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ABSTRACT. Let G be a complete, weakly σ-distributive lattice group and X be a set of the power of the continuum. Under the continuum hypothesis it is proved that there does not exist a non-trivial G-valued (sub)measure on the algebra of all subsets of X that assigns the measure θ to each singleton of X.

Introduction

Does there exist on the class of all subsets of a given set X a finite non trivial measure that assigns measure 0 to each singleton of X? It is evident that no such measure can exist if X is countable. It is shown in [1] that under the assumption of the continuum hypothesis no such measure can exist if X has the power of the continuum.

In 1986 Riečanová [4] raised the above question for Stone algebra valued measures. The aim of this note is to strengthen and generalize the results of [4] for vector lattice and lattice group valued measures and submeasures. The theory of vector lattice valued measures was developed in the series of papers of Wright in the 1970s (e.g. [9], [10], [11]). Some of his results were extended for ordered group valued measures (e.g. [3], [7]).

Our terminology, notions and notations are used in the sense of [2] and [10].

1. Preliminary results

The range of our measures and submeasures are vector lattices and lattice groups. It is known [1] that a complete lattice group is a commutative group. We recall that a σ-complete lattice group G is said to be weakly σ-distributive if, whenever $a \geqq a_{ij} \downarrow \theta \ (j \to \infty)$, $i = 1, 2, \ldots, n, \ldots$, then

$$\bigwedge \left\{ \bigvee_{i=1}^{\infty} a_{i\Phi(i)} \mid \Phi \colon N \to N \right\} = \theta.$$

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Let $C(S)$ be a space of all continuous real valued functions on a compact Hausdorff space S with the usual linear structure and pointwise order. It is known ([6], [2]) that $C(S)$ is a complete vector lattice iff S is extremally disconnected. Wright ([9], Lemma L) gave a beautiful characteriztion of weak σ-distributivity of $C(S)$; a σ-complete Stone algebra $C(S)$ is weakly σ-distributive iff each σ-meagre subset of S is nowhere dense (a set is σ-meagre if it is a subset of the union of a countable family of closed nowhere dense Baire sets).

There is another form of distributivity: (σ, ∞)-distributivity which turns out to be a strictly stronger condition than weak σ-distributivity ([10]). A σ-complete vector lattice W is weakly (σ, ∞)-distributive if, whenever $\{A_n\}$ $(n = 1, 2, \ldots)$ is a sequence of downward directed non-empty subsets of W such that $\bigcup_{n=1}^{\infty} A_n$ is ordered bounded and $\bigwedge A_n = \theta$ for each n, then

$$\bigwedge \left\{ \bigvee_{n=1}^{\infty} \Phi(n) \mid \Phi \in \prod A_n \right\} = \theta.$$

A σ-complete Stone algebra $C(S)$ is weakly (σ, ∞)-distributive iff every meagre subset of S is nowhere dense (see [10], lemma 2.3).

We define a notion of a lattice group valued submeasure as an analogy of the $C(S)$-valued submeasure from [4]. Let $(\Omega, \mathscr{S})$ be a measurable space and G a lattice group. A map $m\colon \mathscr{S} \to G$ is said to be a (finite) G-valued submeasure if

(i) $m(A) \geqq \theta$ for each $A \in \mathscr{S}$,

(ii) $m(A) \leqq m(B)$ whenever $A \subset B$, $A, B \in \mathscr{S}$,

(iii) $m(A \cup B) \leqq m(A) + m(B)$ for all $A, B \in \mathscr{S}$,

(iv) $\bigwedge_{n=1}^{\infty} m(A_n) = \theta$ whenever $(A_n)_n$ is a monotone decreasing sequence in $\mathscr{S}$ with $\bigcap_{n=1}^{\infty} A_n = \Phi$.

It is easy to see that a G-valued submeasure is continuous from below, i.e. $m(A) = \bigvee m(A_n)$ whenever $A_n \nearrow A$. If we suppose instead of (iii) additivity of m, we call m a G-valued measure. It is clear that in that case

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigvee_{n=1}^{\infty} \left\{ \sum_{k=1}^{n} m(E_k) \right\}$$

whenever (E_n) is a sequence of pairwise disjoint elements of $\mathscr{S}$.

In the end of this part let us point out why the assumptions used in [4] can be formulated in a somewhat more general form. The main result of [4], Theorem 2.1, works with Stone algebra $C(S)$, where S is such that each meagre subset is nowhere dense. When we inspect the proof of that theorem we can find that the set of those $s \in S$ where $\sup f_n(s) < (\bigvee f_n)(s)$, $f_n \in C(S)$, (f_n) is bounded from above, plays the key role and that the set $\{s \in S: \sup f_n(s) < (\bigvee f_n)(s)\}$ is not only meagre but even σ-meagre (lemma K in [9]). The countable union of such sets is σ-meagre again, thus it is sufficient to assume that σ-meagre sets are nowhere dense. According to Wright's results (lemma 2.3 in [10], lemma L in [9]) it means that it is sufficient to assume weak σ-distributivity of $C(S)$ instead of its (σ, ∞)-distributivity, as the author states in [4].

2. Results

In this part the range of m will be a complete, weakly σ-distributive lattice group. We completely abandon the topological methods of [4] and substitute them by the following computational lemma.

Lemma. *Let G be a σ-complete lattice group and (a_{ij}) be a double sequence of elements of G such that $a_{ij} \downarrow \theta$ $(j \to \infty)$ for each $i \in N$. Then to every $b \in G$, $b > \theta$ there exists a bounded sequence (b_{ij}) such that $b_{ij} \downarrow \theta$ $(j \to \infty)$ and such that for every $\Phi: N \to N$*

$$b \wedge \left(\sum_{i=1}^{\infty} a_{i\Phi(i)}\right) \leqq \bigvee_{i=1}^{\infty} b_{i\Phi(i)}.$$

Proof. The following assertion (see lemma 3.3 in [5]) plays an essential role in the proof: If $d, c_1, c_2, \ldots, c_n \in G^+$ and $d \wedge (2^k c_k) \leqq c$ $(k = 1, 2, \ldots, n)$, then

$$d \wedge (c_1 + c_2 + \ldots + c_n) \leqq c.$$

Put $b_{ij} = b \wedge (2^i a_{ij})$ for all $i, j = 1, 2, \ldots$. Evidently $b_{ij} \downarrow \theta$ $(j \to \infty)$ for $i = 1, 2, \ldots$. Let $\Phi: N \to N$ be arbitrary. Plainly $b \wedge (2^i a_{i\Phi(i)}) = b_{i\Phi(i)} \leqq b$ for $i = 1, 2, \ldots$. Applying the above assertion

$$b \wedge (2^i a_{i\Phi(i)}) \leqq \bigvee_{i=1}^{\infty} b_{i\Phi(i)} \quad \text{for} \quad i = 1, 2, \ldots, n$$

implies

$$b \wedge (a_{1\Phi(1)} + a_{2\Phi(2)} + \ldots + a_{n\Phi(n)}) \leqq \bigvee_{i=1}^{\infty} b_{i\Phi(i)} \quad \text{for}$$

$n = 2, 3, \ldots$. Finally

$$b \wedge \left(\sum_{i=1}^{\infty} a_{i\Phi(i)}\right) \leqq \bigvee_{i=1}^{\infty} b_{i\Phi(i)}.$$

Theorem 1. *Let us assume the continuum hypothesis. Let $(\Omega, \mathscr{S})$ be a measurable space and E a set of the power of the continuum. Let G be a complete, weakly* *σ-distributive lattice group. Let m be a G-valued submeasure on $\mathcal{S}$. When $\{A_x\colon x\in E\}$ is a family of pairwise disjoint sets in $\mathcal{S}$ such that $\cup\{A_x\colon x\in F\}\in\mathcal{S}$ for all $F\subset E$, then*

$$m(\cup\{A_x\colon x\in E\}) = \vee\{m(\cup\{A_x\colon x\in I\})\,|\,I\subset E,\quad I \text{ is finite}\}.$$

Proof. Plainly $m(\cup\{A_x\colon x\in E\})$ is an upper bound for the upward directed system $\{m(\cup\{A_x\colon x\in I\})\,|\,I\subset E, I \text{ is finite}\}$. For the reverse inequality we use the Banach—Kuratowski theorem which states that if the continuum hypothesis holds and E is a set of the power of the continuum, then there exists a double sequence (E_{ij}) of subsets of E such that

(i) $E_{ij}\nearrow E\quad (j\to\infty)$

(ii) for all $\Phi\colon N\to N$ $\bigcap_{i=1}^{\infty} E_{i\Phi(i)}$ is a countable set. Let $\Phi\colon N\to N$ be arbitrary and denote the points of $\bigcap_{i=1}^{\infty} E_{i\Phi(i)}$ by $x_1, x_2, \ldots, x_n, \ldots$. By the continuity of m

$$m(A_{x_1}\cup A_{x_2}\cup\ldots\cup A_{x_n}\cup\ldots) = \vee\{m(A_{x_1}\cup\ldots\cup A_{x_n})\colon n=1,2,\ldots\}$$

and evidently

$$\vee\{m(A_{x_1}\cup\ldots\cup A_{x_n})\colon n=1,2,\ldots\}\leqq\vee\{m(\cup\{A_x\colon x\in I\})\colon I\subset E,\ I \text{ is finite}\}$$

Set $b=m(\cup\{A_x\colon x\in E\})$ and

$a=\vee\{m(\cup\{A_x\colon x\in I\})\,|\,I\subset E, I \text{ is finite}\}$. Then

$$b-a\leqq m(\cup\{A_x\colon x\in E\})-m(A_{x_1}\cup A_{x_2}\cup\ldots A_{x_n}\cup\ldots)=$$

$$=m\left(\cup\left\{A_x\colon x\in E-\bigcap_{i=1}^{\infty}E_{i\Phi(i)}\right\}\right)=m\left(\cup\left\{A_x\colon x\in\bigcup_{i=1}^{\infty}(E-E_{i\Phi(i)})\right\}\right)\leqq$$

$$\leqq\sum_{i=1}^{\infty}m(\cup\{A_x\colon x\in E-E_{i\Phi(i)}\}).$$

Define $a_{ij}=m(\cup\{A_x\colon x\in E-E_{ij}\})$. It is easy to verify that $a_{ij}\downarrow\theta\ (j\to\infty)$ for $i=1,2\ \ldots,$ since $\bigcap_{j=1}^{\infty}(\cup\{A_x\colon x\in E-E_{ij}\})=\emptyset$ (A_x are pairwise disjoint and m is continuous at $\emptyset$). So we have $b-a\leqq\sum_{i=1}^{\infty}a_{i\Phi(i)}$ and applying the lemma there exists a bounded double sequence (b_{ij}) in G such that $b_{ij}\downarrow\theta\ (j\to\infty)$ and

$$b-a\leqq b\wedge\sum_{i=1}^{\infty}a_{i\Phi(i)}\leqq\bigvee_{i=1}^{\infty}b_{i\Phi(i)}$$

for all Φ: $N \to N$. Thus $b - a \leqq \inf\left\{\bigvee_{i=1}^{\infty} b_{i\Phi(i)} \mid \Phi: N \to N\right\} = \theta$ according to the weak σ-distributivity of G. This establishes the theorem.

Theorem 2. *Let us assume the continuum hypothesis. Let $(X, \mathscr{S})$ be a measurable space and X a set of the power of the continuum. Let G be a complete, weakly σ-distributive lattice group. Let m be a G-valued submeasure on $\mathscr{S}$ such that $m(\{x\}) = \theta$ for all $x \in X$. If there exists a set $E \in \mathscr{S}$ such that $m(E) > \theta$, then there exists $F \subset X$ such that $F \notin \mathscr{S}$.*

Proof. Let us assume that $E \in \mathscr{S}$ for every $E \subset X$. Then $E = \cup\{\{x\}: x \in E\}$ and by Theorem 1

$$m(E) = \vee\{m(\cup\{\{x\}: x \in I\}): I \subset E, I \text{ is finite}\} = \theta$$

as

$$m(\cup\{\{x\}: x \in I\}) \leqq \sum_{x \in I} m(\{x\}) = \theta.$$

It is possible to extend this result for σ-finite lattice group valued submeasures but it was done in [4]. Actually, part 3 of [4] does not use the fact that the values of m are elements of $C(S)$.

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