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Mathematica Slovaca, Vol. 52 (2002), No. 4, 433--442

Persistent URL: http://dml.cz/dmlcz/129641

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Mathematica Slovaca

ON RELATIVELY UNIFORM CONVERGENCE OF WEIGHTED SUMS OF B-LATTICE VALUED RANDOM ELEMENTS

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(*Communicated by Miloslav Duchoň*)

ABSTRACT. Relatively uniform convergence of weighted sums of random elements taking values in a σ-complete Banach lattice with the σ-property is studied. It is shown that the usual assumptions of independent and identically distributed random elements can be replaced by weaker conditions to obtain a fruitful theory. The results obtained are new even for real valued random elements.

1. Introduction

Random elements in Banach spaces have been intensively studied and many interesting results can be found in literature. On the other hand much less attention has been devoted to Banach lattices and the corresponding order convergence despite that the latter is stronger than the convergence in norm in a number of spaces, e.g. L^p-spaces, $1 \le p < \infty$. In order to get interesting results for random elements in Banach lattices (and, more generally, in vector lattices) the original assumption of regularity is not necessary. It is sufficient to suppose that the lattice is σ-complete with the σ-property.

DEFINITION 1. Let (Z, S, P) be a probability space, B an Archimedean vector lattice. A sequence (X_n) of functions from Z to B *converges to a function* X *almost uniformly* if for every $\varepsilon > 0$ there exists a set $A \in S$ such that $P(A) < \varepsilon$ and (X_n) converges relatively uniformly to X uniformly on $Z - A$, i.e. there exists a sequence (a_n) of real numbers converging to 0 and an element $r \in B$ such that $|X_n(z) - X(z)| \le a_n r$ for each $z \in Z - A$.

2000 Mathematics Subject Classification: Primary 60B12.
Keywords: stochastically dominated, negatively dependent random elements.

This research was supported by VEGA, Grant No. 1/7295/20.

DEFINITION 2. A function $X\colon Z \to B$ is called a *random element* if there exists a sequence (X_n) of countably valued random elements such that (X_n) converges to X almost uniformly.

DEFINITION 3. A vector lattice B is said to *have the σ-property* if for every sequence (u_n) of elements from B there exist an element $u \in B$, $u > 0$, and a sequence (k_n) of positive real numbers such that $|u_n| \le k_n u$ for each n.

PROPOSITION 1. ([2]) *Let B be an Archimedean vector lattice with the σ-property. Then the vector lattice of all random elements is closed with respect to the almost uniform convergence.*

DEFINITION 4. A vector lattice with a monotone norm which is complete with respect to it is called a *Banach lattice.*

PROPOSITION 2. ([2]) *Let B be a Banach lattice. Then each random element is a measurable map from Z to B.*

2. Strong laws of large numbers

DEFINITION 5. A sequence (X_n) of random elements satisfies the *strong law of large numbers* (SLLN) *with centering elements* (m_n) of B and *norming constants* (b_n), $0 < b_n \to \infty$, if there exists an element $a \in B^+$ such that for every $\varepsilon > 0$

$$\lim_{n\to\infty} P\left\{ \bigcap_{k=n}^{\infty} \left\{ z: \ \left| b_k^{-1} \left(\sum_{i=1}^{k} X_i(z) - m_k \right) \right| \le \varepsilon a \right\} \right\} = 1 .$$

If (X_n) satisfies SLLN, then the sum $\left(\sum_{i=1}^{k} X_i(z) - m_k \right) / b_k$ converges to 0 relatively uniformly almost everywhere.

DEFINITION 6. An Archimedean vector lattice B is called *σ-complete* if every non-empty at most countable subset of B which is bounded from above has a supremum.

In [2] the following theorem is proved.

THEOREM 1. *Let B be a σ-complete Banach lattice with the σ-property. If X_n are independent, identically distributed and symmetric random elements in B, then the condition*

$$\sum_{n=1}^{\infty} P\{z: \ |X_1(z)| \le na\}^C < \infty \qquad \textit{for some} \quad a \in B^+ ,$$

is necessary and sufficient for (X_n) *to satisfy the strong law of large numbers with* $m_n = 0$ *and* $b_n = n$ *for each* $n \in \mathbb{N}$.

The aim of this paper is to prove SLLN and related results on convergence of weighted sums under less restrictive conditions on random elements X_n.

DEFINITION 7. Random elements X and Y with values in a Banach lattice B are said to be *negatively dependent* if $P\{X \le x,\ Y \le y\} \le P\{X \le x\}P\{Y \le y\}$ and $P\{X \ge -x,\ Y \ge -y\} \le P\{X \ge -x\}P\{Y \ge -y\}$ for all $x, y \in B^+$.

LEMMA 1. *If* X *and* Y *are negatively dependent, so are their positive parts* X^+ *and* Y^+ *and negative parts* X^- *and* Y^-, *respectively.*

LEMMA 2. ([3]) *If* X *and* Y *are negatively dependent (real valued) random variables, then* $\operatorname{cov}(X,Y) \le 0$, *provided it exists.*

PROPOSITION 3. ([4]) *Let* (A_n) *be a sequence of events. If* $\sum_{n=1}^{\infty} P(A_n) < \infty$, *then* $P\big(\limsup_{n\to\infty} A_n\big) = 0$. *If* $\sum_{n=1}^{\infty} P(A_n) = \infty$ *and* $P(A_n \cap A_m) \le P(A_n)P(A_m)$ *for all* (n,m), $n \ne m$, *then* $P\big(\limsup_{n\to\infty} A_n\big) = 1$.

THEOREM 2. *Let* (X_n) *be a sequence of pairwise negatively dependent random elements in* B. *Let* B *be a* σ*-complete Banach lattice with the* σ*-property stochastically dominated by a random element* X, *i.e.* $P\big(|X_n| \le x\big) \ge P\big(|X| \le x\big)$ *for each* $x \in B^+$. *Let* (a_n) *and* (b_n) *be sequences of positive numbers such that*

$$\sum_{i=1}^{n} a_i = O(na_n), \tag{1}$$

$$b_n/n \uparrow, \qquad b_n/a_n \uparrow \infty, \qquad b_n/na_n \uparrow \infty, \tag{2}$$

$$\sup_{n\in\mathbb{N}} b_{2n}/b_n < \infty. \tag{3}$$

If

$$\sum_{n=1}^{\infty} P\big(a_n|X| \le b_n e\big)^C < \infty \tag{4}$$

for some $e \in B^+$, *then* $\sum_{i=1}^{n} a_i X_i / b_n \to 0$ *relatively uniformly almost everywhere. Moreover, if* $\sum_{i=1}^{n} a_i X_i / b_n \to 0$ *relatively uniformly almost everywhere, then* $\sum_{n=1}^{\infty} P\big(a_n|X_n| \le b_n e\big)^C < \infty$ *for each* $e \in B^+$.

P r o o f . As shown in [2; Theorem 1], we can assume that all X_n take values in a separable Banach space with an order unit norm (i.e. the norm induced by an element $u \in B^+$, $e \le u$). It is well known that the norm convergence and the relatively uniform convergence are equivalent in this case.

Consider now the sequence (X_n^+) of pairwise negatively dependent (Lemma 1) random elements, stochastically dominated by X. It is immediate that $\|X_n^+\|$ are also pairwise negatively dependent and stochastically dominated by $\|X\|$.

Put $Y_n = \|X_n^+\| I(\|X_n^+\| \le c_n)$, where $I(A)$ denotes the characteristic function of the set A and $c_n = b_n/a_n$. Finally put $Z_n = a_n Y_n$. As Z_n are pairwise negatively dependent, we have $\operatorname{cov}(Z_n, Z_m) \le 0$ by Lemma 2. Put $S_n = \sum_{i=1}^{n} Z_i$ and $k(n) = 2^n$. We have for each $\varepsilon > 0$

$$\begin{aligned}\sum_{n=1}^{\infty} P\big(|S_{k(n)} - E(S_{k(n)})|/b_{k(n)} > \varepsilon\big) &\le 1/\varepsilon^2 \sum_{n=1}^{\infty} E\big(S_{k(n)} - E(S_{k(n)})\big)^2 / b_{k(n)}^2 \\ &\le 1/\varepsilon^2 \sum_{n=1}^{\infty} \big(1/b_{k(n)}\big)^2 \sum_{j=1}^{k(n)} E(Z_j^2) \\ &\le 1/\varepsilon^2 \sum_{j=1}^{\infty} E(Z_j^2) \sum_{n=l(j)}^{\infty} (1/b_{k(n)})^2 ,\end{aligned}$$

where $l(j) = \min\{n : \ k(n) \ge j\}$.

As $b_n/n \uparrow$, we have

$$\sum_{n=l}^{\infty} \big(1/b_{k(n)}\big)^2 \le \big(1/b_{k(l)}\big)^2 \sum_{n=l}^{\infty} \big(k(l)/k(n)\big)^2 = 4/3\big(1/b_{k(l)}\big)^2$$

and hence we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty} P\big(|S_{k(n)} - E(S_{k(n)})|/b_{k(n)} > \varepsilon\big) \\ &\quad\le 4/3\varepsilon^2 \sum_{j=1}^{\infty} E(Z_j^2)/b_j^2 = 4/3\varepsilon^2 \sum_{j=1}^{\infty} \big(E(Y_j)/c_j\big)^2 \\ &\quad\le C \sum_{n=1}^{\infty} \big(E(\|X\| I(\|X\| \le c_n))/c_n\big)^2 \\ &\quad= C \sum_{n=1}^{\infty} (1/c_n)^2 \sum_{k=1}^{n} E\big(\|X\|^2 I(c_{k-1} < \|X\| \le c_k)\big) \\ &\quad= C \sum_{k=1}^{\infty} E\big(\|X\|^2 I(c_{k-1} < \|X\| \le c_k)\big) \sum_{n=k}^{\infty} (1/c_n)^2\end{aligned}$$

$$\le K\sum_{k=1}^{\infty} kE(\|X\|^2 I(c_{k-1} < \|X\| \le c_k))/c_k^2$$
$$\le K\sum_{k=1}^{\infty} kP(c_{k-1} < \|X\| \le c_k)$$
$$= K\sum_{k=1}^{\infty} P(a_k\|X\| > b_k)$$
$$= K\sum_{k=1}^{\infty} P(a_k\|X\| \le b_k e)^C < \infty .$$

Hence $(S_{k(n)} - E(S_{k(n)}))/b_{k(n)} \to 0$ almost everywhere.

As for each natural k there exists a natural number n such that $k(n) \le k \le k(n+1)$, using the inequalities

$$\begin{aligned}(S_{k(n)} - E(S_{k(n)}))/b_{k(n+1)} - (E(S_{k(n+1)}) - E(S_{k(n)}))/b_{k(n+1)} \\ \le (S_k - E(S_k))/b_k \\ \le (S_{k(n+1)} - E(S_{k(n+1)}))/b_{k(n)} + (E(S_{k(n+1)}) - E(S_{k(n)}))/b_{k(n)}\end{aligned}$$

and (3) it suffices to prove that $(E(S_{k(n+1)}) - E(S_{k(n)}))/b_{k(n)} \to 0$ in order to obtain that $(S_k - E(S_k))/b_k \to 0$. To prove this, we observe that because of (3) it suffices to prove that $E(S_{k(n)})/b_{k(n)} \to 0$. We have

$$\begin{aligned}E(S_{k(n)})/b_{k(n)} &= \sum_{j=1}^{k(n)} E(Z_j)/b_{k(n)} \\ &= \sum_{j=1}^{k(n)} a_j(E(\|X_j^+\| I(\|X_j^+\| \le c_j)))/b_{k(n)} \\ &\le \sum_{j=1}^{k(n)} b_j P(\|X\| > c_j)/b_{k(n)} + \sum_{j=1}^{k(n)} a_j(E(\|X\| I(\|X\| \le c_j)))/b_{k(n)}.\end{aligned}$$

The first summand goes to zero by Kronecker lemma. As for the second one, consider $k(n) > m \ge 1$. We have

$$\sum_{j=1}^{k(n)} a_j\big(E(\|X\|I(\|X\| \le c_j))\big)/b_{k(n)}$$
$$\le C\sum_{j=1}^{k(n)} a_j\big(E(\|X\|I(\|X\| \le c_m))\big)/b_{k(n)}$$
$$+\, C\sum_{j=1}^{k(n)} a_j\big(E(\|X\|I(c_m < \|X\| \le c_{k(n)}))\big)/b_{k(n)}$$
$$\le Cc_m/b_{k(n)}\sum_{j=1}^{k(n)} a_j + C\sum_{j=1}^{k(n)} a_j \sum_{k=m+1}^{k(n)} 1/b_{k(n)}E(\|X\|I(c_{k-1} < \|X\| \le c_k))$$
$$\le Cc_m/b_{k(n)}\sum_{j=1}^{k(n)} a_j + C\sum_{j=1}^{k(n)} a_j \sum_{k=m+1}^{k(n)} k/k(n)a_{k(n)}c_kP(c_{k-1} < \|X\| \le c_k)$$

which goes to zero because of (1), (2) and (4).

The inspection of the proof that $E(S_{k(n)})/b_{k(n)}$ goes to zero reveals that it can also be used to prove $E(S_n)/b_n \to 0$. All we need to do is to replace $k(n)$ by n. Putting together that both $(S_n - E(S_n))/b_n$ and $E(S_n)/b_n$ go to zero we obtain that

$$\sum_{j=1}^{n} a_jY_j/b_n = \sum_{j=1}^{n} a_j\|X_j^+\|I(\|X_j^+\| \le c_j)/b_n \to 0$$

almost everywhere. Since

$$\sum_{n=1}^{\infty} P(\|X_n^+\| \ne Y_n) = \sum_{n=1}^{\infty} P(\|X_n^+\| > c_n) \le \sum_{n=1}^{\infty} P(\|X_n\| > c_n) < \infty,$$

we obtain $\sum_{i=1}^{n} a_i\|X_i^+\|/b_n \to 0$ almost everywhere. If the proof is repeated with X_n^- instead of X_n^+, we get $\sum_{i=1}^{n} a_i\|X_i^-\|/b_n \to 0$ and consequently $\sum_{i=1}^{n} a_i\|X_i\|/b_n \to 0$ almost everywhere. It follows that $\sum_{i=1}^{n} a_iX_i/b_n \to 0$ in norm and hence $\sum_{i=1}^{n} a_iX_i/b_n \to 0$ relatively uniformly almost everywhere owing to the above mentioned property of the order-unit norm.

Necessity can be proved as follows. We have

$$a_nX_n/b_n = \sum_{i=1}^{n} a_iX_i/b_n - b_{n-1}/b_n\sum_{i=1}^{n-1} a_iX_i/b_{n-1}$$

and thus $a_n X_n / b_n$ goes to 0 in norm almost everywhere. Since $\|X_n^+\|$ are pairwise negatively dependent, it follows that $a_n \|X_n^+\| / b_n \to 0$ implies $P\left(\limsup_{n\to\infty} \|X_n^+\| > 1/2c_n\right) = 0$ and thus $\sum_{n=1}^{\infty} P(\|X_n^+\| > 1/2c_n) < \infty$ by Proposition 3. Because of the inequality

$$\sum_{n=1}^{\infty} P\big(a_n \|X_n\| \leq b_n e\big)^C \leq \sum_{n=1}^{\infty} P\big(a_n X_n^+ \leq 1/2 b_n e\big)^C + \sum_{n=1}^{\infty} P\big(a_n X_n^- \leq 1/2 b_n e\big)^C$$

we obtain

$$\sum_{n=1}^{\infty} P\big(a_n \|X_n\| \leq b_n e\big)^C < \infty .$$

□

COROLLARY 1. *Let (X_n) be a sequence of pairwise negatively dependent random elements in a σ-complete Banach lattice B with the σ-property stochastically dominated by a random element X, i.e. $P\big(\|X_n\| \leq x\big) \geq P\big(\|X\| \leq x\big)$ for each $x \in B^+$. Then the condition*

$$\sum_{n=1}^{\infty} P\big(\|X\| \leq n^p e\big)^C < \infty \qquad \textit{for some } p > 1 \textit{ and } e \in B^+$$

is necessary and sufficient for (X_n) to satisfy the strong law of large numbers with centering elements $m_n = 0$ and norming constants $b_n = n^p$.

The following corollary can be found in the literature with the stronger assumption that X_n are independent.

COROLLARY 2. *Let (X_n) be a sequence of pairwise negatively dependent (real valued) random variables stochastically dominated by a (real valued) random variable X. Let (a_n) and (b_n) be sequences of positive numbers such that $\sum_{i=1}^{n} a_i = O(n a_n)$, $b_n/n \uparrow$, $b_n/a_n \uparrow \infty$, $b_n/n a_n \uparrow \infty$ and $\sup_{n\in\mathbb{N}} b_{2n}/b_n < \infty$.*

If $\sum_{n=1}^{\infty} P\big(a_n |X| > b_n\big) < \infty$, then $\sum_{i=1}^{n} a_i X_i / b_n \to 0$ almost everywhere.

Moreover, if $\sum_{i=1}^{n} a_i X_i / b_n \to 0$ almost everywhere, then $\sum_{n=1}^{\infty} P\big(a_n |X_n| > b_n\big) < \infty$.

PROPOSITION 4. ([5]) *Let (X_n) be a sequence of random variables. If for every $n \in \mathbb{N}$, $E\big(\|X_n\|^r\big) < M$ for some $r > 0$ and $M \in \mathbb{R}^+$, then there exists a nonnegative random variable X such that $E\big(X^s\big) < \infty$ for every $0 < s < r$ and (X_n) is stochastically dominated by X.*

THEOREM 3. *Let (X_n) be a sequence of pairwise negatively dependent random elements in a σ-complete Banach lattice B with the σ-property and let (a_n) and (b_n) satisfy all the conditions of Theorem 1. Suppose that for each $n \in \mathbb{N}$,*

$$\sum_{k=1}^{\infty} k^{r-1} P(\|X_n\| \le ka)^C \le M$$

for some $r > 1$, $a \in B$ and $M \in \mathbb{R}^+$. Then $\sum_{i=1}^{n} a_i X_i / b_n \to 0$ relatively uniformly almost everywhere.

Proof. The assumption on X_n implies that

$$\begin{aligned} E(\|X_n^+\|^r) &\le 1 + 2^r r \sum_{k=1}^{\infty} k^{r-1} P(\|X_n^+\| > k) \\ &= 1 + 2^r r \sum_{k=1}^{\infty} k^{r-1} P(\|X_n\| \le ka)^C \le M_1 . \end{aligned}$$

Using Proposition 4 we have that there exists a nonnegative random variable X such that $E(X^s) < \infty$ for every $0 < s < r$ and the sequence $\|X_n^+\|$ is stochastically dominated by X. Choose $s = 1$. For each natural number n, let $c_n = b_n / a_n$. Since $c_n > n$ for n sufficiently large, we have $\sum_{n=1}^{\infty} P(X > c_n) < \infty$. Then repeating the proof of Theorem 2 yields the result. □

In what follows the relatively uniform convergence of weighted sums of the type $\sum_{k=1}^{n} A_k X_k$ is studied, where the weights A_k are random variables. It is worth mentioning that no relationship between the random weights A_k and the random elements X_k is supposed and the usual assumption that $A_k X_k$ are independent is replaced by the weaker condition that they are pairwise negatively dependent.

LEMMA 3. *If V is a random element in a σ-complete Banach lattice B with the σ-property and A is a (real valued) random variable, then AV is a random element in B.*

THEOREM 4. *Let (X_n) be a sequence of random elements in a σ-complete Banach lattice B with the σ-property and A_n be a sequence of (real valued) random variables such that $A_n X_n$ are pairwise negatively dependent. Suppose that for each $n \in \mathbb{N}$*

$$\sum_{k=1}^{\infty} k^{1+2/(q-1)} P(|X_n| \le ka)^C \le M$$

for some $q > 1$, $a \in B^+$ *and* $M \in \mathbb{R}^+$. *If moreover*

$$\sum_{n=1}^{\infty} \left(E(|A_n|^{2q})\right)^{1/q} / b_n^2 < \infty \qquad \text{and} \qquad \sum_{j=1}^{n} \left(E(|A_j|^q)\right)^{1/q} = o(b_n)$$

where (b_n) *is a sequence of positive numbers such that* $b_n/n \uparrow \infty$ *and* $\sup_{n\in\mathbb{N}} b_{2n}/b_n < \infty$, *then* $\sum_{k=1}^{n} A_k X_k / b_n \to 0$ *relatively uniformly almost everywhere.*

Proof. Repeating the proof of Theorem 2 word for word with $Z_n = \|(A_n X_n)^+\|$ we find that it only needs to be proved that

$$\sum_{n=1}^{\infty} E(Z_j^2)/b_j^2 < \infty .$$

We have

$$\sum_{j=1}^{\infty} E(Z_j^2)/b_j^2 \le \sum_{j=1}^{\infty} \left(E(|A_j|^{2q})\right)^{1/q} \left(E(\|X_j\|^{2q/(q-1)})\right)^{(q-1)/q} / b_j^2 < \infty$$

owing to the assumptions on A_n and X_n, respectively. Moreover

$$E\left(\sum_{k=1}^{n} Z_k / b_n\right) \le \sum_{k=1}^{n} \left(E(|A_k|^q)\right)^{1/q} \left(E(\|X_k\|^{q/(q-1)})\right)^{(q-1)/q} / b_n \to 0$$

because of Hoelder's inequality and the last assumption of the theorem. The rest of the proof follows immediately. □

I suggest to compare results in this paper with those in [6].

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Received October 1, 2001
Revised January 14, 2002

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