Mathematica Slovaca

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Mathematica Slovaca, Vol. 41 (1991), No. 4, 401--421

Persistent URL: http://dml.cz/dmlcz/129722

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EQUIVALENT ALGORITHMS FOR ESTIMATION IN LINEAR MODEL WITH CONDITION

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ABSTRACT. In the mixed linear model there exist different expressions for an estimator of a given linear function of parameters of the model. It is a welcome possibility how to check the numerical stability of calculation mainly in such cases where the size of the design matrix is large.

It is proved that analogous possibilities exist in the mixed linear model with linear condition on the first order parameters. Explicit formulae are given for the locally and uniformly best linear unbiased estimators of the first order parameters and for minimum norm quadratic estimators of the second order parameters.

Introduction

Let $\mathbf{Y}$ be an *n-dimensional random vector* and $\mathcal{P} = \{P_{\boldsymbol{\beta},\boldsymbol{\vartheta}} \colon \boldsymbol{\beta} \in \mathcal{V},\ \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}\}$ a *class of probability measures with the properties*: the *mean value* $E(\mathbf{Y} \mid \boldsymbol{\beta}, \boldsymbol{\vartheta}) = \mathbf{X}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}$, and the *covariance matrix* $\operatorname{Var}(\mathbf{Y} \mid \boldsymbol{\beta}, \boldsymbol{\vartheta})$ is $\Sigma(\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$. The $n \times k$ *matrix* $\mathbf{X}$ and $n \times n$ *symmetric matrices* $\mathbf{V}_1, \dots, \mathbf{V}_p$ are known. The notation $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\beta} \in \mathcal{V}, \Sigma(\boldsymbol{\vartheta}), \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}})$ is used for this situation.

The set $\mathcal{V}$ is usually supposed to be equal to $\mathbb{R}^k$ (*k-dimensional Euclidean space*); $\underline{\boldsymbol{\vartheta}}$ is an *open set in* $\mathbb{R}^p$ and fulfils the condition: $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\boldsymbol{\vartheta}} \implies \Sigma(\boldsymbol{\vartheta})$ is *positively semidefinite* (p.s.d.); here $'$ denotes a *transposition.* In the following $\boldsymbol{\beta}$ is called the *parameter of the first order* and $\boldsymbol{\vartheta}$ the *parameter of the second order.*

In many situations $\mathcal{V} = \{\mathbf{u} \colon \mathbf{u} \in \mathbb{R}^k,\ \mathbf{b} + \mathbf{B}\mathbf{u} = \mathbf{O}\} \subsetneqq \mathbb{R}^k$, where $\mathbf{B}$ is a $q \times k$ matrix and $\mathbf{b} \in \mathcal{M}(\mathbf{B})$ (*column space of the matrix* $\mathbf{B}$, $\mathcal{M}(\mathbf{B}) = \{\mathbf{B}\mathbf{v} \colon \mathbf{v} \in \mathbb{R}^k\}$). A model of measurement of angles in a plane triangle, i.e. $E\big[(Y_1, Y_2, Y_3)' \mid \boldsymbol{\beta}\big] = \boldsymbol{\beta}$, can serve as an example. Here obviously $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)'$ fulfils the condition $\beta_1 + \beta_2 + \beta_3 - \pi = 0$, thus $\mathbf{B} = (1,1,1)$ and $\mathbf{b} = -\pi$.

If the parameter $\boldsymbol{\beta}$ is expressed as $\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{K}_{\mathbf{B}}\boldsymbol{\gamma}$, where $\boldsymbol{\beta}_0$ is any solution of the equation $\mathbf{b} + \mathbf{B}\boldsymbol{\beta}_0 = \mathbf{O}$ and $\mathbf{K}_{\mathbf{B}}$ is the *matrix of the type* $k \times [k - R(\mathbf{B})]$ possessing the property $\mathcal{M}(\mathbf{K}_{\mathbf{B}}) = \operatorname{Ker} \mathbf{B} = \{\mathbf{u} \colon \mathbf{B}\mathbf{u} = \mathbf{O}\}$, we obtain the model

AMS Subject Classification (1985): Primary 62J05, Secondary 62F10
Key words: Linear model with variance components, Mixed linear model, BLUE, MINQUE

$(\boldsymbol{Y} - \mathbf{X}\beta_0, \mathbf{XK_B}\gamma, \gamma \in \mathbb{R}^{k-R(\mathbf{B})}, \Sigma(\vartheta), \vartheta \in \underline{\vartheta})$. In this model, the tandard formulae can be u ed f r e timators of γ and ϑ and thus no problems occur in determining the ϑ_0-LBLUE (*locally best linear unbiased estimator*) *of ın unbiasedly estimable linear function of* β and ϑ_0-MINQUE (*minim m norm quadratic unbiased estimator*) *of an unbiasedly and invariantly* (*with respect to the first order parameter*) *estimable linear function of* ϑ. (See Chpt. 5 in [1].)

Nevertheless several intere ting facts occur when problems connected with an e timation of the mentioned functions are studi d in the model without reparametrization.

The aim of the present paper is to point out these facts maınly from the point of view of equival nce of different estimatıng pıocedure .

1. Notations and auxiliary statements

In the following $\mathbf{P}_{\mathbf{A}}^{\mathbf{W}}$ denotes the *projection matrix on* $\mathcal{M}(\mathbf{A})$ *with respect to the norm given by the relation* $\|\mathbf{x}\|_{\mathbf{W}} = \sqrt{\mathbf{x}'\mathbf{W}\mathbf{x}}$; thus $\mathbf{W}$ is *positively definıte* (p.d.). It is easy to prove that $\mathbf{P}_{\mathbf{A}}^{\mathbf{W}} = \mathbf{A}(\mathbf{A}'\mathbf{W}\mathbf{A})^{-}\mathbf{A}'\mathbf{W}$ ($^-$ denotes *a generalized inverse* [3]). This expression is well defined when either $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{W})$ or $\mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{A}'\mathbf{W}\mathbf{A})$ even if $\mathbf{W}$ is p.s.d. For $\mathbf{W} = \mathbf{I}$ (*unıt matrix*), the notation $\mathbf{P}_{\mathbf{A}}$ is used. The notation LM (*linear model*) is used for $(\boldsymbol{Y}, \mathbf{X}\beta, \beta \in \mathbb{R}^k, \Sigma(\vartheta), \vartheta \in \underline{\vartheta})$, i.e. for the case $\mathcal{V} = \mathbb{R}^k$.

Lemma 1.1. *Let* $\mathbf{A}$, $\mathbf{B}$ *be* $n \times n$ *symmetric and* p.s.d. *matrices. Then* $\mathcal{M}(\mathbf{A} + \mathbf{B}) = \mathcal{M}(\mathbf{A}, \mathbf{B})$.

Proof see in [3, p. 122].

Lemma 1.2. *In* LM *the following rules can be used:*

R$_1$: *A function* $h(\beta) = \boldsymbol{h}'\beta$, $\beta \in \mathbb{R}^k$, *is linearly unbi sedly estimable ıff* $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}')$.

Proof is obvious.

R$_2$: *If* $h_0(\beta) = 0$, $\beta \in \mathbb{R}^k$, *then the class of all its linear unbiased estimators is* $\mathcal{U}_0 = \{\boldsymbol{L}_0'\boldsymbol{Y} : \boldsymbol{L}_0 \in \mathcal{M}(\mathbf{M_X})\}$, *where* $\mathbf{M_X} = \mathbf{I} - \mathbf{P_X}$.

Proof is ol v˙ us

R$_3$: *A statıstı* $\boldsymbol{L}'\boldsymbol{Y}$ *is* ϑ_0-LBLUE *of its me al e iff the conditı n* $\mathbf{M_X}\Sigma(\vartheta_0)\boldsymbol{L} = \boldsymbol{O}$ *is f ılfilled.*

Proof. The statem nt i a consequence of [2, (i) p. ?57] where the c n dition for the considered ca e is $\forall\{\boldsymbol{L}_0'\boldsymbol{Y} \in \mathcal{U}_0\}\, \mathrm{cov}(\boldsymbol{L}'\boldsymbol{Y}, \boldsymbol{L}'\boldsymbol{Y} \mid \vartheta_0) = 0$. A $\mathcal{U}_0 - \{\boldsymbol{\lambda}'\mathbf{M_X}\boldsymbol{Y} : \boldsymbol{\lambda} \in \mathbb{R}^n\}$, this can be rewritten as $\mathbf{M_X}\Sigma(\vartheta\ )\boldsymbol{L} - \boldsymbol{O}$.

R_4: *A statistic* $\mathbf{L}'\mathbf{Y}$ *is* UBLUE (*uniformly – with respect to* $\vartheta \in \underline{\vartheta}$ *– best linear unbiased estimator*) *of its mean value iff the condition*

$$\mathbf{L} \in \mathrm{Ker}\Big(\sum_{i=1}^{p} \mathbf{V}_i \mathbf{M}_{\mathbf{X}} \mathbf{V}_i\Big)$$

is fulfilled ($\mathrm{Ker}(\cdot)$ *means nullspace*).

Proof cf. [1, p. 203].

R_5: *A function* $f(\boldsymbol{\beta}) = \mathbf{f}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, *has the* UBLUE *iff*

$$\mathbf{f} \in \mathcal{M}\Big[\mathbf{X}'\mathbf{Ker}\Big(\sum_{i=1}^{p} \mathbf{V}_i \mathbf{M}_{\mathbf{X}} \mathbf{V}_i\Big)\Big]$$

(*here* $\mathbf{Ker}\Big(\sum_{i=1}^{p} \mathbf{V}_i \mathbf{M}_{\mathbf{X}} \mathbf{V}_i\Big)$ *denotes a matrix whose columns generate the subspace* $\mathrm{Ker}\Big(\sum_{i=1}^{p} \mathbf{V}_i \mathbf{M}_{\mathbf{X}} \mathbf{V}_i\Big)$).

Proof cf. [1, p 204].

Lemma 1.3. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *be known matrices and* $\mathbf{AXB} = \mathbf{C}$ *an equation for an unknown* $\mathbf{X}$.

(a) *This equation has a solution iff* $\mathbf{AA}^-\mathbf{CB}^-\mathbf{B} = \mathbf{C}$.

(b) *If the condition in* (a) *is fulfilled, then the class of all solutions is*

$$\mathcal{X} = \{\mathbf{A}_0^-\mathbf{C}\mathbf{B}_0^- + \mathbf{Z} - \mathbf{A}_0^-\mathbf{AZBB}_0^- \ :\ \mathbf{Z} \text{ is an arbitrary matrix}\},$$

where $\mathbf{A}_0^-$, $\mathbf{B}_0^-$ *are arbitrary but fixed generalized inverses* (*g-inverses*) *of matrices* $\mathbf{A}$, $\mathbf{B}$.

Proof see in [3, theorem 2.3.2.].

In what follows $\mathbf{A}^+$ denotes the *Moore-Penrose g-inverse of the matrix* $\mathbf{A}$, i e. it is a matrix with the following properties $\mathbf{AA}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{AA}^+ = \mathbf{A}^+$, $\mathbf{AA}^+ = (\mathbf{AA}^+)'$ and $\mathbf{A}^+\mathbf{A} = (\mathbf{A}^+\mathbf{A})'$.

Lemma 1.4. *Let* $\mathbf{A}$ *be any* $n \times k$ *matrix,* $\mathbf{W}$ *be any* $n \times n$ p.s.d. *matrix and let* $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{W})$. *Then*

$$(\mathbf{M_A W M_A})^+ = \begin{cases} \mathbf{W}^{-1} - \mathbf{W}^{-1}\mathbf{A}(\mathbf{A}'\mathbf{W}^{-1}\mathbf{A})^-\mathbf{A}'\mathbf{W}^{-1} & \text{for } \mathbf{W} \text{ p.d.,} \\ \mathbf{W}^+ - \mathbf{W}^+\mathbf{A}(\mathbf{A}'\mathbf{W}^+\mathbf{A})^-\mathbf{A}'\mathbf{W}^+ & \text{otherwise.} \end{cases}$$

If the condition $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{W})$ *is not fulfilled, then*

$$(\mathbf{M_A W M_A})^+ = (\mathbf{W} + \mathbf{AVA}')^+ - (\mathbf{W} + \mathbf{AVA}')^+\mathbf{A}\big[\mathbf{A}'(\mathbf{W} + \mathbf{AVA}')^+\mathbf{A}\big]^- \cdot \cdot \mathbf{A}'(\mathbf{W} + \mathbf{AVA}')^+,$$

where $\mathbf{V}$ *is any* $k \times k$ *matrix with the property* $\mathcal{M}(\mathbf{AVA}') = \mathcal{M}(\mathbf{A})$.

Proof. It is sufficient to verify the four above mentioned properties of the Moore-Penrose g-inverse.

Lemma 1.5. *Let* $\mathbf{N}$ *be an* $n \times n$ p.s.d. *matrix; i.e.* $\exists \{\mathbf{J},\ n \times R(\mathbf{N})$ *matrix*$\}$ $\mathbf{N} = \mathbf{J}\mathbf{J}'$. *Then* $\mathbf{N}^{+} = \mathbf{K}\mathbf{K}'$, *where* $\mathbf{J}'\mathbf{K}\mathbf{K}'\mathbf{J} = \mathbf{I}$.

Proof is obvious.

Definition 1.6. *In* LM *a function* $g(\vartheta) = \mathbf{g}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *is unbiasedly and invariantly estimable by a quadratic estimator if there exists a matrix* $\mathbf{U}$ *possessing the properties*

(a) *unbiasedness* $\forall\{\beta \in \mathbb{R}^k\}\ \forall\{\vartheta \in \underline{\vartheta}\}\ E(\mathbf{Y}'\mathbf{U}\mathbf{Y} \mid \beta, \vartheta) = \mathbf{g}'\vartheta$,

(b) *invariance* $\forall\{\beta \in \mathbb{R}^k\}\ (\mathbf{Y} + \mathbf{X}\beta)'\mathbf{U}(\mathbf{Y} + \mathbf{X}\beta) = \mathbf{Y}'\mathbf{U}\mathbf{Y}$.

Lemma 1.7. *In* LM *a function* $g(\vartheta) = \mathbf{g}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *is unbiasedly and invariantly estimable by an estimator* $\mathbf{Y}'\mathbf{U}\mathbf{Y}$ *iff* $\mathbf{U}\mathbf{X} = \mathbf{O}$, $\mathrm{Tr}(\mathbf{U}\mathbf{V}_i) = g_i$, $i = 1, \dots, p$, *i.e. iff* $\mathbf{g} \in \mathcal{M}(\mathbf{K}^{(\mathbf{I})})$, *where* $\{\mathbf{K}^{(\mathbf{I})}\}_{i,j} = \mathrm{Tr}(\mathbf{M_X}\mathbf{V}_i\mathbf{M_X}\mathbf{V}_j)$, $i, j = 1, \dots, p$.

Proof see in [5].

Lemma 1.8. *The* ϑ_0 − MINQUE *of the function* $g(\cdot)$ *from Lemma* 1.7 *is*

$$\widehat{\mathbf{g}'\vartheta} = \sum_{i=1}^{p} \lambda_i \mathbf{Y}'[\mathbf{M_X}\Sigma(\vartheta_0)\mathbf{M_X}]^{+}\mathbf{V}_i[\mathbf{M_X}\Sigma(\vartheta_0)\mathbf{M_X}]^{+}\mathbf{Y},$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_p)'$ *is a solution of the equation*

$$\left(\mathbf{S}_{[\mathbf{M_X}\Sigma(\vartheta_0)\mathbf{M_X}]^{+}}\right)\boldsymbol{\lambda} = \mathbf{g},$$

$$\left\{\mathbf{S}_{[\mathbf{M_X}\Sigma(\vartheta_0)\mathbf{M_X}]^{+}}\right\}_{i,j} = \mathrm{Tr}\left\{[\mathbf{M_X}\Sigma(\vartheta_0)\mathbf{M_X}]^{+}\mathbf{V}_i[\mathbf{M_X}\Sigma(\vartheta_0)\mathbf{M_X}]^{+}\mathbf{V}_j\right\},$$

$i, j = 1, \dots, p$.
In the case of normal distributions this estimator is ϑ_0 -LMVQUIE, *i.e. locally minimum variance quadratic unbiased invariant estimator.*

Proof see in [5].

2. Estimators of the first order parameters

Let a matrix $\mathbf{K_B} = \mathrm{Ker}(\mathbf{B})$ be of the full rank in columns and $\mathcal{M}(\mathbf{K_B}) = \mathrm{Ker}(\mathbf{B})$, i.e. it is of the type $k \times [k - R(\mathbf{B})]$ ($R(\mathbf{B})$ is the rank of the matrix $\mathbf{B}$).

Let $\mathcal{V} = \{\boldsymbol{u}: \boldsymbol{u} \in \mathbb{R}^k,\ \boldsymbol{b} + \mathbf{B}\boldsymbol{u} = \boldsymbol{O}\}$ and β_0 be an arbitrary but fixed solution of the equation $\boldsymbol{b} + \mathbf{B}\beta_0 = \boldsymbol{O}$. Then it is obvious that models

$$(\mathbf{Y},\ \mathbf{X}\beta,\ \beta \in \mathcal{V},\ \Sigma(\vartheta),\ \vartheta \in \underline{\vartheta}), \tag{2.1}$$

$$(\mathbf{Y} - \mathbf{X}\beta_0,\ \mathbf{X}\mathbf{K}_{\mathbf{B}}\gamma,\ \gamma \in \mathbb{R}^{k-R(\mathbf{B})},\ \Sigma(\vartheta),\ \vartheta \in \underline{\vartheta}) \tag{2.2}$$

and

$$\left(\begin{pmatrix}\mathbf{Y}\\ -\boldsymbol{b}\end{pmatrix},\ \begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}\beta,\ \beta \in \mathcal{V},\ \begin{pmatrix}\Sigma(\vartheta), & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix},\ \vartheta \in \underline{\vartheta}\right) \tag{2.3}$$

are equivalent. The symbol LMC (*linear model with condition*) means any of the models (2.1), (2.2), (2.3).

Lemma 2.1. *In* LMC *a function* $f(\beta) = \boldsymbol{f}'\beta$, $\beta \in \mathcal{V}$, *is unbiasedly estimable iff* $\mathbf{K}'_{\mathbf{B}}\boldsymbol{f} \in \mathcal{M}(\mathbf{K}'_{\mathbf{B}}\mathbf{X}')$.

Proof. As $\beta = \beta_0 + \mathbf{K}_{\mathbf{B}}\gamma$, $\gamma \in \mathbb{R}^{k-R(\mathbf{B})}$, the function $f(\cdot)$ can be written as $f(\beta) = \boldsymbol{f}'\beta_0 + \boldsymbol{f}'\mathbf{K}_{\mathbf{B}}\gamma$. It is unbiasedly estimable iff there exists a vector $\boldsymbol{L} \in \mathbb{R}^n$ and a real number $l \in \mathbb{R}^1$ with the property $\forall\{\gamma \in \mathbb{R}^{k-R(\mathbf{B})}\} E(\boldsymbol{L}'\mathbf{Y} + l \mid \beta_0, \gamma) = \boldsymbol{L}'\mathbf{X}(\beta_0 + \mathbf{K}_{\mathbf{B}}\gamma) + l = \boldsymbol{f}'\beta_0 + \boldsymbol{f}'\mathbf{K}_{\mathbf{B}}\gamma \iff \mathbf{K}'_{\mathbf{B}}\boldsymbol{f} = \mathbf{K}'_{\mathbf{B}}\mathbf{X}'\boldsymbol{L}\ \&\ l = \boldsymbol{f}'\beta_0 - \boldsymbol{L}'\mathbf{X}\beta_0$. Obviously $\exists\{\boldsymbol{L}: \boldsymbol{L} \in \mathbb{R}^n,\ \mathbf{K}'_{\mathbf{B}}\boldsymbol{f} - \mathbf{K}'_{\mathbf{B}}\mathbf{X}'\boldsymbol{L}\} \iff \mathbf{K}'_{\mathbf{B}}\boldsymbol{f} \in \mathcal{M}(\mathbf{K}'_{\mathbf{B}}\mathbf{X}') \implies l = \boldsymbol{f}'\beta_0 - \boldsymbol{L}'\mathbf{X}\beta_0$. □

The equivalence $\forall\{\gamma \in \mathbb{R}^{k-R(\mathbf{B})}\}\boldsymbol{L}'\mathbf{X}\mathbf{K}_{\mathbf{B}}\gamma = \boldsymbol{f}'\mathbf{K}_{\mathbf{B}}\gamma \iff \mathbf{K}'_{\mathbf{B}}\boldsymbol{f} = \mathbf{K}'_{\mathbf{B}}\mathbf{X}'\boldsymbol{L}$ is a consequence of a possibility to change the vector γ in the whole space $\mathbb{R}^{k-R(\mathbf{B})}$. This is impossible in LMC with respect to $\beta \in \mathcal{V} \subsetneq \mathbb{R}^k$. Nevertheless, the following theorem states that the rule R_1 is valid in LMC (2.3.).

Theorem 2.2. *In* LMC *a function* $f(\beta) = \boldsymbol{f}'\beta$, $\beta \in \mathcal{V}$, *is unbiasedly estimable iff* $\boldsymbol{f} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$.

Proof. It is sufficient to prove $\boldsymbol{f} \in \mathcal{M}(\mathbf{X}', \mathbf{B}') \iff \mathbf{K}'_{\mathbf{B}}\boldsymbol{f} \in \mathcal{M}(\mathbf{K}'_{\mathbf{B}}\mathbf{X}')$ with respect to Lemma 2.1. Let $\boldsymbol{f} = \mathbf{X}'\boldsymbol{u} + \mathbf{B}'\boldsymbol{v}$; then $\mathbf{K}'_{\mathbf{B}}\boldsymbol{f} = \mathbf{K}'_{\mathbf{B}}\mathbf{X}'\boldsymbol{u}$ and thus $\mathbf{K}'_{\mathbf{B}}\boldsymbol{f} \in \mathcal{M}(\mathbf{K}'_{\mathbf{B}}\mathbf{X}')$. Let $\mathbf{K}'_{\mathbf{B}}\boldsymbol{f} = \mathbf{K}'_{\mathbf{B}}\mathbf{X}'\boldsymbol{u}$. Then $\boldsymbol{f} \in \{\mathbf{X}'\boldsymbol{u} + \boldsymbol{z} - (\mathbf{K}'_{\mathbf{B}})^-\mathbf{K}'_{\mathbf{B}}\boldsymbol{z}: \boldsymbol{z} \in \mathbb{R}^k\}$ (cf. Lemma 1.3), since $\mathbf{X}'\boldsymbol{u}$ is a particular solution to $\boldsymbol{f}$. Further $\mathcal{M}[\mathbf{I} - (\mathbf{K}'_{\mathbf{B}})^-\mathbf{K}'_{\mathbf{B}}] = \mathcal{M}(\mathbf{B}')$. Thus $\boldsymbol{z} - (\mathbf{K}'_{\mathbf{B}})^-\mathbf{K}'_{\mathbf{B}}\boldsymbol{z} \in \mathcal{M}(\mathbf{B}')$ and $\boldsymbol{f} = \mathbf{X}'\boldsymbol{u} + \mathbf{B}'\boldsymbol{v}$. □

Theorem 2.3. *The class of all linear unbiased estimators of the function* $f_0(\beta) = 0$, $\beta \in \mathcal{V}$, *in* LMC *is*

$$\mathcal{U}_0 = \left\{\boldsymbol{L}'_{01}\mathbf{Y} + \boldsymbol{L}'_{02}(-\boldsymbol{b}): (\boldsymbol{L}'_{01}, \boldsymbol{L}'_{02})' \in \mathcal{M}\left(\mathbf{M}_{\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}}\right)\right\}$$

Thus the rule R_2 *is valid in* LMC (2.3).

P r o o f. With respect to Lemma 1.2 (R_2), the class $\mathcal{U}_0$ in LMC is $\left\{\boldsymbol{L}_0'(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}_0)\colon \boldsymbol{L}_0 \in \mathcal{M}\left(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\right)\right\}$. Let $(\boldsymbol{L}_{01}', \boldsymbol{L}_{02}')' \in \mathcal{M}\left(\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right) \iff \mathbf{X}'\boldsymbol{L}_{01} + \mathbf{B}'\boldsymbol{L}_{02} = \boldsymbol{O} \implies \mathbf{K}_\mathbf{B}'\mathbf{X}'\boldsymbol{L}_{01} = \boldsymbol{O} \iff \boldsymbol{L}_{01} \in \mathcal{M}\left(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\right)$; further $\boldsymbol{L}_{01}'\mathbf{Y} + \boldsymbol{L}_{02}'(-\boldsymbol{b}) = \boldsymbol{L}_{01}'\mathbf{Y} + \boldsymbol{L}_{02}'\mathbf{B}\boldsymbol{\beta}_0 = \boldsymbol{L}_{01}'\mathbf{Y} + (-\boldsymbol{L}_{01}'\mathbf{X})\boldsymbol{\beta}_0 = \boldsymbol{L}_{01}'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0)$. Let $\boldsymbol{L}_0 \in \mathcal{M}\left(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\right) \iff \mathbf{K}_\mathbf{B}'\mathbf{X}'\boldsymbol{L}_0 = \boldsymbol{O} \iff \mathbf{X}'\boldsymbol{L}_0 \in \mathcal{M}(\mathbf{B}') \iff \exists\{\boldsymbol{v} \in \mathbb{R}^q\}\mathbf{X}'\boldsymbol{L}_0 + \mathbf{B}'\boldsymbol{v} = \boldsymbol{O} \iff \binom{\boldsymbol{L}_0}{\boldsymbol{v}} \in \mathcal{M}\left(\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right) \implies \boldsymbol{L}_0'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0) = \boldsymbol{L}_0'\mathbf{Y} + \boldsymbol{v}'\mathbf{B}\boldsymbol{\beta}_0 = \boldsymbol{L}_0'\mathbf{Y} + \boldsymbol{v}'(-\boldsymbol{b})$. □

The following lemma is useful before studing the rule R_3 in LMC (2.3).

Lemma 2.4. *Let* $\mathbf{W}$ *be an* $n \times n$ p.s.d. *matrix and let* $\mathcal{M}(\mathbf{X}) \subset (\mathbf{W})$. *Then*

(a)

$$\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \begin{cases} \mathbf{P}^{\mathbf{W}}_{\mathbf{X}} - \mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^-\mathbf{B}'} & \textit{for} \quad \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}'), \\ \mathbf{P}^{\mathbf{W}}_{\mathbf{X}} - \mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}'} & \textit{otherwise}, \end{cases}$$

where $\mathbf{V}$ *is any* $q \times q$ *matrix with the property* $\mathcal{M}(\mathbf{B}'\mathbf{V}\mathbf{B}) = \mathcal{M}(\mathbf{B}')$.

(b)

$$\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^-\mathbf{B}'} = \mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^-\mathbf{B}'}\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \mathbf{O} \qquad \textit{if} \quad \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}')$$

and

$$\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}'} = \mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}'}\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \mathbf{O} \qquad \textit{otherwise}.$$

(c)

$$\begin{aligned}\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = {} & \mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X} + \mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{X}'\mathbf{W} - \mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X} + \mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}' \cdot \\ & \cdot \left[\mathbf{B}(\mathbf{X}'\mathbf{W}\mathbf{X} + \mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}'\right]^-\mathbf{B}(\mathbf{X}'\mathbf{W}\mathbf{X} + \mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{X}'\mathbf{W}.\end{aligned}$$

P r o o f. The first equality in (a) can be proved directly; as $\mathcal{M}(\mathbf{K}_\mathbf{B}) = \mathcal{M}(\mathbf{M}_{\mathbf{B}'})$, $\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}} = \mathbf{X}\mathbf{M}_{\mathbf{B}'}(\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}\mathbf{X}\mathbf{M}_{\mathbf{B}'})^-\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}$. Now the equality $\mathbf{M}_{\mathbf{B}'}(\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}\mathbf{X}\mathbf{M}_{\mathbf{B}'})^+\mathbf{M}_{\mathbf{B}'} = (\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}\mathbf{X}\mathbf{M}_{\mathbf{B}'})^+$ and the implication $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}'\mathbf{W}\mathbf{X}) \implies (\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}\mathbf{X}\mathbf{M}_{\mathbf{B}'})^+ = (\mathbf{X}'\mathbf{W}$ [illegible] $(\mathbf{X}'\mathbf{W}\mathbf{X}$ [illegible] $\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{W}\mathbf{X})^+$ [illegible] $\mathbf{B}']^-\mathbf{B}'(\mathbf{X}'\mathbf{W}\mathbf{X})^+$ from Lemma 1.4 is to be us[illegible]

$\mathbf{P}^{\mathbf{W}}_{\mathbf{XK_B}} = \mathbf{X}(\mathbf{M_{B'}X'WXM_{B'}})^{+}\mathbf{X'W} = \mathbf{X}(\mathbf{X'WX})^{+}\mathbf{X'W} - \mathbf{X}(\mathbf{X'WX})^{+}\mathbf{B'}[\mathbf{B}(\mathbf{X'WX})^{+}\cdot$
$\cdot\mathbf{X'WX}(\mathbf{X'WX})^{+}\mathbf{B'}]^{-}\mathbf{B}(\mathbf{X'WX})^{+}\mathbf{X'W} = \mathbf{P}^{\mathbf{W}}_{\mathbf{X}} - \mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X'WX})^{+}\mathbf{B'}}$

In the case of the second equality in (a), it is sufficient to prove $R(\mathbf{X}) = R(\mathbf{XK_B}) + R[\mathbf{X}(\mathbf{X'WX}+\mathbf{B'VB})^{-}\mathbf{B'}]$ and $\mathcal{M}(\mathbf{XK_B})\perp_{\mathbf{W}}\mathcal{M}[\mathbf{X}(\mathbf{X'WX}+\mathbf{B'VB})^{-}\mathbf{B'}]$, where $\perp_{\mathbf{W}}$ means the orthogonality with respect to $\mathbf{W}$, i.e. $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x}\perp_{\mathbf{W}}\boldsymbol{y} \Leftrightarrow \boldsymbol{x}'\mathbf{W}\boldsymbol{y} = \boldsymbol{O}$. Let $\mathcal{M}_1 = \mathcal{M}(\mathbf{X})$, $\mathcal{M}_2 = \mathcal{M}(\mathbf{XK_B}) = \mathcal{M}(\mathbf{XM_{B'}})$ and $\mathcal{M}_3 = \mathcal{M}[\mathbf{X}(\mathbf{X'WX}+\mathbf{B'VB})^{-}\mathbf{B'}]$. As $\mathbf{M_{B'}X'WX}(\mathbf{X'WX}+\mathbf{B'VB})^{-}\mathbf{B'} = \mathbf{M_{B'}}(\mathbf{X'WX} + \mathbf{B'VB})(\mathbf{X'WX} + \mathbf{B'VB})^{-}\mathbf{B'} = \mathbf{M_{B'}B'} = \mathbf{O}$, $\mathcal{M}_2\perp_{\mathbf{W}}\mathcal{M}_3$. To prove $R(\mathbf{X}) = R(\mathbf{XK_B}) + R[\mathbf{X}(\mathbf{X'WX} + \mathbf{B'VB})^{-}\mathbf{B'}]$ we proceed as follows:
$\mathbf{P}^{\mathbf{W}}_{\mathbf{XK_B}} = \mathbf{P}^{\mathbf{W}}_{\mathbf{XM_{B'}}} = \mathbf{XM_{B'}}(\mathbf{M_{B'}X'WXM_{B'}})^{+}\mathbf{M_{B'}X'W} = \mathbf{X}[\mathbf{M_{B'}}(\mathbf{X'WX} + \mathbf{B'VB})\cdot$
$\cdot\mathbf{M_{B'}}]^{+}\mathbf{X'W} = \mathbf{X}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{X'W} - \mathbf{X}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{B'}[\mathbf{B}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{B'}]^{+}\mathbf{B}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{X'W}$ (Lemma 1.4 is used)

$$\mathbf{WX}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{X'W} = \mathbf{WP}^{\mathbf{W}}_{\mathbf{XK_B}} + \mathbf{WM}_3,$$

where
$\mathbf{M}_3 = \mathbf{X}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{B'}[\mathbf{B}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{B'}]^{+}\mathbf{B}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{X'W}$.
Both matrices $\mathbf{WP}^{\mathbf{W}}_{\mathbf{XK_B}}$, $\mathbf{WM}_3$ are p.s.d. and $\left(\mathbf{WP}^{\mathbf{W}}_{\mathbf{XK_B}}\right)'\mathbf{W}^{+}\mathbf{WM}_3 = \mathbf{O}$ (it is a consequence of $\mathcal{M}_2\perp_{\mathbf{W}}\mathcal{M}_3$); thus with respect to Lemma 1.1, we have $R[\mathbf{WX}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{X'W}] = R\left(\mathbf{WP}^{\mathbf{W}}_{\mathbf{XK_B}} + \mathbf{WM}_3\right) = R\left(\mathbf{WP}^{\mathbf{W}}_{\mathbf{XK_B}}, \mathbf{WM}_3\right) = R\left(\mathbf{WP}^{\mathbf{W}}_{\mathbf{XK_B}}\right) + R(\mathbf{WM}_3)$. Further $R[\mathbf{WX}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{X'W}] = R(\mathbf{X})$, $R\left(\mathbf{WP}^{\mathbf{W}}_{\mathbf{XK_B}}\right) = R(\mathbf{XK_B})$ and $R(\mathbf{M}_3) = R(\mathbf{WM}_3) = R[\mathbf{X}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{B'}]$.
The last three equalities are consequences of the following relations, cf. Lemma 1.5: $\mathbf{X'WX} + \mathbf{B'VB} = \mathbf{JJ'}$, $(\mathbf{X'WX} + \mathbf{B'VB})^{+} = \mathbf{KK'}$, $\mathcal{M}(\mathbf{X'W}) \subset \mathcal{M}(\mathbf{J}) \Leftrightarrow \exists\{\mathbf{F}: \mathbf{X'W} = \mathbf{JF}\}$, thus $\mathbf{WXKK'X'W} = \mathbf{F'J'KK'JF} = \mathbf{F'F} \Longrightarrow R[\mathbf{WX}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{X'W}] = R(\mathbf{F'}) \geq R(\mathbf{F'J}) = R(\mathbf{X'W}) \geq R(\mathbf{X'WW}^{+}) = R(\mathbf{X'})$; the inequality $R[\mathbf{WX}(\mathbf{X'WX} + \mathbf{B'VB})^{+}\mathbf{X'W}] \leq R(\mathbf{X})$ is obvious.
Similarly $R\left(\mathbf{WP}^{\mathbf{W}}_{\mathbf{XK_B}}\right) = R(\mathbf{WXK_B}) \geq R(\mathbf{W}^{+}\mathbf{WXK_B}) = R(\mathbf{XK_B}) \geq R(\mathbf{WXK_B})$ and $R(\mathbf{M}_3) \geq R(\mathbf{WM}_3) \geq R(\mathbf{W}^{+}\mathbf{WM}_3) = R(\mathbf{M}_3)$ (here the implication $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\mathbf{W}) = \mathcal{M}(\mathbf{W}^{+}) \implies \mathbf{W}^{+}(\mathbf{W}^{+})^{+}\mathbf{X} = \mathbf{X}$ was used).

The statement (b) is a consequence of the equalities $\mathbf{K'_B X'WX}(\mathbf{X'WX})^{-}\mathbf{B'} = \mathbf{K'_B B'} = \mathbf{O}$ and $\mathbf{K'_B X'WX}(\mathbf{X'WX} + \mathbf{B'VB})^{-}\mathbf{B'} = \mathbf{K'_B}(\mathbf{X'WX} + \mathbf{B'VB})(\mathbf{X'WX} + \mathbf{B'VB})^{-}\mathbf{B'} = \mathbf{K'_B B'} = \mathbf{O}$, respectively.

(c) is implied by the equality $(\mathbf{M_{B'}X'WXM_{B'}})^{+} = [\mathbf{M_{B'}}(\mathbf{X'WX} + \mathbf{B'VB})\mathbf{M_{B'}}]^{+}$ and by the last statement of Lemma 1.4. ❑

Theorem 2.5. *In* LMC (2.3) *the rule* R_3 *is valid.*

P r o o f. With respect to Lemma 1.2, the rule R_3 in LMC (2.2) states: a statistic $\boldsymbol{L}_1'(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0) + \boldsymbol{f}'\boldsymbol{\beta}_0$ is the $\boldsymbol{\vartheta}_0$-LBLUE of the function $f(\boldsymbol{\beta}) = \boldsymbol{f}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}$, where $\mathbf{K}_\mathbf{B}'\boldsymbol{f} = \mathbf{K}_\mathbf{B}'\mathbf{X}'\boldsymbol{L}_1$ ($\implies E[\boldsymbol{L}_1'(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0) + \boldsymbol{f}'\boldsymbol{\beta}_0 \mid \boldsymbol{\beta}] = \boldsymbol{L}_1'(\mathbf{X}\boldsymbol{\beta} - \mathbf{X}\boldsymbol{\beta}_0) + \boldsymbol{f}'\boldsymbol{\beta}_0 = \boldsymbol{L}_1'\mathbf{X}\mathbf{K}_\mathbf{B}\boldsymbol{\gamma} + \boldsymbol{f}'\boldsymbol{\beta}_0 = \boldsymbol{f}'(\mathbf{K}_\mathbf{B}\boldsymbol{\gamma} + \boldsymbol{\beta}_0) = \boldsymbol{f}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}$) iff

$$\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1 = \boldsymbol{O}. \tag{A}$$

The rule R_3 in LMC (2.3) states: a statistic $\boldsymbol{L}_1'\boldsymbol{Y} + \boldsymbol{L}_2'(-\boldsymbol{b})$ is the $\boldsymbol{\vartheta}_0$-LBLUE of the same function $f(\boldsymbol{\beta}) = \boldsymbol{f}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}$, where $\boldsymbol{f} = \mathbf{X}'\boldsymbol{L}_1 + \mathbf{B}'\boldsymbol{L}_2$ (cf. Theorem 2.2) ($\implies E[\boldsymbol{L}_1'\boldsymbol{Y} + \boldsymbol{L}_2'(-\boldsymbol{b}) \mid \boldsymbol{\beta}] = \boldsymbol{L}_1'\mathbf{X}\boldsymbol{\beta} + \boldsymbol{L}_2'\mathbf{B}\boldsymbol{\beta} = \boldsymbol{f}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}$) iff

$$\mathbf{M}_{\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}}\begin{pmatrix}\Sigma(\boldsymbol{\vartheta}_0), & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\begin{pmatrix}\boldsymbol{L}_1\\ \boldsymbol{L}_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{O}\\ \boldsymbol{O}\end{pmatrix}. \tag{B}$$

Let (B) be valid, i.e.

$$\begin{pmatrix}\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}', & -\mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{B}'\\ -\mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}', & \mathbf{I} - \mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{B}'\end{pmatrix}\begin{pmatrix}\Sigma(\boldsymbol{\vartheta}_0), & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\begin{pmatrix}\boldsymbol{L}_1\\ \boldsymbol{L}_2\end{pmatrix}$$
$$= \begin{pmatrix}[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}']\Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1\\ -\mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}'\,\Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1\end{pmatrix} = \begin{pmatrix}\boldsymbol{O}\\ \boldsymbol{O}\end{pmatrix}$$

Let (A) be valid, i.e. (cf. Lemma 2.4 (c), where $\mathbf{W} = \mathbf{I}$, $\mathbf{V} = \mathbf{I}$) $\{\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}' + \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{B}']^-\mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}'\}\Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1 = \boldsymbol{O}$. Let ① $= \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}'$, ② $= \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{B}']^- \cdot \cdot\mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}'$. As the matrices ① and ② are symmetric and p.s.d., we have:

$$\begin{aligned}\text{(A)} &\iff \Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1 \perp \mathcal{M}(① + ②) \iff \Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1 \perp \mathcal{M}(①, ②)\\ &\iff \Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1 \perp \mathcal{M}(①)\ \&\ \Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1 \perp \mathcal{M}(②)\\ &\iff \text{(B)}.\end{aligned}$$

The equivalence $\Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1 \perp \mathcal{M}(① + ②) \iff \Sigma(\boldsymbol{\vartheta}_0)\boldsymbol{L}_1 \perp \mathcal{M}(①, ②)$ is a consequence of Lemma 1.1 (The implication (B) $\implies$ (A) is obvious.) □

The rule R_4 in LMC (2.2) states: a statistic $\boldsymbol{L}_1'(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0) + \boldsymbol{f}'\boldsymbol{\beta}_0$, where $\mathbf{K}_\mathbf{B}'\boldsymbol{f} = \mathbf{K}_\beta'\mathbf{X}'\boldsymbol{L}_1$, is the UBLUE of its mean value $f(\boldsymbol{\beta}) = \boldsymbol{f}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}$, iff $\boldsymbol{L}_1 \in \operatorname{Ker}\left(\sum_{i=1}^p \mathbf{V}_i\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{V}_i\right)$. The corresponding statistic in LMC (2.3) is $\boldsymbol{L}_1'\boldsymbol{Y} + \boldsymbol{L}_2'(-\boldsymbol{b})$, where $\boldsymbol{L}_1'\mathbf{X} + \boldsymbol{L}_2'\mathbf{B} = \boldsymbol{f}'$.
The question is whether

$$\boldsymbol{L}_1 \in \operatorname{Ker}\left(\sum_{i=1}^p \mathbf{V}_i\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{V}_i\right)$$
$$\iff \begin{pmatrix}\boldsymbol{L}_1\\ \boldsymbol{L}_2\end{pmatrix} \in \operatorname{Ker}\left[\sum_{i=1}^p \begin{pmatrix}\mathbf{V}_i, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\mathbf{M}_{\begin{pmatrix}\mathbf{X}\\ \mathbf{B}\end{pmatrix}}\begin{pmatrix}\mathbf{V}_i, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\right]. \tag{2.4}$$

Theorem 2.6.

(a) *The equivalence* (2.4) *is valid; thus the rule* R_4 *holds in* LMC (2.3), *i.e.* $\boldsymbol{L}_1'\boldsymbol{Y}+\boldsymbol{L}_2'(-\boldsymbol{b})$ *is the* UBLUE *of its mean value iff*

$$\boldsymbol{L}_1 \in \operatorname{Ker}\Big(\sum_{i=1}^{p} \mathbf{V}_i\{\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{X}' + \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{B}']^{-}\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{X}'\}\mathbf{V}_i\Big).$$

(b) *If* $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}')$, *then* $\boldsymbol{L}_1'\boldsymbol{Y}+\boldsymbol{L}_2'(-\boldsymbol{b})$ *is the* UBLUE *of its mean value iff*

$$\boldsymbol{L}_1 \in \operatorname{Ker}\Big(\sum_{i=1}^{p} \mathbf{V}_i\mathbf{M}_{\mathbf{X}}\mathbf{V}_i + \sum_{i=1}^{p} \mathbf{V}_i\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{B}'}\mathbf{V}_i\Big) \subset \operatorname{Ker}\Big(\sum_{i=1}^{p} \mathbf{V}_i\mathbf{M}_{\mathbf{X}}\mathbf{V}_i\Big).$$

(c)

$$\operatorname{Ker}\Big(\sum_{i=1}^{p} \mathbf{V}_i\{\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{X}' + \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{B}']^{-}\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{X}'\}\mathbf{V}_i\Big) = \operatorname{Ker}\Big\{\sum_{i=1}^{p} \mathbf{V}_i[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{X}']\mathbf{V}_i\Big\}.$$

Proof. As the matrices $\mathbf{V}_i\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{V}_i$, $i=1,\dots,p$, are p.s.d., we have

$$\boldsymbol{L}_1 \in \operatorname{Ker}\Big(\sum_{i=1}^{p} \mathbf{V}_i\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{V}_i\Big) \iff \boldsymbol{L}_1 \perp \mathcal{M}\left(\mathbf{V}_i\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{V}_i\right), \quad i=1,\dots,p$$
$$\iff \boldsymbol{L}_1 \perp \mathcal{M}(\mathbf{V}_i\{\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{X}' + \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{B}'\cdot [\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{B}']^{-}\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{X}'\}\mathbf{V}_i), \quad i=1,\dots,p.$$

Similarly

$$\begin{pmatrix}\boldsymbol{L}_1\\ \boldsymbol{L}_2\end{pmatrix} \in \operatorname{Ker}\left[\sum_{i=1}^{p}\begin{pmatrix}\mathbf{V}_i, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\begin{pmatrix}\mathbf{V}_i, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\right]$$
$$\iff \boldsymbol{L}_1 \in \operatorname{Ker}\Big\{\sum_{i=1}^{p} \mathbf{V}_i[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{X}']\mathbf{V}_i\Big\}$$
$$\iff \boldsymbol{L}_1 \perp \mathcal{M}\{\mathbf{V}_i[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^{-}\mathbf{X}']\mathbf{V}_i\}, \quad i=1,\dots,p.$$

The matrix $\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}} = \begin{pmatrix} \mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}', & -\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}' \\ -\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}', & \mathbf{I}-\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}' \end{pmatrix}$ is p.s.d., therefore

$$\mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'] \subset \mathcal{M}[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}']$$
$$\Longrightarrow \mathcal{M}[\mathbf{V}_i\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'] \subset \mathcal{M}\{\mathbf{V}_i[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}']\mathbf{V}_i\}$$

(the matrix $\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'$ is p.s.d. since it is a diagonal submatrix of a p.s.d. matrix). This inclusion implies the equivalence

$$\begin{aligned} &\boldsymbol{L}_1 \perp \mathcal{M}(\mathbf{V}_i\{\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}' \\ &\quad +\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}']^-\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'\}\mathbf{V}_i) \\ &\qquad \Longleftrightarrow \boldsymbol{L}_1 \perp \mathcal{M}\{\mathbf{V}_i[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}']\mathbf{V}_i\} \end{aligned}$$

which proves (a) and (c).

If $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}')$, then, with respect to Lemma 2.4. (a),

$$\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \mathbf{M}_\mathbf{X} + \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{B}'}.$$

If the relationship

$$\begin{aligned} &\mathrm{Ker}\Big(\sum_{i=1}^p \mathbf{V}_i\mathbf{M}_\mathbf{X}\mathbf{V}_i + \sum_{i=1}^p \mathbf{V}_i\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{B}'}\mathbf{V}_i\Big) \\ &\quad = \Big[\mathcal{M}\Big(\sum_{i=1}^p \mathbf{V}_i\mathbf{M}_\mathbf{X}\mathbf{V}_i + \sum_{i=1}^p \mathbf{V}_i\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{B}'}\mathbf{V}_i\Big)\Big]^\perp \\ &\quad = \Big[\mathcal{M}\Big(\sum_{i=1}^p \mathbf{V}_i\mathbf{M}_\mathbf{X}\mathbf{V}_i, \sum_{i=1}^p \mathbf{V}_i\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{B}'}\mathbf{V}_i\Big)\Big]^\perp \\ &\qquad\qquad \subset \Big[\mathcal{M}\Big(\sum_{i=1}^p \mathbf{V}_i\mathbf{M}_\mathbf{X}\mathbf{V}_i\Big)\Big]^\perp = \mathrm{Ker}\Big(\sum_{i=1}^p \mathbf{V}_i\mathbf{M}_\mathbf{X}\mathbf{V}_i\Big) \end{aligned}$$

are taken into account, then (b) is proved. □

Lemma 2.7. *In* LMC

(a) $\mathcal{M}[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'] = \mathcal{M}(\mathbf{M}_\mathbf{X}) \oplus \mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}']$,
(b) $\mathcal{M}(\mathbf{X}) = \mathcal{M}(\mathbf{X}\mathbf{K}_\mathbf{B}) \oplus \mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}']$.

P r o o f.

(a) $\mathcal{M} = \mathcal{M}[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'] = \{[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'](\mathbf{X}\boldsymbol{u}+\boldsymbol{k}_{\mathbf{X}'}): \boldsymbol{u} \in \mathbb{R}^k,\ \boldsymbol{k}_{\mathbf{X}'} \in \mathrm{Ker}(\mathbf{X}')\}$ since $\mathcal{M}(\mathbf{X}) \oplus \mathrm{Ker}(\mathbf{X}') = \mathbb{R}^n$. Thus $\mathcal{M} = \{\mathbf{X}\boldsymbol{u}+\boldsymbol{k}_{\mathbf{X}'} -$

$\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B}-\mathbf{B}'\mathbf{B})\boldsymbol{u}\colon \boldsymbol{u}\in \mathbb{R}^k,\ \boldsymbol{k}_{\mathbf{X}'}\in \mathrm{Ker}(\mathbf{X}')\} = \mathrm{Ker}(\mathbf{X}')\oplus \mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'\mathbf{B}] = \mathcal{M}(\mathbf{M}_{\mathbf{X}})\oplus\mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}']$. The equality $\mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'\mathbf{B}] = \mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}']$ is implied by the following relations: $\mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'] \supset \mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'\mathbf{B}] \supset \mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'] = \mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}']$ since for any matrix $\mathbf{A}$ we have $\mathcal{M}(\mathbf{A}) = \mathcal{M}(\mathbf{A}\mathbf{A}')$.

(b) Let $\mathbf{T} = \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'$ and $\mathbf{U} = \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\cdot \mathbf{B}']^-\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'$. Then $\mathbf{P}_{\mathbf{X}\mathbf{K}_{\mathbf{B}}} = \mathbf{T}-\mathbf{U}$ (cf. Lemma 2.4 for $\mathbf{W} = \mathbf{I}$ and $\mathbf{V} = \mathbf{I}$) and $\mathbf{T} = \mathbf{P}_{\mathbf{X}\mathbf{K}_{\mathbf{B}}} + \mathbf{U}$. Both matrices $\mathbf{P}_{\mathbf{X}\mathbf{K}_{\mathbf{B}}}$, $\mathbf{U}$ are p.s.d. and $\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}' = \mathbf{O} \implies \mathcal{M}(\mathbf{P}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}})\perp\mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'] \implies \mathcal{M}(\mathbf{P}_{\mathbf{X}\mathbf{K}_{\mathbf{B}}})\perp\mathcal{M}(\mathbf{U}) \iff \mathcal{M}(\mathbf{X}\mathbf{K}_{\mathbf{B}})\perp\mathcal{M}(\mathbf{U})$.

Thus $\mathcal{M}\left(\mathbf{P}_{\mathbf{X}\mathbf{K}_{\mathbf{B}}}+\mathbf{U}\right) = \mathcal{M}\left(\mathbf{P}_{\mathbf{X}\mathbf{K}_{\mathbf{B}}},\mathbf{U}\right) = \mathcal{M}(\mathbf{X}\mathbf{K}_{\mathbf{B}})\oplus \mathbf{M}(\mathbf{U}) = \mathcal{M}(\mathbf{T})$. The equality $\mathcal{M}(\mathbf{T}) = \mathcal{M}(\mathbf{X})$ is implied by the following: $\mathcal{M}(\mathbf{X}') \subset \mathcal{M}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B}) = \mathcal{M}[(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^+]$, where $(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^+$ can be expressed as $\mathbf{J}\mathbf{J}'$; thus $\exists\{\mathbf{F}\colon \mathbf{X}' = \mathbf{J}\mathbf{F}\} \implies \mathcal{M}(\mathbf{X}) = \mathcal{M}(\mathbf{F}'\mathbf{J}') = \mathcal{M}(\mathbf{F}'\mathbf{J}'\mathbf{J}\mathbf{F}) \subset \mathcal{M}(\mathbf{F}'\mathbf{J}'\mathbf{J}) = \mathcal{M}(\mathbf{F}'\mathbf{J}'\mathbf{J}\mathbf{J}'\mathbf{J}\mathbf{F}) = \mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'] \subset \mathcal{M}(\mathbf{F}'\mathbf{J}') = \mathcal{M}(\mathbf{X})$. The equality $\mathcal{M}(\mathbf{U}) = \mathcal{M}[\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}']$ can be proved by an analogous consideration from the inclusion $\mathcal{M}[\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'] \subset \mathcal{M}[\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}']$ $(\Longleftarrow \mathcal{M}(\mathbf{B}\mathbf{J}\mathbf{J}'\mathbf{X}') \subset \mathcal{M}(\mathbf{B}\mathbf{J}) = \mathcal{M}(\mathbf{B}\mathbf{J}\mathbf{J}'\mathbf{B}'))$.

Remark 2.8. Lemma 2.7 (a) and Theorem 2.6 show that the conditions on UBLUE are stronger in LMC than in LM. It is implied by the inclusion $\mathrm{Ker}\left\{\sum_{i=1}^{p}\mathbf{V}_i[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}']\mathbf{V}_i\right\} \subset \mathrm{Ker}\left(\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{\mathbf{X}}\mathbf{V}_i\right)$. When $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}')$, then the statement is obvious directly from Theorem 2.6 (b); further from Lemma 2.4 (a)

$$\begin{aligned}
&\mathbf{M}_{\mathbf{X}\mathbf{K}_{\mathbf{B}}} = \mathbf{M}_{\mathbf{X}} + \mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'}\\
&\implies \mathcal{M}\left(\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{\mathbf{X}\mathbf{K}_{\mathbf{B}}}\mathbf{V}_i\right)\\
&\qquad = \mathcal{M}\left(\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{\mathbf{X}}\mathbf{V}_i + \mathbf{V}_i\mathbf{P}_{\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'}\mathbf{V}_i\right) \supset \mathcal{M}\left(\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{\mathbf{X}}\mathbf{V}_i\right)\\
&\iff \mathrm{Ker}\left(\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{\mathbf{X}\mathbf{K}_{\mathbf{B}}}\mathbf{V}_i\right) \subset \mathrm{Ker}\left(\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{\mathbf{X}}\mathbf{V}_i\right).
\end{aligned}$$

Theorem 2.9. *In* LMC *a function* $f(\boldsymbol{\beta}) = \boldsymbol{f}'\boldsymbol{\beta}$, $\boldsymbol{\beta}\in\mathcal{V}$, *can be estimated by* UBLUE *iff* $\boldsymbol{f} \in \mathcal{M}\left(\mathbf{X}'\mathrm{Ker}\left\{\sum_{i=1}^{p}\mathbf{V}_i[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}']\mathbf{V}_i\right\}, \mathbf{B}'\right)$.

Proof. As $f(\cdot)$ is unbiasedly estimable $\boldsymbol{f} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$ (cf. Theorem 2.2), i.e. $\exists \{\boldsymbol{L}_1 \in \mathbb{R}^n, \boldsymbol{L}_2 \in \mathbb{R}^q\}$ $\boldsymbol{f} = \mathbf{X}'\boldsymbol{L}_1 + \mathbf{B}'\boldsymbol{L}_2$ and $\boldsymbol{L}_1'\boldsymbol{Y} + \boldsymbol{L}_2'(-\boldsymbol{b})$ is an unbiased estimator. It is UBLUE with respect to Theorem 2.6 iff $\boldsymbol{L}_1 \in \operatorname{Ker}\Big\{\sum_{i=1}^{p} \mathbf{V}_i[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}']\mathbf{V}_i\Big\} \Longrightarrow \boldsymbol{f} = \mathbf{X}'\boldsymbol{L}_1 + \mathbf{B}'\boldsymbol{L}_2 \in \mathcal{M}\Big(\mathbf{X}'\operatorname{Ker}\Big\{\sum_{i=1}^{p} \mathbf{V}_i[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}']\mathbf{V}_i\Big\}, \mathbf{B}'\Big)$. □

Remark 2.10. In LMC, the condition $\boldsymbol{b} + \mathbf{B}\boldsymbol{\beta} = \boldsymbol{O}$ enlarges the class of the linear functions of $\boldsymbol{\beta}$ which are uniformly best linearly estimable by adding $\mathcal{M}(\mathbf{B}')$ but simultaneously it reduces this class with respect to Remark 2.8. Compare R_5 in Lemma 1.2.

Lemma 2.11. *Let in* $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\beta} \in \mathbb{R}^k, \Sigma(\boldsymbol{\vartheta}), \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}})$, $h(\boldsymbol{\beta}) = \boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, *be a function with the property* $\boldsymbol{h} \in \mathcal{M}(\mathbf{X}')$. *Then the* $\boldsymbol{\vartheta}_0$-LBLUE *of it can be calculated by the following equivalent (i.e. the same with probability one) expressions.*

(1) $\widehat{\boldsymbol{h}'\boldsymbol{\beta}} = \boldsymbol{h}'\left[(\mathbf{X}')^-_{m[\Sigma(\boldsymbol{\vartheta}_0)]}\right]'\boldsymbol{Y}$,

(2) $\widehat{\boldsymbol{h}'\boldsymbol{\beta}} = \boldsymbol{h}'(\mathbf{X}'\mathbf{M}\mathbf{X})^-\mathbf{X}'\mathbf{M}\boldsymbol{Y}$,

where $\mathbf{M} = (\Sigma(\boldsymbol{\vartheta}_0) + \mathbf{X}\mathbf{U}\mathbf{X}')^+ + \mathbf{K}$, $R(\mathbf{X}'\mathbf{M}\mathbf{X}) = R(\mathbf{X}')$,
$\mathbf{U}$, $\mathbf{K}$ *are arbitrary matrices with properties* $\mathcal{M}(\Sigma(\boldsymbol{\vartheta}_0), \mathbf{X}) = \mathcal{M}(\Sigma(\boldsymbol{\vartheta}_0) + \mathbf{X}\mathbf{U}'\mathbf{X}') = \mathcal{M}(\Sigma(\boldsymbol{\vartheta}_0) + \mathbf{X}\mathbf{U}\mathbf{X}')$ *and* $\Sigma(\boldsymbol{\vartheta}_0)\mathbf{K}'\mathbf{X} = \mathbf{O}$, $\mathbf{X}'\mathbf{K}\mathbf{X} = \mathbf{O}$,

(3) $\widehat{\boldsymbol{h}'\boldsymbol{\beta}} = \boldsymbol{h}'\mathbf{C}_3\boldsymbol{Y} = \boldsymbol{h}'\mathbf{C}_2'\boldsymbol{Y}$,

where

$$\begin{pmatrix} \Sigma(\boldsymbol{\vartheta}_0), & \mathbf{X} \\ \mathbf{X}', & \mathbf{O} \end{pmatrix}^- = \begin{pmatrix} \mathbf{C}_1, & \mathbf{C}_2 \\ \mathbf{C}_3, & -\mathbf{C}_4 \end{pmatrix},$$

(4) $\widehat{\boldsymbol{h}'\boldsymbol{\beta}} = \boldsymbol{h}'\left[\mathbf{X}'(\Sigma(\boldsymbol{\vartheta}_0) + \mathbf{X}\mathbf{X}')^-\mathbf{X}\right]^-\mathbf{X}'(\Sigma(\boldsymbol{\vartheta}_0) + \mathbf{X}\mathbf{X}')^-\boldsymbol{Y}$

$\Big($*a special choice of* $(\mathbf{X}')^-_{m[\Sigma(\boldsymbol{\vartheta}_0)]}\Big)$,

(5) $\widehat{\boldsymbol{h}'\boldsymbol{\beta}} = \boldsymbol{h}'\left\{(\Sigma(\boldsymbol{\vartheta}_0) + \mathbf{X}\mathbf{X}')^-\mathbf{X}\left[\mathbf{X}\mathbf{X}'(\Sigma(\boldsymbol{\vartheta}_0) + \mathbf{X}\mathbf{X}')^-\mathbf{X}\right]^-\mathbf{X}\right\}'\boldsymbol{Y}$

$\Big($*a special choice of* $(\mathbf{X}')^-_{\pi[\Sigma(\boldsymbol{\vartheta}_0)]}\Big)$.

Proof. See in [4] and [1, p. 161].

Theorem 2.12. *Let in* $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\beta} \in \mathcal{V}, \Sigma(\boldsymbol{\vartheta}), \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}})$, $\mathcal{V} = \{\boldsymbol{u}: \boldsymbol{b} + \mathbf{B}\boldsymbol{u} = \boldsymbol{O}\}$, $f(\boldsymbol{\beta}) = \boldsymbol{f}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}$, *be a function with the property* $\boldsymbol{f} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$. *Then the* $\boldsymbol{\vartheta}_0$-LBLUE *of it can be calculated by the expressions given in Lemma* 2.11, *where* $\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}$ *is substituted for* $\mathbf{X}$, $\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix}$ *for* $\Sigma(\vartheta_0)$ *and* $\begin{pmatrix} \mathbf{Y} \\ -\boldsymbol{b} \end{pmatrix}$ *for* $\mathbf{Y}$.

Proof.

(1) is a consequence of Theorem 2.2 and definition of the ϑ_0-LBLUE.

(2) Let $\mathbf{M} = \left[\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} + \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \mathbf{U}(\mathbf{X}', \mathbf{B}')\right]^- + \mathbf{K}$,

where $R\left((\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\right) = R(\mathbf{X}', \mathbf{B}')$.

Then the system $(\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\widehat{\boldsymbol{\beta}} = (\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \mathbf{Y} \\ -\boldsymbol{b} \end{pmatrix}$ is solvable.

Furthermore the relationship

$$\left[\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} + \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \mathbf{U}(\mathbf{X}', \mathbf{B}')\right] \mathbf{M}' \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}$$
$$= \left[\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} + \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \mathbf{U}(\mathbf{X}', \mathbf{B}')\right] \cdot$$
$$\cdot \left(\left\{\left[\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} + \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \mathbf{U}(\mathbf{X}', \mathbf{B}')\right]^-\right\}' + \mathbf{K}'\right) \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}$$

implies

$$\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} \mathbf{M}' \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} - \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \mathbf{U}(\mathbf{X}', \mathbf{B}')\mathbf{M}' \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \mathbf{Q}.$$

Hence, if $\mathbf{Z}$ is an arbitrary matrix such that $\mathcal{M}(\mathbf{Z}) \subset \mathrm{Ker}(\mathbf{X}', \mathbf{B}')$, then

$$(\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix}\mathbf{Z} = \mathbf{Q}'(\mathbf{X}', \mathbf{B}')\mathbf{Z} = \mathbf{O},$$

which means with respect to Theorem 2.3 and Theorem 2.5 that $(\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \mathbf{Y} \\ -\boldsymbol{b} \end{pmatrix}$ is the ϑ_0-LBLUE of its mean value $(\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{V}$; thus

$$(\boldsymbol{L}_1', \boldsymbol{L}_2')\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\left[(\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\right]^-(\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \mathbf{Y} \\ -\boldsymbol{b} \end{pmatrix}$$

is the ϑ_0-LBLUE of the function

$$f(\boldsymbol{\beta}) = (\boldsymbol{L}_1', \boldsymbol{L}_2')\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\left[(\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\right]^-(\mathbf{X}', \mathbf{B}')\mathbf{M}\begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix}\boldsymbol{\beta}$$
$$= (\boldsymbol{L}_1'\mathbf{X} + \boldsymbol{L}_2'\mathbf{B})\boldsymbol{\beta} = \boldsymbol{f}'\boldsymbol{\beta}, \quad \boldsymbol{\beta} \in \mathcal{V}.$$

(Cf. [4] and Theorem 5.3.2 in [1], where an analogous consideration in LM is made.)

(3) is a consequence of the properties of the *Pandora-Box matrix*

$$\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} & \vdots & \mathbf{X} \\ \mathbf{O}, & \mathbf{O} & \vdots & \mathbf{B} \\ \hdashline \mathbf{X}', & \mathbf{B}' & \vdots & \mathbf{O} \end{pmatrix}^{-} = \begin{pmatrix} \mathbf{C}_1 & \vdots & \mathbf{C}_2 \\ \hdashline \mathbf{C}_3 & \vdots & -\mathbf{C}_4 \end{pmatrix}$$

cf. [4] or Theorem 5.5.6 in [1] which states that

$$\mathbf{C}_2 = (\mathbf{X}', \mathbf{B}')^{-}_{m\left[\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix}\right]} \quad \text{and} \quad \mathbf{C}_3' = (\mathbf{X}', \mathbf{B}')^{-}_{m\left[\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix}\right]}.$$

(4) and (5) can be obtained by a special choice of the matrix $(\mathbf{X}', \mathbf{B}')^{-}_{m\left[\begin{pmatrix} \Sigma(\vartheta_0), & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix}\right]}$ (in detail see in [1] Lemma 2.1.20).

Remark 2.13. An estimation of the first order parameter β in LMC can be proceed with respect to Theorem 2.12, Lemma 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.5, Theorem 2.6 and Theorem 2.9, in several different ways. When a numerical calculation is large, i.e. the numbers n, k, q are large, then the possibility to obtain the same results in different ways is welcome from the point of view of checking the numerical stability. An analogous possibility would be welcome in the estimation of the second order parameter.

3. Estimators of the second order parameters

The simplest kind of an estimator of a function $g(\vartheta) = \mathbf{g}'\vartheta$, $\vartheta \in \underline{\vartheta}$, is $\mathbf{Y}'\mathbf{U}\mathbf{Y}$, where the matrix $\mathbf{U}$ fulfils conditions for unbiasedness, i.e. $\forall\{\beta \in \mathcal{V}\}\ \forall\{\vartheta \in \underline{\vartheta}\} E(\mathbf{Y}'\mathbf{U}\mathbf{Y} \mid \beta, \vartheta) = \mathbf{g}'\vartheta$ and invariance, i.e. $\forall\{\beta \in \mathcal{V}\}\ (\mathbf{Y} + \mathbf{X}\beta)'\mathbf{U}(\mathbf{Y} + \mathbf{X}\beta) = \mathbf{Y}'\mathbf{U}\mathbf{Y}$.

In LMC, two problems arise from the point of view of equivalent algorithms. The first one is connected with an existence of the unbiased and invariant estimator. With respect to Lemma 1.7, the matrix $\mathbf{K}^{(I)}$ in LMC is given by the relations

$$\{\mathbf{K}^{(I)}\}_{i,j} = \mathrm{Tr}\left(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}} \mathbf{V}_i \mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}} \mathbf{V}_j\right), \quad i, j = 1, \ldots, p$$

and the problem is if

$$\mathbf{g} \in \mathcal{M}(\mathbf{K}^{(I)}) \iff \mathbf{g} \in \mathcal{M}(\widetilde{\mathbf{K}}^{(I)}),$$

where

$$\{\widetilde{\mathsf{K}}^{(\mathsf{I})}\}_{i,j} = \mathrm{Tr}\left[\mathsf{M}_{\binom{\mathsf{X}}{\mathsf{B}}}\begin{pmatrix}\mathsf{V}_i, & \mathsf{O}\\ \mathsf{O}, & \mathsf{O}\end{pmatrix}\mathsf{M}_{\binom{\mathsf{X}}{\mathsf{B}}}\begin{pmatrix}\mathsf{V}_j, & \mathsf{O}\\ \mathsf{O}, & \mathsf{O}\end{pmatrix}\right], \quad i,j = 1,\dots,p.$$

The other problem is connected with the expression for the estimator. With respect to Lemma 1.8, the MINQUE or the ϑ_0-LMVQUIE in the case of normality is

$$\sum_{i=1}^{p}\lambda_i(\boldsymbol{Y}-\mathsf{X}\boldsymbol{\beta}_0)'\left(\mathsf{M}_{\mathsf{X}\mathsf{K}_\mathsf{B}}\Sigma_0\mathsf{M}_{\mathsf{X}\mathsf{K}_\mathsf{B}}\right)^+\mathsf{V}_i\left(\mathsf{M}_{\mathsf{X}\mathsf{K}_\mathsf{B}}\Sigma_0\mathsf{M}_{\mathsf{X}\mathsf{K}_\mathsf{B}}\right)^+(\boldsymbol{Y}-\mathsf{X}\boldsymbol{\beta}_0),$$

where $\boldsymbol{\beta}_0$ is any solution to $\boldsymbol{b}+\mathsf{B}\boldsymbol{\beta}_0 = \boldsymbol{O}$. The question is whether this estimator can be calculated from the expression

$$\sum_{i=1}^{p}\lambda_i(\boldsymbol{Y}',-\boldsymbol{b}')\left[\mathsf{M}_{\binom{\mathsf{X}}{\mathsf{B}}}\begin{pmatrix}\Sigma(\vartheta_0), & \mathsf{O}\\ \mathsf{O}, & \mathsf{O}\end{pmatrix}\mathsf{M}_{\binom{\mathsf{X}}{\mathsf{B}}}\right]^+\begin{pmatrix}\mathsf{V}_i, & \mathsf{O}\\ \mathsf{O}, & \mathsf{O}\end{pmatrix}\cdot$$
$$\cdot\left[\mathsf{M}_{\binom{\mathsf{X}}{\mathsf{B}}}\begin{pmatrix}\Sigma(\vartheta_0), & \mathsf{O}\\ \mathsf{O}, & \mathsf{O}\end{pmatrix}\mathsf{M}_{\binom{\mathsf{X}}{\mathsf{B}}}\right]^+\begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{b}\end{pmatrix}.$$

Theorem 3.1. *Let the matrix* $\mathsf{K}^{(\mathsf{I})}$ *be given by the relations* $\{\mathsf{K}^{(\mathsf{I})}\}_{i,j} = \mathrm{Tr}\left(\mathsf{M}_{\mathsf{X}\mathsf{K}_\mathsf{B}}\mathsf{V}_i\mathsf{M}_{\mathsf{X}\mathsf{K}_\mathsf{B}}\mathsf{V}_j\right)$, $i,j = 1,\dots,p$, *where* $\mathsf{X},\mathsf{B},\mathsf{V}_1,\dots,\mathsf{V}_p$ *are matrices from* LMC. *Let the matrix* $\widetilde{\mathsf{K}}^{(\mathsf{I})}$ *be given by the relations*

$$\{\widetilde{\mathsf{K}}^{(\mathsf{I})}\}_{i,j} = \mathrm{Tr}\left[\mathsf{M}_{\binom{\mathsf{X}}{\mathsf{B}}}\begin{pmatrix}\mathsf{V}_i, & \mathsf{O}\\ \mathsf{O}, & \mathsf{O}\end{pmatrix}\mathsf{M}_{\binom{\mathsf{X}}{\mathsf{B}}}\begin{pmatrix}\mathsf{V}_j, & \mathsf{O}\\ \mathsf{O}, & \mathsf{O}\end{pmatrix}\right], \quad i,j = 1,\dots,p.$$

Then $\mathsf{M}(\mathsf{K}^{(\mathsf{I})}) = \mathsf{M}(\widetilde{\mathsf{K}}^{(\mathsf{I})})$.

P r o o f. The (i,j)th element of the matrix $\widetilde{\mathsf{K}}\mathsf{I}$ can be rewritten as

$$\{\mathsf{K}^{(\mathsf{I})}\}_{i,j} = \mathrm{Tr}\{[\mathsf{I}-\mathsf{X}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{X}']\mathsf{V}_i[\mathsf{I}-\mathsf{X}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{X}']\mathsf{V}_j\}$$

and with respect to Lemma 2.4 (c) with $\mathsf{W} = \mathsf{I}$ and $\mathsf{V} = \mathsf{I}$,
$\{\mathsf{K}^{(\mathsf{I})}\}_{i,j} = \mathrm{Tr}\big(\{\mathsf{I}-\mathsf{X}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{X}'+\mathsf{X}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{B}'[\mathsf{B}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{B}']^-\cdot$
$\cdot\mathsf{B}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{X}'\}\mathsf{V}_i\{\mathsf{I}-\mathsf{X}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})\mathsf{X}'+\mathsf{X}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{B}'[\mathsf{B}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{B}']^-\cdot$
$\cdot\mathsf{B}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{X}'\}\mathsf{V}_j\big)$.
Let us denote $\mathsf{S}_1 = \mathsf{I}-\mathsf{X}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{X}'$ and $\mathsf{S}_2 = \mathsf{S}_1+\mathsf{X}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{B}'[\mathsf{B}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{B}']^-\mathsf{B}(\mathsf{X}'\mathsf{X}+\mathsf{B}'\mathsf{B})^-\mathsf{X}'$.

As $\mathcal{M}\{\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}']^-\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'\} \subset \mathcal{M}(\mathbf{S}_1)$ (cf. Lemma 2.7 (a)) and $\mathbf{S}_2 - \mathbf{S}_1$ is obviously p.s.d., we have with respect to Lemma 1.1: $\mathcal{M}(\mathbf{S}_2) = \mathcal{M}(\mathbf{S}_1 + (\mathbf{S}_2 - \mathbf{S}_1)) = \mathcal{M}(\mathbf{S}_1, \mathbf{S}_2 - \mathbf{S}_1) = \mathcal{M}(\mathbf{S}_1)$. Now we use the spectral decomposition of the matrices $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively.

$$\mathbf{S}_1 = \mathbf{Q}_1\boldsymbol{\lambda}_1\mathbf{Q}_1', \quad \mathbf{Q}_1'\mathbf{Q}_1 = \mathbf{I}_{r,r}, \quad r = R(\mathbf{S}_1) = R(\mathbf{S}_2), \quad \boldsymbol{\lambda}_1 \text{ is p.d.},$$
$$\mathbf{S}_2 = \mathbf{Q}_2\boldsymbol{\lambda}_2\mathbf{Q}_2', \quad \mathbf{Q}_2'\mathbf{Q}_2 = \mathbf{I}_{r,r}, \quad \boldsymbol{\lambda}_2 \text{ is p.d.}$$

Thus the matrix $\widetilde{\mathbf{K}}\mathbf{l}$ can be expressed as $\widetilde{\mathbf{V}}'(\mathbf{S}_1 \otimes \mathbf{S}_1)\widetilde{\mathbf{V}} = \widetilde{\mathbf{V}}'(\mathbf{Q}_1 \otimes \mathbf{Q}_1)(\boldsymbol{\lambda}_1 \otimes \boldsymbol{\lambda}_1)(\mathbf{Q}_1' \otimes \mathbf{Q}_1')\widetilde{\mathbf{V}}$, where $\widetilde{\mathbf{V}} = (\text{vec}(\mathbf{V}_1), \dots, \text{vec}(\mathbf{V}_p))$ and the matrix $\mathbf{K}^{(\mathbf{l})}$ as $\widetilde{\mathbf{V}}'(\mathbf{S}_2 \otimes \mathbf{S}_2)\widetilde{\mathbf{V}} = \widetilde{\mathbf{V}}'(\mathbf{Q}_2 \otimes \mathbf{Q}_2)(\boldsymbol{\lambda}_2 \otimes \boldsymbol{\lambda}_2)(\mathbf{Q}_2' \otimes \mathbf{Q}_2')\widetilde{\mathbf{V}}$. As $\mathcal{M}(\mathbf{S}_1) = \mathcal{M}(\mathbf{Q}_1) = \mathcal{M}(\mathbf{S}_2) = \mathcal{M}(\mathbf{Q}_2)$ and the matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are of the full rank in columns, there exists a regular $r \times r$ matrix $\mathbf{R}$ such that $\mathbf{Q}_2 = \mathbf{Q}_1\mathbf{R}$. Thus

$$\mathbf{K}^{(\mathbf{l})} = \widetilde{\mathbf{V}}'(\mathbf{Q}_1 \otimes \mathbf{Q}_1)(\mathbf{R} \otimes \mathbf{R})(\boldsymbol{\lambda}_2 \otimes \boldsymbol{\lambda}_2)(\mathbf{R}' \otimes \mathbf{R}')(\mathbf{Q}_1' \otimes \mathbf{Q}_1')\widetilde{\mathbf{V}}.$$

As the matrices $\boldsymbol{\lambda}_1 \otimes \boldsymbol{\lambda}_1$ and $(\mathbf{R} \otimes \mathbf{R})(\boldsymbol{\lambda}_2 \otimes \boldsymbol{\lambda}_2)(\mathbf{R}' \otimes \mathbf{R}')$ are p.d., we have $\mathbf{M}(\widetilde{\mathbf{K}}\mathbf{l}) = \mathcal{M}(\widetilde{\mathbf{V}}'(\mathbf{Q}_1 \otimes \mathbf{Q}_1)) = \mathcal{M}(\mathbf{K}^{(\mathbf{l})})$. □

Theorem 3.2. *Let* $\boldsymbol{g} \in \mathcal{M}(\mathbf{K}^{(\mathbf{l})})$ *in* LMC *and let*

$$\tau_1(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0) = \sum_{i=1}^{p}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0)'\lambda_i(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\Sigma_0\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}})^+\mathbf{V}_i(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\Sigma_0\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}})^+(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0),$$

where

$$\left(\mathbf{S}_{(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\Sigma_0\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}})^+}\right)\boldsymbol{\lambda} = \boldsymbol{g}. \tag{3.1}$$

Let

$$\tau_2(\mathbf{Y}, -\boldsymbol{b}) = \sum_{i=1}^{p}(\mathbf{Y}', -\boldsymbol{b}')\lambda_i\left[\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\begin{pmatrix}\Sigma_0, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right]^+\begin{pmatrix}\mathbf{V}_i, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\cdot$$
$$\left[\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\begin{pmatrix}\Sigma_0, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right]^+\begin{pmatrix}\mathbf{Y}\\ -\boldsymbol{b}\end{pmatrix}$$

where

$$\mathbf{S}_{(*)^+}\boldsymbol{\lambda} = \boldsymbol{g}, \quad (*)^+ = \left[\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\begin{pmatrix}\Sigma_0, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right]^+. \tag{3.2}$$

Then

(a) $\tau_1(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0)$ *does not depend on the choice of* $\boldsymbol{\beta}_0 \in \mathcal{V}$.
(b) $\tau_1(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0) = \tau_2(\mathbf{Y}, -\boldsymbol{b})$ *with probability one.*

Proof.

(a) Let $\beta_{01}, \beta_{02} \in \mathcal{V}$, $\beta_{01} \neq \beta_{02}$. Then $\beta_{02} - \beta_{01} \in \mathcal{M}(\mathbf{K_B})$ and $\mathbf{X}(\beta_{02} - \beta_{01}) \in \mathcal{M}(\mathbf{XK_B})$. As $\left(\mathbf{M}_{\mathbf{XK_B}}\Sigma_0\mathbf{M}_{\mathbf{XK_B}}\right)^+ = \left(\mathbf{M}_{\mathbf{XK_B}}\Sigma_0\mathbf{M}_{\mathbf{XK_B}}\right)^+\mathbf{M}_{\mathbf{XK_B}}$, we have

$$\left(\mathbf{M}_{\mathbf{XK_B}}\Sigma_0\mathbf{M}_{\mathbf{XK_B}}\right)^+[\mathbf{Y} - \mathbf{X}\beta_{01} - (\mathbf{Y} - \mathbf{X}\beta_{02})]$$
$$= \left(\mathbf{M}_{\mathbf{XK_B}}\Sigma_0\mathbf{M}_{\mathbf{XK_B}}\right)^+\mathbf{M}_{\mathbf{XK_B}}\mathbf{X}(\beta_{02} - \beta_{01}) = \mathbf{O}.$$

(b) With respect to (a) β_0 can be chosen as $\mathbf{B}^-(-\boldsymbol{b})$ with an arbitrary g-inverse $\mathbf{B}^-$. thus $\mathbf{Y} - \mathbf{X}\beta_0 = (\mathbf{I}, -\mathbf{XB}^-)\begin{pmatrix}\mathbf{Y}\\ -\boldsymbol{b}\end{pmatrix}$ and

$$t_1 = (\mathbf{Y} - \mathbf{X}\beta_0)'\left(\mathbf{M}_{\mathbf{XK_B}}\Sigma_0\mathbf{M}_{\mathbf{XK_B}}\right)^+\mathbf{V}_i\left(\mathbf{M}_{\mathbf{XK_B}}\Sigma_0\mathbf{M}_{\mathbf{XK_B}}\right)^+(\mathbf{Y} - \mathbf{X}\beta_0)$$
$$= (\mathbf{Y}', -\boldsymbol{b}')\begin{pmatrix}\mathbf{I}\\ -(\mathbf{B}^-)'\mathbf{X}'\end{pmatrix}\left(\mathbf{M}_{\mathbf{XK_B}}\Sigma_0\mathbf{M}_{\mathbf{XK_B}}\right)^+\mathbf{V}_i\left(\mathbf{M}_{\mathbf{XK_B}}\Sigma_0\mathbf{M}_{\mathbf{XK_B}}\right)^+\cdot$$
$$\cdot(\mathbf{I}, -\mathbf{XB}^-)\begin{pmatrix}\mathbf{Y}\\ -\boldsymbol{b}\end{pmatrix}.$$

The corresponding term of $\tau_2(\mathbf{Y}, -\boldsymbol{b})$ is

$$t_2 = (\mathbf{Y}', -\boldsymbol{b}')\left[\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\begin{pmatrix}\Sigma_0, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right]^+\begin{pmatrix}\mathbf{V}_i, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\cdot$$
$$\cdot\left[\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\begin{pmatrix}\Sigma_0, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right]^+\begin{pmatrix}\mathbf{Y}\\ -\boldsymbol{b}\end{pmatrix}.$$

Let us denote $① = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}'$, $② = -\mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{B}'$, $\mathbf{S} = \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}'$ and $(*) = \mathbf{M}_{\mathbf{XK_B}}\Sigma_0\mathbf{M}_{\mathbf{XK_B}}$.

Then

$$\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\begin{pmatrix}\Sigma_0, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}} = \begin{pmatrix}①\\ ②'\end{pmatrix}\Sigma_0(①, ②).$$

Let us denote

$$\left[\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\begin{pmatrix}\Sigma_0, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right]^+ = \begin{pmatrix}Ⓐ, & Ⓑ\\ Ⓑ', & Ⓒ\end{pmatrix}.$$

Then

$$t_2 = (\mathbf{Y}', -\boldsymbol{b}')\begin{pmatrix}Ⓐ\\ Ⓑ'\end{pmatrix}\mathbf{V}_i(Ⓐ, Ⓑ)\begin{pmatrix}\mathbf{Y}\\ -\boldsymbol{b}\end{pmatrix}$$

and

$$t_1 = (\mathbf{Y}', -\boldsymbol{b}') \begin{pmatrix} (*)^+ \\ -(\mathbf{B}^-)'\mathbf{X}'(*)^+ \end{pmatrix} \mathbf{V}_i((*)^+, -(*)^+\mathbf{X}\mathbf{B}^-) \begin{pmatrix} \mathbf{Y} \\ -\boldsymbol{b} \end{pmatrix}.$$

If there exists © such that the equality

$$\begin{pmatrix} (*)^+, & -(*)^+\mathbf{X}\mathbf{B}^- \\ -(\mathbf{B}^-)'\mathbf{X}'(*)^+, & © \end{pmatrix} = \left[\begin{pmatrix} ① \\ ②' \end{pmatrix} \Sigma_0(①, ②) \right]^+$$

is valid, then (b) will be proved.

Let © $= (\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^-$. Let

$$\mathbf{E} = \begin{pmatrix} ① \\ ②' \end{pmatrix} \Sigma_0(①, ②) \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} (*)^+, & -(*)^+\mathbf{X}\mathbf{B}^- \\ -(\mathbf{B}^-)'\mathbf{X}'(*)^+, & (\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^- \end{pmatrix}.$$

Then it must be proved:
(1) $\mathbf{E} = \mathbf{EGE}$, (2) $\mathbf{G} = \mathbf{GEG}$, (3) $\mathbf{EG} = \mathbf{G}'\mathbf{E}'$, (4) $\mathbf{GE} = \mathbf{E}'\mathbf{G}'$.
As $\mathbf{E}$ and $\mathbf{G}$ are symmetric, the property (4) is implied by the property (3).

(1)

$$\mathbf{EGE} = \begin{pmatrix} ① \\ ②' \end{pmatrix} \Sigma_0(①, ②)\mathbf{G} \begin{pmatrix} ① \\ ②' \end{pmatrix} \Sigma_0(①, ②).$$

The term

$$\Sigma_0(①, ②)\mathbf{G} \begin{pmatrix} ① \\ ②' \end{pmatrix} \Sigma_0$$

can be expressed as

$$\Sigma_0①(*)^+①\Sigma_0 - \Sigma_0①(*)^+\mathbf{X}\mathbf{B}^-②'\Sigma_0 - \Sigma_0②(\mathbf{B}^-)'\mathbf{X}'(*)^+①\Sigma_0 + \Sigma_0②(\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^-②'\Sigma_0.$$

As $\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{X}\mathbf{B}^-\mathbf{B} = \mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{X}$, since $\mathcal{M}(\mathbf{K}_\mathbf{B}) = \mathcal{M}(\mathbf{I} - \mathbf{B}^-\mathbf{B})$ and $\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{X}(\mathbf{I} - \mathbf{B}^-\mathbf{B}) = \mathbf{O}$, we have $(*)^+\mathbf{X}\mathbf{B}^-②' = -(*)^+\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{X}\mathbf{B}^-\mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}' = -(*)^+\mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^-\mathbf{X}'$.
Thus

$$\begin{aligned} &\Sigma_0(①, ②)\mathbf{G} \begin{pmatrix} ① \\ ②' \end{pmatrix} \Sigma_0 \\ &\quad = \Sigma_0\left[(\mathbf{I} - \mathbf{S})(*)^+(\mathbf{I} - \mathbf{S}) + (\mathbf{I} - \mathbf{S})(*)^+\mathbf{S} + \mathbf{S}(*)^+(\mathbf{I} - \mathbf{S}) + \mathbf{S}(*)^+\mathbf{S}\right]\Sigma_0 \\ &\quad = \Sigma_0(*)^+\Sigma_0 \end{aligned}$$

and

$$\mathbf{EGE} = \begin{pmatrix} \textcircled{1}\Sigma_0(*)^+\Sigma_0\textcircled{1}, & \textcircled{1}\Sigma_0(*)^+\Sigma_0\textcircled{2} \\ \textcircled{2}'\Sigma_0(*)^+\Sigma_0\textcircled{1}, & \textcircled{2}'\Sigma_0(*)^+\Sigma_0\textcircled{2} \end{pmatrix}.$$

Further
$\textcircled{1}\Sigma_0(*)^+\Sigma_0\textcircled{1} = \textcircled{1}[\Sigma_0 - \mathbf{X}\mathbf{K_B}(\mathbf{K'_B}\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K_B})^+\mathbf{K'_B}\mathbf{X}']\textcircled{1} = \textcircled{1}\Sigma_0\textcircled{1} - [\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}']\mathbf{X}\mathbf{K_B}(\mathbf{K'_B}\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K_B})^+\mathbf{K'_B}\mathbf{X}'[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'] = \textcircled{1}\Sigma_0\textcircled{1}$,
since
$\mathbf{K'_B}\mathbf{X}'[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'] = \mathbf{K'_B}\mathbf{X}' - \mathbf{K'_B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B}-\mathbf{B}'\mathbf{B})(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}' = \mathbf{K'_B}\mathbf{B}'\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}' = \mathbf{O}$.
$\textcircled{1}\Sigma_0(*)^+\Sigma_0\textcircled{2} = \textcircled{1}[\Sigma_0 - \mathbf{X}\mathbf{K_B}(\mathbf{K'_B}\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K_B})^+\mathbf{K'_B}\mathbf{X}']\textcircled{2} = \textcircled{1}\Sigma_0\textcircled{2}$, since
$\textcircled{1}\mathbf{X}\mathbf{K_B} = [\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}']\mathbf{X}\mathbf{K_B} = \mathbf{X}\mathbf{K_B} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B}-\mathbf{B}'\mathbf{B})\mathbf{K_B} = \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'\mathbf{B}\mathbf{K_B} = \mathbf{O}$.
$\textcircled{2}'\Sigma_0(*)^+\Sigma_0\textcircled{2} = \textcircled{2}'[\Sigma_0 - \mathbf{X}\mathbf{K_B}(\mathbf{K'_B}\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K_B})^+\mathbf{K'_B}\mathbf{X}']\textcircled{2} = \textcircled{2}'\Sigma_0\textcircled{2}$, since
$\textcircled{2}'\mathbf{X}\mathbf{K_B} = -\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'\mathbf{X}\mathbf{K_B} = -\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B}-\mathbf{B}'\mathbf{B})\mathbf{K_B} = -\mathbf{B}\mathbf{K_B} + \mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'\mathbf{B}\mathbf{K_B} = \mathbf{O}$. Thus $\mathbf{EGE} = \mathbf{E}$.

(2)

$$\begin{pmatrix} (*)^+, & -(*)^+\mathbf{X}\mathbf{B}^- \\ -(\mathbf{B}^-)'\mathbf{X}'(*)^+, & (\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^- \end{pmatrix} \begin{pmatrix} \textcircled{1} \\ \textcircled{2}' \end{pmatrix} \Sigma_0(\textcircled{1}, \textcircled{2}) \cdot$$
$$\cdot \begin{pmatrix} (*)^+, & -(*)^+\mathbf{X}\mathbf{B}^- \\ -(\mathbf{B}^-)'\mathbf{X}'(*)^+, & (\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^- \end{pmatrix} = \begin{pmatrix} \textcircled{11}, & \textcircled{12} \\ \textcircled{21}, & \textcircled{22} \end{pmatrix},$$

$$\textcircled{11} = (*)^+\textcircled{1}\Sigma_0\textcircled{1}(*)^+ - (*)^+\mathbf{X}\mathbf{B}^-\textcircled{2}'\Sigma_0\textcircled{1}(*)^+ - (*)^+\textcircled{1}\Sigma_0\textcircled{2}(\mathbf{B}^-)'\mathbf{X}'(*)^+ + (*)^+\mathbf{X}\mathbf{B}^-\textcircled{2}'\Sigma_0\textcircled{2}(\mathbf{B}^-)'\mathbf{X}'(*)^+.$$

As

$$(*)^+\mathbf{X}\mathbf{B}^-\textcircled{2}' = -(*)^+\mathbf{M}_{\mathbf{X}\mathbf{K_B}}\mathbf{X}\mathbf{B}^-\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}' = -(*)^+\mathbf{S},$$

$\textcircled{11}$ can be rewritten as
$(*)^+[(\mathbf{I}-\mathbf{S})\Sigma_0(\mathbf{I}-\mathbf{S}) + \mathbf{S}\Sigma_0(\mathbf{I}-\mathbf{S}) + \Sigma_0\mathbf{S} + \mathbf{S}\Sigma_0\mathbf{S}](*)^+ = (*)^+\Sigma_0(*)^+ = [\Sigma_0^+ \cdot \cdot\Sigma_0\Sigma_0^+ - \Sigma_0^+\mathbf{X}\mathbf{K_B}(\mathbf{K'_B}\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K_B})^+\mathbf{K'_B}\mathbf{X}'\Sigma_0^+\Sigma_0\Sigma_0^+][\mathbf{I} - \mathbf{X}\mathbf{K_B}(\mathbf{K'_B}\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K_B})^+ \cdot \cdot\mathbf{K'_B}\mathbf{X}'\Sigma_0^+ = (*)^+$.
$\textcircled{12} = -(*)^+\textcircled{1}\Sigma_0\textcircled{1}(*)^+\mathbf{X}\mathbf{B}^- + (*)^+\mathbf{X}\mathbf{B}^-\textcircled{2}'\Sigma_0\textcircled{1}(*)^+\mathbf{X}\mathbf{B}^- + (*)^+\textcircled{1}\Sigma_0\textcircled{2}(\mathbf{B}^-)' \cdot \cdot\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^- - (*)^+\mathbf{X}\mathbf{B}^-\textcircled{2}'\Sigma_0\textcircled{2}(\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^- = -(*)^+\big[(\mathbf{I}-\mathbf{S})\Sigma_0(\mathbf{I}-\mathbf{S}) + \mathbf{S}\Sigma_0(\mathbf{I}-\mathbf{S}) + (\mathbf{I}-\mathbf{S})\Sigma_0\mathbf{S} + \mathbf{S}\Sigma_0\mathbf{S}\big](*)^+\mathbf{X}\mathbf{B}^- = -(*)^+\Sigma_0(*)^+\mathbf{X}\mathbf{B}^- = -(*)^+\mathbf{X}\mathbf{B}^-$.
$\textcircled{22} = (\mathbf{B}^-)'\mathbf{X}'(*)^+\textcircled{1}\Sigma_0\textcircled{1}(*)^+\mathbf{X}\mathbf{B}^- - (\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^-\textcircled{2}'\Sigma_0\textcircled{1}(*)^+\mathbf{X}\mathbf{B}^- - (\mathbf{B}^-)' \cdot \cdot\mathbf{X}'(*)^+\textcircled{1}\Sigma_0\textcircled{2}(\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^- + (\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^-\textcircled{2}'\Sigma_0\textcircled{2}(\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^- = (\mathbf{B}^-)'\mathbf{X}'(*)^+\big[(\mathbf{I}-\mathbf{S})\Sigma_0(\mathbf{I}-\mathbf{S}) + \mathbf{S}\Sigma_0(\mathbf{I}-\mathbf{S}) + (\mathbf{I}-\mathbf{S})\Sigma_0\mathbf{S} + \mathbf{S}\Sigma_0\mathbf{S}\big](*)^+\mathbf{X}\mathbf{B}^- = (\mathbf{B}^-)'\mathbf{X}'(*)^+\Sigma_0(*)^+\mathbf{X}\mathbf{B}^- = (\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^-$.
$\textcircled{21}$ is obviously equal to $-(\mathbf{B}^-)'\mathbf{X}'(*)^+$.
Thus $\mathbf{GEG} = \mathbf{G}$.

(3)

$$\begin{pmatrix} \textcircled{1} \\ \textcircled{2}' \end{pmatrix} \Sigma_0(\textcircled{1}, \textcircled{2}) \begin{pmatrix} (*)^+, & -(*)^+\mathbf{X}\mathbf{B}^- \\ -(\mathbf{B}^-)'\mathbf{X}'(*)^+, & (\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^- \end{pmatrix} = \begin{pmatrix} \textcircled{\alpha\alpha}, & \textcircled{\alpha\beta} \\ \textcircled{\beta\alpha}, & \textcircled{\beta\beta} \end{pmatrix}.$$

$\textcircled{\alpha\alpha} = \textcircled{1}\Sigma_0\textcircled{1}(*)^+ - \textcircled{1}\Sigma_0\textcircled{2}(\mathbf{B}^-)'\mathbf{X}'(*)^+$
$= (\mathbf{I}-\mathbf{S})\Sigma_0(\mathbf{I}-\mathbf{S})(*)^+ + (\mathbf{I}-\mathbf{S})\Sigma_0\mathbf{S}(*)^+ = (\mathbf{I}-\mathbf{S})\Sigma_0(*)^+$
$= [\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'][\Sigma_0\Sigma_0^+ - \Sigma_0\Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B}(\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B})^+\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+]$
$= \Sigma_0\Sigma_0^+ - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'$,
since $\mathbf{X}'\Sigma_0\Sigma_0^+ = \mathbf{X}'$ and
$[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}']\Sigma_0\Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B}$
$= [\mathbf{X} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B}-\mathbf{B}'\mathbf{B})]\mathbf{K}_\mathbf{B} = \mathbf{O}$.
Thus $\textcircled{\alpha\alpha} = \textcircled{\alpha\alpha}'$. Further $\textcircled{\alpha\beta} = -\textcircled{1}\Sigma_0\textcircled{1}(*)^+\mathbf{X}\mathbf{B}^- + \textcircled{1}\Sigma_0\textcircled{2}(\mathbf{B}^-)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^-$
$= -(\mathbf{I}-\mathbf{S})\Sigma_0(\mathbf{I}-\mathbf{S})(*)^+\mathbf{X}\mathbf{B}^- - (\mathbf{I}-\mathbf{S})\Sigma_0\mathbf{S}(*)^+\mathbf{X}\mathbf{B}^- = -(\mathbf{I}-\mathbf{S})\Sigma_0(*)^+\mathbf{X}\mathbf{B}^-$;
$\textcircled{\beta\alpha} = \textcircled{2}'\Sigma_0\textcircled{1}(*)^+ - \textcircled{2}'\Sigma_0\textcircled{2}(\mathbf{B}^-)'\mathbf{X}'(*)^+$
$= \textcircled{2}'\Sigma_0(\mathbf{I}-\mathbf{S})(*)^+ + \textcircled{2}'\Sigma_0\mathbf{S}(*)^+ = \textcircled{2}'\Sigma_0(*)^+$
$= -\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'\Sigma_0[\Sigma_0^+ - \Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B}(\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B})^+\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+]$
$= -\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'$;
$\textcircled{\alpha\beta}' = -(\mathbf{B}^-)'\mathbf{X}(*)^+\Sigma_0(\mathbf{I}-\mathbf{S})$
$= -(\mathbf{B}^-)'\mathbf{X}'[\Sigma_0^+ - \Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B}(\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B})^+\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+]\Sigma_0[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}']$
$= -(\mathbf{B}^-)'\mathbf{X}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'] = -(\mathbf{B}^-)'\mathbf{B}'\mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'$.

If $\mathbf{B}^+$ is chosen for $\mathbf{B}^-$, then $(\mathbf{B}^+)'\mathbf{B}'\mathbf{B} = \mathbf{B}$ $(\Longleftarrow \mathbf{B}'(\mathbf{B}^+)^+\mathbf{B}^+ = \mathbf{B}')$. Thus $\textcircled{\beta\alpha} = \textcircled{\alpha\beta}'$.
$\textcircled{\beta\beta} = -\textcircled{2}'\Sigma_0\textcircled{1}(*)^+\mathbf{X}\mathbf{B}^+ + \textcircled{2}'\Sigma_0\textcircled{2}(\mathbf{B}^+)'\mathbf{X}'(*)^+\mathbf{X}\mathbf{B}^+$
$= -\textcircled{2}'[\Sigma_0\textcircled{1} + \Sigma_0\mathbf{S}](*)^+\mathbf{X}\mathbf{B}^+ = -\textcircled{2}'\Sigma_0(*)^+\mathbf{X}\mathbf{B}^+$
$= \mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'\Sigma_0[\Sigma_0^+ - \Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B}(\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B})^+\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+]\mathbf{X}\mathbf{B}^+$
$= \mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'\mathbf{X}\mathbf{B}^+ - \mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B}-\mathbf{B}'\mathbf{B})\mathbf{K}_\mathbf{B}(\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{K}_\mathbf{B})^+\cdot$
$\cdot\mathbf{K}_\mathbf{B}'\mathbf{X}'\Sigma_0^+\mathbf{X}\mathbf{B}^+ = \mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{X}'\mathbf{X}\mathbf{B}^+ = \mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B}-\mathbf{B}'\mathbf{B})\mathbf{B}^+$
$= \mathbf{B}\mathbf{B}^+ - \mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'\mathbf{B}\mathbf{B}^+ = \mathbf{B}\mathbf{B}^+ - \mathbf{B}(\mathbf{X}'\mathbf{X}+\mathbf{B}'\mathbf{B})^-\mathbf{B}' = \textcircled{\beta\beta}'$.
Thus $\mathbf{E}\mathbf{G} = \mathbf{G}'\mathbf{E}'$, i.e. $\mathbf{G}\mathbf{E} = \mathbf{E}'\mathbf{G}'$.

The choice of the vector $\boldsymbol{\lambda}$ in $\tau_1(\boldsymbol{Y} - \mathbf{X}\beta_0)$ from (3.1) and in $\tau_2(\boldsymbol{Y}, -\boldsymbol{b})$ from (3.2), respectively, is a consequence of the unbiasedness of the estimators, cf. Lemma 1.8. □

R e m a r k 3.3. With respect to Theorem 3.2 there exist different expressions for the ϑ_0-MINQUE in LMC which gave the same estimator of the function $g(\cdot)$. This is a welcome possibility to check the numerical stability of the calculation.

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Received July 10, 1989

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