r-p-1} (-1)^{q+1} \mathrm{d}_T^q \left(\sum_{v=1}^{r} L_{x_{p+q+1}^i x_v^j} \, \mathrm{d}x_v^j \right).$$

Lemma 1 is proved.

The form ω_L being a π_{r-1}^{2r-1}-semibasic 1-form on $T^{2r-1}M$ determines a fibre morphism $\mathcal{L}_L \colon T^{2r-1}M \to T^*T^{r-1}M$ over $\mathrm{id}_{T^{r-1}M}$ such that $\mathcal{L}_L(h)(Y) = \omega_L(X)$, where $X \in T_h T^{2r-1}M$ and $T\pi_{r-1}^{2r-1}(X) = Y$. In the induced chart $(x_0^i, \dots, x_{r-1}^i, z_i^0, \dots, z_i^{r-1})$ on $T^*T^{r-1}M$ the map $\mathcal{L}_L$ (the Legendre transformation of L) is of the form

$$\bar{x}_{p-1}^i = x_{p-1}^i , \qquad z_i^{p-1} = R_i^p(x_0^i, \dots, x_{2r-p}^i) .$$

Then adding the equations

$$\mathrm{d}\bar{x}_{p-1}^i = \mathrm{d}x_{p-1}^i , \tag{9}$$

$$\mathrm{d}z_i^{p-1} = \mathrm{d}R_i^p = \sum_{u=0}^{2r-p-1} \frac{\partial R_i^p}{\partial x_u^j} \mathrm{d}x_u^j + (-1)^{r-p+1} L_{x_r^i x_r^j} \, \mathrm{d}x_{2r-p}^j ,$$

which follow from (5) we get the tangent prolongation $T\mathcal{L}_L$ of the Legendre transformation of L.

Denote by $p_{2r-2}^{2r-1}, \dots, p_0^{2r-1}$ the following submersions from $T^*T^{r-1}M$:

$p_s^{2r-1} \colon T^*T^{r-1}M \to \left(V\pi_{2r-2-s}^{r-1}\right)^*$,

$$p_s^{2r-1}(h) = h|_{V\pi_{2r-2-s}^{r-1}} = (x_0^i, \dots, x_{r-1}^i, z_i^{2r-1-s}, \dots, z_i^{r-1}),$$
$$s = r, \dots, 2r-2 .$$

$p_{r-1}^{2r-1} \colon T^*T^{r-1}M \to T^{r-1}M$,

$$p_{r-1}^{2r-1}(h) = (x_0^i, \dots, x_{r-1}^i) .$$

$p_k^{2r-1} = \pi_k^{r-1} \cdot p_{r-1}^{2r-1} \colon T^*T^{r-1}M \to T^kM , \qquad k = 0, \dots, r-2 .$

The equations (9) immediately give

Lemma 2. *Let L be a Lagrangian of order r. Then the Legendre transformation $\mathcal{L}_L\colon T^{2r-1}M \to T^*T^{r-1}M$ is a fibre morphism from (π_u^{2r-1}) into (p_u^{2r-1}) for $u = 0, \dots, 2r-2$.*

Let us recall some notions of geometry on $T^*T^{r-1}M$. There is the canonical Liouville from $\lambda = \sum_{p=0}^{r-1} z_i^p \, \mathrm{d}x_p^i$ on $T^*T^{r-1}M$ that is an 1-form such that $\lambda(X) = z\big(Tp_{r-1}^{2r-1}X\big)$, $X \in T_z(T^*T^{r-1}M)$. Then $\mathrm{d}\lambda = \sum_{p=0}^{r-1} \mathrm{d}z_i^p \wedge \mathrm{d}x_p^i$ is the canonical symplectic form.

Let K be a given connection on (p_{2r-2}^{2r-1}) having the horizontal projection of the form

$$\begin{aligned} h_K = \sum_{p=0}^{r-1} \mathrm{d}x_p^i \otimes \partial/\partial x_p^i + \sum_{p=1}^{r-1} \mathrm{d}z_i^p \otimes \partial/\partial z_i^p \\ + \left(\sum_{p=1}^{r-1} K_{pi}^j \, \mathrm{d}z_j^p + \sum_{p=0}^{r-1} \bar{K}_{ij}^p \, \mathrm{d}x_p^j \right) \otimes \partial/\partial z_i^0 . \end{aligned} \tag{10}$$

We are interested in the connection $\mathrm{Orth}_{\mathrm{d}\lambda} K$, the horizontal vectors X of which are orthogonal to all K-horizontal vectors Y on $T^*T^{r-1}M$ according to the form $\mathrm{d}\lambda$, i.e. $\mathrm{d}\lambda(X,Y) = 0$. It is easy to verify that the connection $\mathrm{Orth}_{\mathrm{d}\lambda} K$ is a connection on (p_0^{2r-1}) with the horizontal projection

$$\begin{aligned} h_{\mathrm{Orth}_{\mathrm{d}\lambda} K} = \mathrm{d}x_0^i \otimes \partial/\partial x_0^i - \sum_{p=1}^{r-1} K_{pj}^i \, \mathrm{d}x_0^j \otimes \partial/\partial x_p^i \\ + \sum_{p=0}^{r-1} \bar{K}_{ji}^p \, \mathrm{d}\bar{x}_0^j \otimes \partial/\partial z_i^p . \end{aligned} \tag{11}$$

Return to the fundamental equation (3) of the Lagrangian formalism. Recall that the 2-form $\mathrm{d}\omega_L = \sum_{p=1}^{r} \mathrm{d}R_i^p \wedge \mathrm{d}x_{p-1}^i$ is symplectic if and only if the forms $\mathrm{d}R_i^p$, $\mathrm{d}x_{p-1}^i$, $p = 1, \dots, r$, are independent in any induced chart. Then by virtue of the relation (9) $\mathrm{d}\omega_L$ is symplectic if and only if $\det\big(L_{x_r^i x_r^j}\big) \neq 0$. In this case the Lagrangian L is called regular.

In what follows, we shall consider the Lagrangian L to be regular. Now the equation (3) $i_X \, \mathrm{d}\omega_L = -\mathrm{d}E$ has a unique solution $X = S_L$ called the mechanical system of the Lagrangian L. By Lemma 1 S_L is the semispray on $T^{2r-1}M$ which satisfies the equation (6), i.e.

$$b^i = (-1)^{r+1}\tilde{L}^{is}\left(\sum_{u=0}^{2r-2}\frac{\partial R^1_s}{\partial x^j_u}x^j_{u+1} + Lx^s_0\right), \qquad L_{x^i_r x^s_r}\tilde{L}^{sj} = \delta^j_i . \tag{12}$$

We can find relations between the connections ${}^{2r-2}\Gamma_S$ and ${}^0\Gamma_S$ on $T^{2r-1}M$ determined by the mechanical system $S = S_L$. In this case the connection $\mathrm{Orth}_{\mathrm{d}\omega_L}\,{}^{2r-2}\Gamma_S$ which is $\mathrm{d}\omega_L$-orthogonal to ${}^{2r-2}\Gamma_S$ is a connection on (π^{2r-1}_0). To specify some properties of the connections ${}^0\Gamma_S$ and $\mathrm{Orth}_{\mathrm{d}\omega_L}\,{}^{2r-2}\Gamma_S$ we use the Legendre transformation $\mathcal{L}_L$. By (9) it is clear that $\mathcal{L}_L$ is a local symplectic isomorphism of the symplectic manifolds $(T^{2r-1}M, \mathrm{d}\omega_L)$ and $(T^*T^{r-1}M, \mathrm{d}\lambda)$.

Let K and ${}^0H$ be the connections on $T^*T^{r-1}M$, which are the images of the connections ${}^{2r-2}\Gamma_S$ and ${}^0\Gamma_S$ under the Legendre transformation $\mathcal{L}_L$. Recall that the difference ${}^0H - {}^0\bar{H}$ of two connections on (p^{2r-1}_0) is a p^{2r-1}_0-semibasic Vp^{2r-1}_0-value form on $T^*T^{r-1}M$. We will deduce that the form ${}^0H$-$\mathrm{Orth}_{\mathrm{d}\lambda}K$ is a p^{2r-1}_0-semibasic Vp^{2r-1}_1-value 1-form on $T^*T^{r-1}M$.

Let (10) be the expression of the horizontal projection of K. Let

$$O_H = \mathrm{d}x^0_0 \otimes \partial/\partial x^i_0 + \sum_{p=1}^{r-1} H^i_{pj}\,\mathrm{d}x^j_0 \otimes \partial/\partial x^i_p + \sum_{p=1}^{r-1} \bar{H}^p_{ij}\,\mathrm{d}x^j_0 \otimes \partial/\partial z^p_i$$

be the horizontal projection of the connection ${}^0H$. Then by (11) the form ${}^0H$-$\mathrm{Orth}_{\mathrm{d}\lambda}K$ is a p^{2r-1}_0-semibasic and just Vp^{2r-1}_1-value form if $H^i_{1j} = -K^i_{1j}$ and $H^i_{2j} \neq -K^i_{2j}$. To prove these relations we will find the local functions H^i_{1j}, K^i_{1j}, H^i_{2j}, K^i_{2j}.

By (1) the horizontal projection of the connection ${}^{2r-2}\Gamma_{S_L}$ is

$$\sum_{p=0}^{2r-2}\mathrm{d}x^i_p \otimes \partial/\partial x^i_p + \frac{1}{2r}\sum_{p=1}^{2r-1}\frac{\partial b^i}{\partial x^j_p}\,p\,\mathrm{d}x^j_{p-1} \otimes \partial/\partial x^i_{2r-1},$$

where the local functions b^i are defined by (12). In another way, ${}^{2r-2}\Gamma_{S_L}$ is given by the equation

$$\mathrm{d}x^i_{2r-1} = \sum_{p=0}^{2r-2}\Gamma^{ip}_j\,\mathrm{d}x^j_p, \qquad \Gamma^{ip}_j = \frac{1}{2r}(p+1)\frac{\partial b^i}{\partial x^j_{p+1}}.$$

Then the equation of the connection K is

$$
\begin{aligned}
\mathrm{d}z_i^0 = \sum_{s=0}^{2r-4} \frac{\partial R_i^1}{\partial x_s^j}\,\mathrm{d}x_s^j + \frac{\partial R_i^1}{\partial x_{2r-3}^j}\,\mathrm{d}x_{2r-3}^j + \frac{\partial R_i^1}{\partial x_{2r-2}^j}\,\mathrm{d}x_{2r-2}^j \\
+ (-1)^r L_{x_r^i x_r^j} \sum_{p=0}^{2r-2} \Gamma_v^{jp}\,\mathrm{d}x_p^v .
\end{aligned}
\tag{13}
$$

Using the equations

$$
\begin{aligned}
\mathrm{d}x_{2r-3}^j &= (-1)^r \tilde{L}^{jw}\left(\mathrm{d}z_w^2 - \sum_{s=0}^{2r-4} \frac{\partial R_w^3}{\partial x_s^k}\,\mathrm{d}x_s^k\right) \\
\mathrm{d}x_{2r-2}^j &= (-1)^{r-1} \tilde{L}^{jw}\Bigg[\mathrm{d}z_w^1 - \sum_{s=0}^{2r-4} \frac{R_w^2}{x_s^k}\,\mathrm{d}x_s^k \\
&\qquad - \frac{\partial R_w^2}{\partial x_{2r-3}^v}(-1)^r \tilde{L}^{vt}\left(\mathrm{d}z_t^2 - \sum_{s=0}^{2r-4} \frac{\partial R_t^3}{\partial x_s^k}\,\mathrm{d}x_s^k\right)\Bigg],
\end{aligned}
$$

which follow from (9), we deduce from (13) that

$$
\begin{aligned}
K_{1i}^j &= \left(\frac{\partial R_i^1}{\partial x_{2r-2}^q} + (-1)^r L_{x_r^i x_r^u} \Gamma_q^{u,2r-2}\right)(-1)^{r-1} \tilde{L}^{qj} \\
K_{2i}^j &= \left(\frac{\partial R_i^1}{\partial x_{2r-3}^q} + (-1)^r L_{x_r^i x_r^u} \Gamma_q^{u,2r-3}\right)(-1)^r \tilde{L}^{qj} \\
&\quad + \left(\frac{\partial R_i^1}{\partial x_{2r-2}^q} + (-1)^r L_{x_r^i x_r^u} \Gamma_q^{u,2r-2}\right)(-1)^{r-1} \tilde{L}^{qw}\left(-\frac{\partial R_w^2}{\partial x_{2r-3}^v}(-1)^r \tilde{L}^{vj}\right).
\end{aligned}
$$

By (2) and (9) the horizontal projection of the connection ${}^0H$ is of the form

$$
\mathrm{d}x_0^i \otimes \partial/\partial x_0^i + (2r-1)! \sum_{s=1}^{r-1} B_{2r-1,s}^i\, \partial/\partial x_s^i + \sum_{p=1}^{r-1} \bar{H}_{ij}^p\, \mathrm{d}x_0^j \otimes \partial/\partial z_i^p .
$$

Since $B_{2r-1,1}^i = \dfrac{1}{(2r)!} A_1^i = \dfrac{1}{(2r)!} \dfrac{\partial b^i}{\partial x_{2r-1}^j}\,\mathrm{d}x_0^j$,

$$
H_{1j}^i = \frac{(2r-1)!}{(2r)!} \frac{\partial b^i}{\partial x_{2r-1}^j} = \frac{1}{2r} \frac{\partial b^i}{\partial x_{2r-1}^j} .
$$

Quite analogously we deduce that

$$H^i_{2j} = \frac{1}{r}\left(\frac{\partial b^i}{\partial x^j_{2r-2}}\frac{1}{2r-1} + \frac{\partial b^i}{\partial x^u_{2r-1}}\frac{1}{2r}\frac{\partial b^u}{\partial x^j_{2r-1}}\right).$$

Using $\Gamma^{ip}_j = \frac{1}{2r}(p+1)\frac{\partial b^i}{\partial x^j_{p+1}}$, (12) and

$$R^p_i = \sum_{q=0}^{r-p}(-1)^{q+1}\sum_{u=1}^{q} L_{x^i_{p+q}x^{i_1}_{p_1}\dots x^{i_n}_{p_n}} \sum_{t_1+\dots+t_u=q} A_{t_1\dots t_u}x^{i_1}_{p_1+1}\dots x^{i_n}_{p_u+t_u}$$

we obtain

$$-K^i_{1i} = +\frac{1}{2r}\tilde{L}^{jq}\left(L_{x^i_r x^q_{r-1}} - L_{x^i_{r-1}x^q_r} - rL_{x^i_r x^{i_1}_{p_1}x^q_r}x^{i_1}_{p_1+1}\right) = H^j_{1i}.$$

By a little more complicated calculation it can be shown that $H^i_{2j} \neq -K^i_{2i}$. Therefore the form ${}^0H$-$\mathrm{Orth}_{\mathrm{d}\lambda} K$ is p^{2r-1}_0-semibasic with values in Vp^{2r-1}_1. Then using the symplectic isomorphism $\mathcal{L}_L$ we get the following proposition:

PROPOSITION. *Let L be a regular Lagrangian. Then the connections ${}^0\Gamma$ and ${}^{2r-2}\Gamma$ determined by the mechanical system S_L of L are such that the form ${}^0\Gamma$-$\mathrm{Orth}_{\mathrm{d}\omega_L}{}^{2r-2}\Gamma$ is π^{2r-1}_0-semibasic with values in $V\pi^{2r-1}_1$.*

Remarks.

1. If $r = 1$, then ${}^0\Gamma = {}^{2\cdot1-2}\Gamma$ and the Proposition asserts that ${}^0\Gamma = \mathrm{Orth}_{\mathrm{d}\omega_L}{}^0\Gamma$, i.e. that the connection ${}^0\Gamma$ is Lagrangian, compare with [4, 5].

2. In [3] large families γ^k_s of natural operators ϕ of first order from the space of all semisprays S on T^nM into the space of connections on (π^n_k), $k = 0,\dots,n-1$, have been constructed. The simplest of them, from the point of their coordinate expression view, are the connections $(n-k)!\,\tau^{n-k}_S J_{n-k}$. In the case $n = 3$, it is proved in [5] that there are no other first order natural operators. If S_L is the mechanical system determined by a regular Lagrangian L of order r and if $\Gamma \in \gamma^k_{S_L}$, then the connection $\mathrm{Orth}_{\mathrm{d}\omega_{L_S}}\Gamma$ is a connection on (π^{2r-1}_{2r-2-k}). For $r = 2$, [5] shows that $\mathrm{Orth}_{\mathrm{d}\omega_{L_S}}\Gamma$ does not belong to $\gamma^{2r-2-k}_{S_L}$. Our proposition demonstrates what properties the connections Γ and $\mathrm{Orth}_{\mathrm{d}\omega_L}\Gamma$ only have. The operators ϕ do not determine all of the connections induced by L. Every regular Lagrangian L determines other families of connections $\mathrm{Orth}_{\mathrm{d}\omega_L}\Gamma$, where Γ is a connection belonging to $\gamma^k_{S_L}$, $k = 0,\dots,2r-2$.

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