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EDGE-DOMATICALLY FULL GRAPHS

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ABSTRACT. Edge-domatic number $ed(G)$ of a graph G is the maximum number of classes of a partition of the edge set of G into dominating edge sets. If $ed(G) = \delta_e(G) + 1$, where $\delta_e(G)$ is the minimum degree of an edge in G, the graph G is called edge-domatically full. In the paper edge-domatically full graphs are characterized.

We consider finite undirected graphs without loops and multiple edges.

The domatic number of a graph was introduced by E. J. Cockayne and S. T. Hedetniemi in [1]. A dominating set in a graph G is a subset D of the vertex set $V(G)$ of G with the property that for each $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. A domatic partition of G is a partition of $V(G)$, all of whose classes are dominating sets in G. The maximum number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$.

An analogous concept was defined in [4]. A dominating edge set in a graph G is a subset D of the edge set $E(G)$ of G with the property that for each $e \in E(G) - D$ there exists an edge $f \in D$ having a common end vertex with e. An edgedomatic partition of G is a partition of $E(G)$, all of whose classes are dominating edge sets in G. The maximum number of classes of an edge-domatic partition of G is called the edge-domatic number of G and denoted by $ed(G)$.

The authors of the concept of the domatic number have defined also the concept of a domatically full graph. A graph G is called domatically full, if $d(G) = \delta(G) + 1$, where $\delta(G)$ is the minimum degree of a vertex of G; this is the maximum possible value of the domatic number. We can introduce an analogous concept concerning the edge-domatic number.

The degree $\delta(e)$ of an edge e of G is the number of edges of G which have a common end vertex with e. The minimum of $\delta(e)$ for all edges e of G will be denoted by $\delta_e(G)$. A graph in which all edges have the same degree will be called edge-regular.

A graph G is called edge-domatically full if $ed(G) = \delta_e(G) + 1$. (Obviously always $ed(G) \leqq \delta_e(G) + 1$.)

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In our considerations we shall use the concepts of a multiple cover of a graph and an odd graph.

The concept of a double cover of a graph was introduced by D. A. Waller in [3]; it can be easily generalized. Let G be a graph, let k be a positive integer. To each vertex $v \in V(G)$ we assign a set $S(v)$ in such a way that $|S(v)| = k$ for each $v \in V(G)$ and $S(v_1) \cap S(v_2) = \emptyset$ for $v_1 \neq v_2$. Let H be a graph with the vertex set $S = \bigcup_{v \in V(G)} S(v)$ and with the following structure:

(1) Each set $S(v)$ is independent in H.

(2) If v_1, v_2 are adjacent vertices of G, then $S(v_1) \cup S(v_2)$ induces a linear subgraph of H.

(3) If v_1, v_2 are non-adjacent vertices in G, then $S(v_1) \cup S(v_2)$ is an independent set in H.

Then H is called a k-cover of G.

For $k = 1$ the unique k-cover of G is G itself. The k-covers of G for all k are called multiple covers of G.

For a given number k there exist various non-isomorphic k-covers of G. One of them is the graph consisting of k disjoint copies of G.

Odd graphs were introduced by H. M. Mulder in the book [2]. Let k be an integer, $k \geqq 2$. Let $\mathcal{M}_k$ be the family of all subsets of the number set $\{1, \ldots, 2k - 1\}$ which have the cardinality $k - 1$. The odd graph O_k is the graph whose vertex set is $\mathcal{M}_k$ and in which two vertices are adjacent if and only if they are disjoint (as sets).

The graph O_k is regular of degree k. The graph O_2 is the triangle, the graph O_3 is the Petersen graph.

We shall define a certain analogy of odd graphs. Let k, n be integers, $0 \leqq k \leqq n/2 - 1$. By $\mathcal{M}_k^n$ we denote the family of all subsets of the number set $\{1, \ldots, n\}$ which have the cardinality k. The $\binom{n}{k}$-bigraph B_k^n is the bipartite graph with bipartition classes $\mathcal{M}_k^n$, $\mathcal{M}_{n-k-1}^n$ in which a vertex $X \in \mathcal{M}_k^n$ is adjacent to a vertex $Y \in \mathcal{M}_{n-k-1}^n$ if and only if $X \cap Y = \emptyset$.

The degree of each vertex $X \in \mathcal{M}_k^n$ in B_k^n is $n - k$, the degree of each vertex $Y \in \mathcal{M}_{n-k-1}^n$ is $k + 1$. the graph B_0^n is the star with n edges. The graph B_1^n for $n \geqq 4$ is obtained from the complete graph K_n by inserting one vertex onto each edge. In Fig. 1 we see the graph B_1^4.

Now we shall prove a lemma.

Lemma 1. *Let a connected finite graph G be edge-regular. Then either G is regular, or G is a bipartite graph and any two vertices of the same bipartition class of G have the same degree.*

Proof. Let G be edge-regular. Then there exists a non-negative integer r such that each edge of G has common end vertices with exactly r edges. Let e

be an edge of G, let v_1, v_2 be its end vertices. Then $r = \delta(v_1) + \delta(v_2) + 1$, where δ denotes the degree of a vertex. Now let v_3 be a vertex adjacent to v_2. We have $r = \delta(v_2) + \delta(v_3) + 1$; together with the previous equality this yields $\delta(v_3) = \delta(v_1)$. We can proceed by induction (because G is finite and connected); we prove that $\delta(x) = \delta(v_1)$ for each vertex whose distance from v_1 is even and $\delta(x) = \delta(v_2)$ for each x whose distance from v_1 is odd. This can be realized in two ways. If $\delta(v_1) = \delta(v_2)$, then all vertices of G have this degree and G is regular. If $\delta(v_1) \neq \delta(v_2)$, then the vertex set of G is partitioned into two sets V_1, V_2 such that each vertex of V_1 has the degree $\delta(v_1)$, each vertex of V_2 has the degree $\delta(v_2)$ and each edge joins a vertex of V_1 with a vertex of V_2; the graph G is bipartite and any two vertices of the same bipartition class of G have the same degree. □

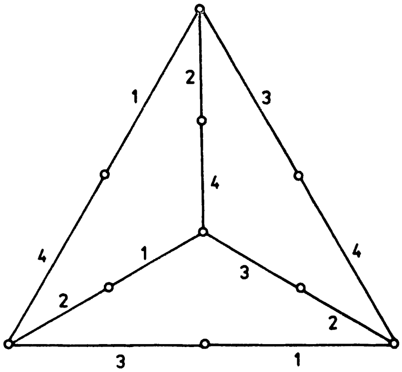

Fig. 1

The following lemma will concern multiple covers of graphs.

Lemma 2. *Let H be a multiple cover of a graph G. Then $ed(H) \geqq ed(G)$.*

Proof. The graph H is a k-cover of G for some k. Let $ed(G) = d$, let $\{D_1, \ldots, D_d\}$ be an edge-domatic partition of G with d classes. We shall construct a partition $\{D'_1, \ldots, D'_d\}$ of $E(H)$ in the following way. If x, y are adjacent vertices in G and the edge joining them is in D_i, $1 \leq i \leq d$, then all edges of H joining a vertex of $S(x)$ with a vertex of $S(y)$ will be in D'_i. From the definition of the k-cover it is easy to see that $\{D'_1, \ldots, D'_d\}$ is an edge-domatic partition of H and thus $ed(H) \geqq ed(G)$. □

Note that the equality need not occur. The circuit C_{12} of length 12 is a 3-cover of the circuit C_4 of length 4 and we have $ed(C_4) = 2$, $ed(C_{12}) = 3$.

Now we prove a theorem.

Theorem 1. *Let G be a finite connected edge-regular graph of degree r. Then the following two assertions are equivalent:*

(i) *G is edge-domatically full.*

(ii) *G is a multiple cover of* $O_{r/2+1}$ or of B_{s-1}^{r+1} *for some s,* $1 \leqq s \leqq r/2 - 1$.

Proof. (i) ⇒ (ii). Suppose that G is edge-domatically full and therefore $ed(G) = r + 1$. Let $\{D_1, \ldots, D_{r+1}\}$ be an edge-domatic partition of G with $r + 1$ classes. For each vertex x of G let $F(x)$ be the subset of the number set $\{1, \ldots, r + 1\}$ consisting of all numbers i such that x is adjacent to no edge from D_i. Obviously $|F(x)| = r + 1 - \delta(x)$. According to Lemma 1 either the graph G is regular, or G is bipartite and any two vertices of the same bipartition class of G have the same degree. In the first case the graph G, being edge-regular of degree r, is regular of degree $q = r/2 + 1$. In the second case its vertices have degrees $s, r + 2 - s$, where $s \leqq r/2$. In the first case we consider the family $\mathscr{M}_{r/2+1}$ of all subsets of the number set $\{1, \ldots, r + 1\}$ which have the cardinality $r/2$. We partition the vertex set $V(G)$ of G into the sets $S(X)$ for $X \in \mathscr{M}_{r/2+1}$ in such a way that each $S(X)$ is the set of all vertices $x \in V(G)$ such that $F(x) = X$. Let X, Y be two sets from $\mathscr{M}_{r/2+1}$, let $x \in S(X)$, $y \in S(Y)$ and let x, y be adjacent in G. Let the edge joining x and y in G be in D_j for some $j \in \{1, \ldots, r + 1\}$. As G is edge-domatically full, the edge xy has a common end vertex with exactly one edge from each class of $\{D_1, \ldots, D_{r+1}\}$ other than its own one and with no edge from its own class. Therefore $X \cup Y = \{1, \ldots, r + 1\} - \{j\}$, $X \cap Y = \emptyset$. Consider further $x \in S(X)$. If Y is such a set from $\mathscr{M}_{r/2}$ that $X \cap Y = \emptyset$, then $|X \cup Y| = r$ and there exists exactly one $j \in \{1, \ldots, r + 1\}$ such that $j \notin X \cup Y$. Therefore there exists an edge $e \in D_j$ incident with x and its other end vertex is in $S(Y)$. Such an edge is exactly one; otherwise there would be two edges of the same class D_j with a common end vertex, which is impossible. Therefore any two sets X, Y from $\mathscr{M}_{r/2+1}$ such that $X \cap Y = \emptyset$ have the property that each vertex of $S(X)$ is adjacent to exactly one vertex of $S(Y)$ and each vertex of $S(Y)$ is adjacent to exactly one vertex of $S(X)$. This implies $|S(X)| = |S(Y)|$. As G is connected, for any two sets X, Y from $\mathscr{M}_{r/2+1}$ there exists a path connecting a vertex $x \in S(X)$ with a vertex $y \in S(Y)$; let this path consist of the vertices $x = u_0, u_1, \ldots, u_p = y$. For each $i = 0, 1, \ldots, p$ let $U_i = F(u_i)$. Then $|S(U_i)| = |S(U_{i+1})|$ for $i = 0, 1, \ldots, p - 1$ and hence $|S(X)| = |S(Y)|$. Therefore all sets $S(X)$ for $X \in \mathscr{M}_{r/2+1}$ have the same cardinality k and G is a k-cover of $O_{r/2+1}$. In the case when G is not regular, we consider the sets $\mathscr{M}^{r+1}_{r+1-s}$, $\mathscr{M}^{r+1}_{s-1}$ instead of $\mathscr{M}_{r/2+1}$. Analogously as in the preceding case we construct the sets $S(X)$ for all $X \in \mathscr{M}^{r+1}_{r+1-s} \cup \mathscr{M}^{r+1}_{s-1}$. For any $X \in \mathscr{M}^{r+1}_{r+1-s}$ and $Y \in \mathscr{M}^{r+1}_{s-1}$ such that $X \cap Y = \emptyset$ each vertex of $S(X)$ is adjacent to exactly one vertex of $S(Y)$ and each vertex of $S(Y)$ is adjacent to exactly one vertex of $S(X)$. If $X \cap Y \neq \emptyset$, then no edges exist between $S(X)$ and $S(Y)$. This can be proved analogously to the preceding case. Also there are no edges between the sets $S(X_1)$, $S(X_2)$, where both X_1, X_2 belong to the same of the families $\mathscr{M}^{r+1}_{r+1-s}$, $\mathscr{M}^{r+1}_{s-1}$; this would contradict the edge-regularity of G. Therefore G is a multiple cover of B^{r+1}_{s-1}.

(ii) ⇒ (i). In [5] it was proved that $ed(O_k) = 2k - 1$. As O_k is edge-regular of degree $2k - 2$, it is edge-domatically full. Analogously as in the proof in [5] we shall prove that B_k^n is edge-domatically full for each k, n such that $0 \leqq k \leqq \leqq n/2 - 1$. If $X \in \mathscr{M}_k^n$, $Y \in \mathscr{M}_{n-k-1}^n$ are adjacent vertices in B_k^n, then $\{1, \ldots, n\} - (X \cup Y)$ is a one-element set. We define the partition $\{D_1, \ldots, D_n\}$ in such a way that the edge joining X and Y is in D_j, where $\{j\} = \{1, \ldots, n\} - (X \cup Y)$. The vertex X is incident with all edges from D_i for $i \in Y$ and the vertex Y is incident with edges from D_i for $i \in X$. Hence the edge joining X and Y has common end vertices with edges from all D_i for $i \neq j$; the partition $\{D_1, \ldots, D_n\}$ is edge-domatic. We have $ed(B_k^n) = n$ and B_k^n is edge-domatically full. Now if H is a multiple cover of $O_{r/2+1}$ or of B_{s-1}^{r+1}, then $d(H) \geqq r + 1$ according to Lemma 2. On the other hand, the minimum degree of an edge of a multiple cover of a graph is equal to the minimum degree of an edge of the original graph. Thus $\delta_e(H) = r$ and $ed(H) = r + 1$; the graph H is edge-domatically full. □

Now we shall consider interrelations between edge-domatically full graphs and domatically full ones.

Theorem 2. *Let G be a regular finite connedted graph, let H be the graph obtained from* G *by inserting a vertex onto each edge. The graph* G *is domatically full if and only if H is edge-domatically full.*

Proof. As G is regular, all vertices have the same degree r. Evidently then each edge of H has the degree r. If x, y are two adjacent vertices of G, then by $u(x, y)$ we denote the vertex of H inserted onto the edge joining x and y in G. If H is edge-domatically full, then $ed(H) = r + 1$ and there exists an edge-domatic partition $\mathscr{D}' = \{D'_1, \ldots, D'_{r+1}\}$ of H with $r + 1$ classes. Let x be a vertex of G. In H the vertex x is incident with edges from all classes of $\mathscr{D}'$ except one; let this class be D'_j. If y is a vertex adjacent to x in G, then necessarily the edge joining y with $u(x, y)$ in H is in D'_j. We construct a partition $\mathscr{D} = \{D_1, \ldots, D_{r+1}\}$ of $V(G)$ such that $x \in D_j$ if and only if x is incident with no edge from D'_j in H. For each $i \neq j$ there exists an edge of D'_i incident with x; it joins x with $u(x, y)$ in H for some vertex y of G adjacent to x in G. The vertex y is incident with no edge from D'_i; otherwise the edge joining y with $u(x, y)$ in H would have common end vertices with two edges of D'_i. Hence $y \in D_i$. As x and i were chosen arbitrarily, the partition $\mathscr{D}$ is a domatic partition of G; we have $d(G) = r + 1$ and G is domatically full.

Now suppose that G is domatically full. There exists a domatic partition $\mathscr{D} = \{D_1, \ldots, D_{r+1}\}$ of G with $r + 1$ classes. We construct a partition $\mathscr{D}' = \{D'_1, \ldots, D'_{r+1}\}$ of the edge set of H. Let e be an edge of H. Then e joins x with $u(x, y)$, where x, y are vertices of G adjacent in G. Let j be the number such that $y \in D_j$; then e will be in D'_j. Now let $i \neq j$. If $x \in D_i$, then the edge joining y and $u(x, y)$ is in D'_i and has the common end vertex $u(x, y)$ with e in H. If $x \notin D_i$, then there exists a vertex $z \in D_i$ adjacent to x in G. Then the edge joining x and $u(x, z)$ is in D'_i and has the common end vertex x with e. Therefore $\mathscr{D}'$ is an edge-domatic partition of H and $ed(H) = r + 1$; the graph H is edge-domatically full. □

The following lemma follows immediately from the definition of the multiple cover of a graph.

Lemma 3. *Let G, H' be two finite connected graphs, let G' be the graph obtained from G by inserting one vertex onto each edge. The graph H' is a multiple cover of G' if and only if there exists a multiple cover H of G such that H' is obtained from H by inserting one vertex onto each edge.* □

With the help of this lemma we prove a theorem concerning domatically full graphs.

Theorem 3. *Let G be a finite connected regular graph of degree r. Then the following two assertions are equivalent:*

(I) *G is domatically full.*

(II) *G is a multiple cover of the complete graph K_r.*

Proof. (I) $\Rightarrow$ (II). Let G be domatically full. If $r = 2$, then G is a circuit of length divisible by 3 and thus a multiple cover of the circuit of length 3, which is the complete graph K_3. Thus suppose $r \geqq 3$. Let G' be the graph obtained from G by inserting one vertex onto each edge. According to Theorem 2 the graph G' is edge-domatically full. According to Theorem 1 the graph G' is a multiple cover of an odd graph or of an $\binom{n}{k}$-bigraph. As G' contains vertices of degree 2, it is a multiple cover of B_1^{r+1}. The graph B_1^{r+1} is obtained from the complete graph K_{r+1} by inserting one vertex onto each edge. According to Lemma 3 the graph G' is obtained from a multiple cover of K_{r+1} by inserting one vertex onto each edge. As in G' no two vertices of degree 2 are adjacent, this multiple cover is uniquely determined as the graph obtained from G' by substituting all paths of length 2 with inner vertices of degree 2 by paths of length 1; this graph is G. Therefore G is a multiple cover of K_{r+1}.

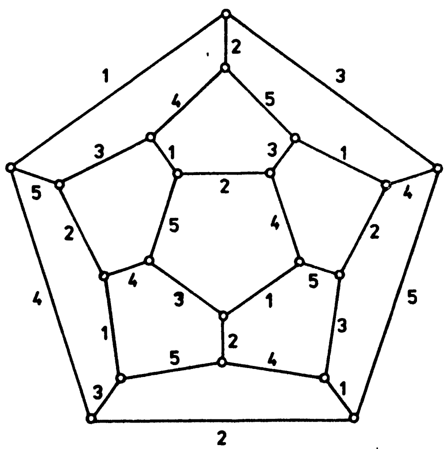

Fig. 2

(II) $\Rightarrow$ (I). Let G be a multiple cover of K_{r+1}. Then it is easy to see that the sets $S(x)$ for $x \in V(K_{r+1})$ form a domatic partition of G with $r + 1$ classes. Hence $d(G) = r + 1$ and G is domatically full. □

In Fig. 1 we see the graph B_1^4, in Fig. 2 the graph of the regular dodecahedron which is a 2-cover of the Petersen graph O_2. In both these graphs the edges are labelled in such a way that the sets of equally labelled edges form an edge-domatic partition with the maximum number of edges.

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