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ON THE COMPLETION OF A LATTICE BY ENDS

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Stimulated by Leader's and Finkelstein's [3] topological considerations, Arnow [1] defined the notion of a system of ends of a lattice.

Let L be a lattice. To each system of ends E there corresponds a lattice L_E. The main results of [1] are as follows (cf. [1], Theorem 1.1 and Theorem 1.2):

(A) The lattice L_E is conditionally complete.

(B) There exists an injection f of the lattice L into L_E and this mapping f is onto L_E if and only if L is conditionally complete.

Let us denote by $U(A)$ $(L(A))$ the set of all upper (lower) bounds of a subset $A \subseteq L$ in L. Let $d(L)$ be the conditional Dedekind completion of L (i.e., $d(L)$ is the system of all sets $L(U(A))$ where A is a nonempty and upper bounded subset of L. Cf., e.g., Birkhoff [2], p. 126) and let f_1 be the natural injection of L into $d(L)$. In this note it will be shown that for each system of ends E, the lattices L_E and $d(L)$ coincide up to isomorphisms leaving L fixed, i.e., that there is an isomorphism φ of $d(L)$ onto L_E such that $\varphi(f_1(x)) = f(x)$ is valid for each $x \in L$. In particular, if E_1 and E_2 are two systems of ends on L, then L_{E_1} is isomorphic to L_{E_2}.

Let $\mathscr{L}$ be the class of all lattices. A mapping t: $\mathscr{L} \to \mathscr{L}$ will be said to be a c-mapping, if it fulfils the following conditions for each $L \in \mathscr{L}$: (a) $t(L)$ is conditionally complete; (b) there exists an injection f_t of L into $t(L)$ having the property that f_t is an epimorphism if and only if L is conditionally complete. Two c-mappings t_1 and t_2 will be called equivalent if there exists an isomorphism ψ of $t_2(L)$ onto $t_1(L)$ and injections f_{t_1}, f_{t_2} into $t_1(L)$, $t_2(L)$, respectively, such that $\psi(f_{t_2}(x)) = f_{t_1}(x)$ for each $x \in L$. It is easy to verify that there exists a proper class of nonequivalent c-pappings (cf. also Example 4 below).

1. Preliminaries

Let us recall some definitions and results from [1] and [3]. Let $(L, \vee, \wedge)$ be a lattice. Suppose that there is defined a binary relation $\ll$ on L satisfying the following conditions:

A$_1$. If $a \ll b$, then $a \leq b$.

A$_2$. If $a \ll b \leq c$ or $a \leq b \ll c$, then $a \ll c$.

A_3. If $a \ll b$ and $c \ll d$, then $a \vee c \ll b \vee d$ and $a \wedge c \ll b \wedge d$.

A_4. If $a \ll c$, then there exists an element $b \in L$ such that $a \ll b \ll c$.

A_5. For each $b \in L$ there exist elements a and c in L such that $a \ll b \ll c$.

A_6. If $x \ll a$ implies $x \ll b$, then $a \leq b$.

A_7. If $a \ll x$ implies $b \ll x$, then $b \leq a$.

Then the structure $(L, \vee, \wedge, \ll)$ is said to be a regular lattice (cf. [1]).

Next we suppose that L is a regular lattice.

Let a, a' be elements of L with the property $a \ll a'$. The set $\{x \in L: a \ll x \ll a'\}$ will be denoted by (a, a') and called a cell from L. Denote by S the set of all cells from L.

It can be easily verified that the following assertions hold for each cell (a, a'), (b, b') from S (cf. [1]).

(a) If $(a, a') \cap (b, b') \neq \emptyset$, then $(a, a') \cap (b, b') = (a \vee b, a' \wedge b')$.

(b) $(a, a') \cap (b, b') \neq \emptyset$ if and only if $a \ll b'$ and $b \ll a'$.

Let S_1 be a subset of S. We say that a cell $(x, x') \in S$ clings to S_1 if $(a, a') \cap (x, x') \neq \emptyset$ for each cell $(a, a') \in S_1$ (cf. [3]).

Define a binary relation on S as follows: for each (a, a'), $(b, b') \in S$ we put

$$(a, a') \Subset (b, b') \quad \text{if} \quad b \ll a \text{ and } \quad a' \ll b'.$$

For subsets A and B of L, $A \ll B$ means that $a \ll b$ for each $a \in A$, $b \in B$. Let A, A' be nonempty subsets of L such that $A \ll A'$. Denote

$$A \times A' = \{(a, a') \in S: a \in A, a' \in A'\}.$$

Suppose that A and A' are nonempty subsets of L with $A \ll A'$. The set $A \times A'$ is said to be an end from S if the following conditions are fulfilled (cf. [3]):

E_1. If (a, a'), $(b, b') \in A \times A'$, then there exists a cell $(c, c') \in A \times A'$ such that $(c, c') \Subset (a, a') \cap (b, b')$.

E_2. If (a, a'), $(b, b') \in S$ such that (a, a') clings to $A \times A'$ and $(a, a') \Subset (b, b')$, then $(b, b') \in A \times A'$.

The condition E_1 is equivalent to $(a, a') \cap (b, b') \neq \emptyset$ for each (a, a'), $(b, b') \in A \times A'$.

From the definition it follows that if $A \times A'$ and $B \times B'$ are ends from S with $A \times A' \subseteq B \times B'$, then $A \times A' = B \times B'$ (each end is maximal with respect to the set inclusion). The set of all ends from S will be denoted by L_E.

Now we shall describe the construction of the completion of a lattice L by ends (cf. [1]).

Let $\leq$ be a binary relation on L_E defined in the following way: $A \times A' \leq B \times B'$ iff $A \subseteq B$. Then L_E is partially ordered by $\leq$, moreover, L_E is a conditionally complete lattice. The set $N^x = \{(y, y') \in S: x \in (y, y')\} \in L_E$ for each $x \in L$. The mapping $f(x) = N^x$ is an isomorphism from the lattice L into L_E and the mapping f is onto L_E if and only if L is conditionally complete. We shall call L_E the completion of L by ends.

2. The relation between $d(L)$ and L_E

Let L be a regular lattice. In this paragraph it will be shown that the conditional Dedekind completion $d(L)$ is isomorphic with the completion L_E by ends.

Let $x \in L$ and $z \in d(L)$. Denote

$$L(z)=\{a\in L\colon a\leq z\},\quad U(z)=\{a\in L\colon a\geq z\};$$
$$A_x=\{a\in L\colon a\ll x\},\quad A'_x=\{a\in L\colon a\gg x\};$$
$$A(z)=\cup A_x(x\in L(z)),\quad A'(z)=\cup A'_x(x\in U(z)).$$

The sets $L(z)$ and $U(z)$ are non-void. From A_5 we infer that A_x, A'_x and so $A(z)$, $A'(z)$ are non-void as well. Choose arbitrary $a\in A(z)$, $a'\in A'(z)$. Then there exist $x\in L(z)$, $x'\in U(z)$ such that $a\ll x$, $x'\ll a'$. By A_2 from $x\leq x'$ it follows that $a\ll a'$, and thus $A(z)\ll A'(z)$.

1. *A cell* $(x, x')\in S$ *clings to* $A(z)\times A'(z)$ *if and only if* $x\in L(z)$, $x'\in U(z)$.

Proof. Assume that (x, x') clings to $A(z)\times A'(z)$, u is an arbitrary element of $U(z)$ and that $a'\in L$ with the property $u\ll a'$. Then we have $a'\in A'(z)$. The hypothesis implies that $(x, x')\cap(a, a')\neq\emptyset$ for any $a\in A(z)$. By using (b) we obtain $x\ll a'$. We have shown that $u\ll a'$ implies $x\ll a'$. Hence according to A_7 $x\leq u$. From this it follows that $x\leq z$, i.e., $x\in L(z)$. It can be verified in an analogous manner that $x'\in U'(z)$.

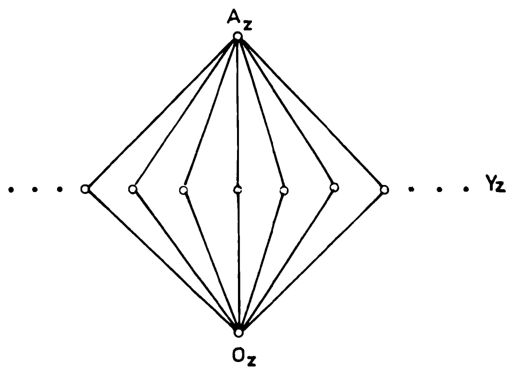

Fig. 1

Conversely, let (x, x') be a cell from S such that $x\in L(z)$, $x'\in U(z)$ and let (a, a') be an arbitrary cell belonging to $A(z)\times A'(z)$. There exists an element $x_1\in L(z)$ such that $a\ll x_1$. Since $x_1\leq x'$, by A_2 $a\ll x'$ holds. In a similar way we get $x\ll a'$. By using (b) we obtain $(x, x')\cap(a, a')\neq\emptyset$, which implies that (x, x') clings to $A(z)\times A'(z)$.

2. $A(z)\times A'(z)\in L_E$.

Proof. First, we intend to show that the condition E_1 is satisfied. Assume that

(a, a'), $(b, b')\in A(z)\times A'(z)$. From 1 we infer that $(a, a')\cap(b, b')\neq\emptyset$. Then with respect to (a), $(a, a')\cap(b, b')=(a\vee b, a'\wedge b')$. There exist elements $x\in L(z)$, $y\in L(z)$ with $a\ll x$, $b\ll y$. Hence by A_3 we have $a\vee b\ll x\vee y$. By A_4 there exists an element $c\in L$ with $a\vee b\ll c\ll x\vee y$. Since $x\vee y\in L(z)$, we conclude $c\in A(z)$. Similarly we prove the existence of an element $c'\in A'(z)$ having the property $c'\ll a'\wedge b'$. Therefore $c\ll c'$, $(c, c')\in A(z)\times A'(z)$ and $(c, c')\subset(a, a')\cap(b, b')$.

There remains to be shown that the condition E_2 holds. Suppose that (x, x'), $(y, y')\in S$, $(x, x')\Subset(y, y')$ and that (x, x') clings to $A(z)\times A'(z)$. Whence $y\ll x$, $x'\ll y'$ and from 1 we deduce $x\in L(z)$, $x'\in U(z)$. Then $y\in A(z)$, $y'\in A'(z)$ and thus $(y, y')\in A(z)\times A'(z)$.

Next we show that every end from S can be written in the form $A(z)\times A'(z)$.

3. *Let $B\times B'\in L_E$. Then there exists an element $z\in d(L)$ such that $B\times B'=A(z)\times A'(z)$.*

Proof. $B(B')$ is a nonempty upper (lower) bounded subset of L. It is clear that $\sup L(U(B))=\inf U(B)$. This element from $d(L)$ will be denoted by z. Hence $L(z)=L(U(B))$ and $U(z)=U(B)$.

It is enough to verify that $B\times B'\subseteq A(z)\times A'(z)$. Assume that $(b, b')\in B\times B'$. By E_1 there exists a cell $(x, x')\in B\times B'$ such that $(x, x')\subset(b, b')$, i.e., $b\ll x$, $x'\ll b'$. We claim that $b\in A(z)$, since $x\in B\subseteq L(z)$. Similarly we obtain that $b'\in A'(z)$. Consequently, $(b, b')\in A(z)\times A'(z)$ and so $B\times B'\subseteq A(z)\times A'(z)$. The validity of equality follows from the maximality of ends with respect to the set inclusion.

4. *Let z_1, $z_2\in d(L)$. Then $z_1\leq z_2$ if and only if $A(z_1)\subseteq A(z_2)$.*

Proof. Suppose that $z_1\leq z_2$ and that $a\in A(z_1)$. Hence there exists an element $x\in L(z_1)$, with $a\ll x$. The assumption implies $L(z_1)\subseteq L(z_2)$. Then $x\in L(z_2)$ and so $a\in A(z_2)$. Thus $A(z_1)\subseteq A(z_2)$ holds.

Conversely, let $A(z_1)\subseteq A(z_2)$, $x\in L(z_1)$ and $u\in U(z_2)$. Suppose that a is an arbitrary element of L with $a\ll x$. As $a\in A(z_1)$, according to the assumption we obtain $a\in A(z_2)$. There exists $a_2\in A(z_2)$ with $a\ll a_2\leq u$. Using A_2 we get $a\ll u$. Then by A_6 $x\leq u$ is valid. Hence $x\leq z_2$, i.e., $x\in L(z_2)$. We have seen that $L(z_1)\subseteq L(z_2)$, and thus $z_1\leq z_2$, as desired.

From the statement 4 it immediately follows

5. *$z_1=z_2$ if and only if $A(z_1)=A(z_2)$.*

Let φ be a mapping from $d(L)$ into L_E defined by the rule

$$\varphi(z)=A(z)\times A'(z).$$

By summarizing, we infer from 1—5 that φ is an isomorphism from the lattice $d(L)$ onto L_E. Hence the following Theorem is valid:

6. Theorem. *The lattices $d(L)$ and L_E are isomorphic.*

7. *$\varphi(f_1(x))=f(x)$ for each $x\in L$.*

Proof. Let $x \in L$. We identify x and $f_1(x)$. We have to show that $A(x) \times A'(x) = N^x$. It is sufficient to prove the inclusion $A(x) \times A'(x) \subseteq N^x$. Let $(y, y') \in A(x) \times A'(x)$. Hence $y \preccurlyeq x_1$ for some $x_1 \in L(x)$. From $x \in L(x)$ and $y \preccurlyeq x_1 \leqslant x$ according to A_2 it follows $y \preccurlyeq x$. In an analogical way we obtain $x \preccurlyeq y'$. Therefore $(y, y') \in N^x$.

Every lattice can be considered as a regular lattice if the relation $\leqslant$ is taken as the relation $\preccurlyeq$. There are regular lattices (for instance the chain $(R, \leqslant)$ of all real numbers with the natural order $\leqslant$) with respect to the relation $\preccurlyeq$ equal to $<$.

On the other hand there exist regular lattices $(L, \leqslant, \preccurlyeq)$ such that $(L, \leqslant)$ is a chain and that the relation $\preccurlyeq$ is different from both relations $\leqslant$ and $<$ (Example 1).

8. *Let $(L, \leqslant, \preccurlyeq)$ be a regular lattice and let $(L, \leqslant)$ be a chain $x, y \in L$, $x \neq y$. Then $x \preccurlyeq y$ if and only if $x < y$.*

Proof. Let $x \preccurlyeq y$. Then A_1 and the assumption imply $x < y$.

Conversely, let there exist elements $x, y \in L$ such that $x < y$, $x \not\preccurlyeq y$. Hence according to A_5 and A_6 there is an element $a \in L$ having the property $a \preccurlyeq y$, $a \not\preccurlyeq x$. We have two possibilities: $x \leqslant a \leqslant y$ or $a < x$. Suppose that $x \leqslant a \leqslant y$. Since $x \leqslant a \preccurlyeq y$, by A_2 we obtain $x \preccurlyeq y$, a contradiction. Now let $a < x$. From $a \preccurlyeq y$ and A_4 it follows that there exists $b \in L$ with $a \preccurlyeq b \preccurlyeq y$. Hence $x \leqslant b \leqslant y$ or $b < x$. In the same way as above we obtain $x \preccurlyeq y$ or $a \preccurlyeq x$, respectively, contrary to suppositions. The proof is complete.

If we suppose in 8 that $(L, \leqslant)$ is a lattice, the assertion fails in general (Example 2).

3. Examples

Example 1. Let $(L, \leqslant)$ be a chain and let $(L, \leqslant, <)$ be a regular lattice. Pick out any $a \in L$. Define a relation $\preccurlyeq_a$ on L in the following way: put $a \preccurlyeq_a a$ and $x \preccurlyeq_a y$ iff $x < y$. Then $(L, \leqslant, \preccurlyeq_a)$ is a regular lattice. The relation $\preccurlyeq_a$ coincides neither with $\leqslant$ nor with $<$.

Example 2. Let $(R, \leqslant)$ be the chain of all real numbers with the natural order $\leqslant$. Suppose that the lattice $(L, \leqslant)$ is the direct product of lattices R_i, $L = \Pi R_i$ $(i \in I)$ where $R_i = (R, \leqslant)$ for each $i \in I$. Let i be a fixed element of I. Define $x \preccurlyeq_i y$ on L to mean $x(i) < y(i)$ and $x(k) \leqslant y(k)$ for each $k \in I$, $k \neq i$. Hence $(L, \leqslant, \preccurlyeq_i)$ is a regular lattice. Let x, y be elements of L such that $x(j) = 0$ for each $j \in I$ and $y(i) = 0$, $y(k) = 1$ for each $k \in I$, $k \neq i$. Therefore $x < y$ but $x \not\preccurlyeq_i y$.

The following example shows that the systems of ends can be different on the same lattice.

Example 3. Let $(L, \leqslant)$ be a chain and let $(L, \leqslant, <)$ be a regular lattice. Take $a, b \in L$, $a \neq b$. By Example 1 $(L, \leqslant, \preccurlyeq_a)$ and $(L, \leqslant, \preccurlyeq_b)$ are regular lattices. The systems of all cells (ends) will be denoted by S_a and S_b (L_{E_a} and L_{E_b}), respectively.

Hence $N^a = \{(x, y) \in S_a \colon x \preccurlyeq_a a \preccurlyeq_a y\} \in L_{E_a}$ and a cell (a, a) belongs to the end N^a. On the other hand $N^a \notin L_{E_b}$, since $a \not\preccurlyeq_b a$. Hence $L_{E_a} \neq L_{E_b}$ is valid.

There exists a proper class of nonequivalent c-mappings.

Example 4. Let $d(L)$ be the conditional Dedekind completion of the lattice L. We may suppose that $L \subseteq d(L)$. Take an element $z \in d(L) - L$. Let α be an infinite cardinal and $D_z(\alpha)$ the α-diamant in the picture.

Denote by Y_z the set of all mutually incomparable elements of $D_z(\alpha)$. We suppose that card $Y_z = \alpha$. Let us form the set $f_\alpha(L) = L \cup (\cup D_z(\alpha)\ (z \in d(L) - L))$. Define a partial order $\leqslant$ on $f_\alpha(L)$ by putting:

if $t_1, t_2 \in L$, then $t_1 \leqslant t_2$ iff $t_1 \leqslant t_2$ in L,

if $t_1, t_2 \in D_z(\alpha)$, then $t_1 \leqslant t_2$ iff $t_1 \leqslant t_2$ in $D_z(\alpha)$,

if $t_1 \in L$, $t_2 \in D_z(\alpha)$, then $t_1 \leqslant t_2$ $(t_2 \leqslant t_1)$ iff $t_1 \leqslant z$ $(z \leqslant t_1)$ in $d(L)$,

if $t_1 \in D_{z_1}(\alpha)$, $t_2 \in D_{z_2}(\alpha)$, then $t_1 \leqslant t_2$ iff $z_1 \leqslant z_2$ in $d(L)$.

Therefore $f_\alpha(L)$ turns out to be a conditionally complete lattice. The mapping $f_\alpha \colon \mathscr{L} \to \mathscr{L}$ is a c-mapping. If $\beta > \alpha$, the mappings f_α and f_β fail to be equivalent. We conclude that the class $\{f_\alpha\}$ of nonequivalent c-mappings is a proper class.

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О ПОПОЛНЕНИИ СТРУКТУР КОНЦАМИ

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Резюме

Понятие пополнения структуры концами определил Б. Й. Арнов. В этой статье доказано, что пополнение структуры L при помощи концов изоморфно условному дедекиндову пополнению структуры L.