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Milan Paštéka

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CONVERGENCE OF SERIES AND SUBMEASURES OF THE SET OF POSITIVE INTEGERS

MILAN PAŠTÉKA

ABSTRACT. We introduce the notion of compact submeasure and shaw a conection between this notion and a convergence of infinite series.

Denote by $\mathbb{N}$ the set of all positive integers and by $P(\mathbb{N})$ the system of all subsets of $\mathbb{N}$.

A function m: $P(\mathbb{N}) \to [0, \infty)$ is said to be a submeasure if for any two sets A, $B \in P(\mathbb{N})$ there holds:

(i) $$A \subseteq B \Rightarrow m(A) \leq m(B)$$

(ii) $$m(A \cup B) \leq m(A) + m(B)$$

If m is a submeasure, let $Z(m)$ denote the system of all sets $A \in P(\mathbb{N})$ satisfying the condition $m(A) = 0$. In [1] it is proved that ilf m is the upper asymptotic density then the infinite series $\sum_{n=1}^{\infty} a_n$, with nonnegative elements, converges if and only if for every $A \in Z(m)$ we have $\sum_{n \in A} a_n < \infty$. The aim of our paper is to show that this result can be extended to a broader class of submeasures for which the system $Z(m)$ can be smaller than the system of all sets $A \in \mathbb{N}$ with asymptotic density 0.

A submeasure m: $P(\mathbb{N}) \to [0, \infty)$ is said to be compact if and only if

(iii) $$m(\{a\}) = 0 \text{ for every } a \in \mathbb{N}$$

(iv) For every $\varepsilon > 0$ there exists a decomposition $A_1 \cup \ldots \cup A_k = \mathbb{N}$ such that $m(A_i) < \varepsilon$, $i = 1, 2, \ldots, k$.

Theorem. Let m be a compact submeasure. Then the infinite series with

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nonnegative elements $\sum_{n=1}^{\infty} a_n$ converges if and only if for every set $A \in Z(m)$ there holds

$$\sum_{n \in A} a_n < \infty$$

Proof. The necessity of the condition is evident. Assume that

$$\sum_{n=1}^{\infty} a_n = \infty, \qquad a_n \geq 0,\ n = 1, 2, \ldots$$

For $A \in P(\mathbb{N})$ put

$$\mathscr{S}(A) = \sum_{n \in A} a_n$$

According to (iv) we obtain that there exists a decomposition

$$A_1^{(1)} \cup \ldots \cup A_{k_1}^{(1)} = \mathbb{N} \tag{1}$$

such that $m(A_i) < 1$, $i = 1, 2, \ldots, k_1$. From the divergence of the series $\sum_{n=1}^{\infty} a_n$ we deduce that there exists an index i_0 such that

$$\mathscr{S}(A_{i_0}^{(1)}) = \infty \tag{2}$$

Put $A^{(1)} = A_{i_0}^{(1)}$. Again (iv) implies that there exists a decomposition

$$A_1^{(2)} \cup \ldots \cup A_{k_2}^{(2)} = \mathbb{N}$$

such that $m(A_j) < \frac{1}{2}$, $j = 1, 2, \ldots, k_2$. Using the equality

$$A^{(1)} = (A^{(1)} \cap A_1^{(2)}) \cup \ldots \cup (A^{(1)} \cap A_{k_2}^{(2)})$$

and (2) we obtain that there exist j_0 such that

$$\mathscr{S}(A^{(1)} \cap A_{j_0}^{(2)}) = \infty$$

Let us denote $A^{(2)} = A^{(1)} \cap A_{j_0}^{(2)}$. From (i) it follows that $m(A^{(2)}) < \frac{1}{2}$. By induction we can construct a sequence of sets

$$A^{(1)} \supset A^{(2)} \supset \ldots \supset A^{(n)} \supset \ldots$$

such that

$$\mathscr{S}(A^{(n)}) = \infty, \qquad n = 1, 2, \ldots \tag{3}$$

and

$$m(A^{(n)}) < \frac{1}{n}, \qquad n = 1, 2, \ldots \tag{4}$$

According to (3) it is easy to see that there exists a sequence of positive integers $m_1, m_2, \ldots$ such that

$$\begin{aligned} &\mathscr{S}(A^{(1)} \cap \{1, 2, \ldots, m_1\}) \geq 1 \\ &\quad \vdots\ \vdots\ \vdots \\ &\mathscr{S}(A^{(n)} \cap \{m_{n-1} + 1, \ldots, m_n\}) \geq 1 \end{aligned} \tag{5}$$

for $n = 1, 2, \ldots$. Let us put $m_0 = 0$ and

$$B^n = A^{(n)} \cap \{m_{n-1} + 1, \ldots, m_n\}, \qquad n = 1, 2, \ldots$$

If

$$B = \bigcup_{n=1}^{\infty} B^n$$

then, by virtue of (5), we have

$$\sum_{j \in B} a_j = \mathscr{S}(B) = \infty$$

The sets B^n, $n = 1, 2, \ldots$ are finite. Consequently it follows from (iii) and (ii) that $m(B^n) = 0$, $n = 1, 2, \ldots$. It is trivial that

$$B^n \cup B^{n+1} \cup \ldots \subseteq A^{(n)}, \qquad n = 1, 2, \ldots.$$

And therefore for $n = 1, 2, \ldots$ there holds according to (i) and (4)

$$m(B) \leq m(B^1 \cup \ldots \cup B^n) + m(A^{(n+1)}) \leq \frac{1}{n+1}$$

Thus, for $n \to \infty$ we have $m(B) = 0$. The proof is completed.

In paper [2], the measure density of a set $A \in P(\ )$ has been introduced in the following way:

Let the symbol $a + \langle d \rangle$ denote the arithmetic sequence $\{a + nd,\ n = 0, 1, 2, \ldots\}$.

For two sets B_1, B_2 let the symbol $B_1 \dot{\subset} B_2$ denote that the set

$$B_1 \backslash B_2$$

is finite. We shall write $B_1 \doteq B_2$ instead of the fact $B_1 \dot{\subset} B_2$ and $B_2 \dot{\subset} B_1$. It is easy to see that $B_1 \doteq B_2$ if and only if the sets B_1 and B_2 differ at most in a finite number of elements.

Let $\mathscr{D}_0$ be the system of all subsets $S \subseteq \mathbb{N}$ such that there exists a finite number of arithmetic sequences $a_1 + \langle d_1 \rangle$ ..., $a_k + \langle d_k \rangle$ such that

$$S \doteq a_1 + \langle d_1 \rangle \cup \dots \cup a_k + \langle d_k \rangle$$

Now we introduce on $\mathscr{D}_0$ a real function Δ in the following way: For every disjoint union of arithmetic sequences

$$S = a_1 + \langle d_1 \rangle \cup \dots \cup a_k + \langle d_k \rangle$$

we put $\Delta(S) = \dfrac{1}{d_1} + \dots + \dfrac{1}{d_k}$ and for each $S' \doteq S$ we put $\Delta(S') = \Delta(S)$.

If $A \in P(\mathbb{N})$ then the value

$$\mu(A) = \inf\{\Delta(S); \qquad A \dot{\subset} S \wedge S \in \mathscr{D}_0\}$$

will be called the measure density of the set A.

In [2, p. 562] it is proved that the measure density has the following properties:

(v) $$A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$$

(vi) $$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

(vii) For each arithmetic sequence $a + \langle d \rangle$ there holds

$$\mu(a + \langle d \rangle) = \frac{1}{d}$$

From (v) and (vi) it follows that μ is a submeasure.

If $a \in \mathbb{N}$ then for every $d \in \mathbb{N}$ we have

$$\{a\} \subseteq a + \langle d \rangle$$

and therefore

$$\mu(\{a\}) = 0$$

It is clear that for every $d \in \mathbb{N}$

$$\mathbb{N} = \langle d \rangle \cup 1 + \langle d \rangle \cup \dots \cup d - 1 + \langle d \rangle$$

Thus according to (vii) we obtain that μ is a compact submeasure.

Consider now the set

$$A = \{n + n!, n = 0, 1, 2, \dots\}$$

It is obvious that the asymptotic density of the set A is 0. Contrary to this fact we prove that

$$\mu(A) = 1 \tag{6}$$

Clearly, $\mu(A) \le 1$. Suppose that $\mu(A) < 1$. Then by definition of μ, there exist such a disjoint system of arithmetic sequences

$$a_1 + \langle d_1 \rangle, \ldots, a_k + \langle d_k \rangle$$

that

$$A \dot{\subset} a_1 + \langle d_1 \rangle \cup \ldots \cup a_k + \langle d_k \rangle \tag{7}$$

and

$$\frac{1}{d_1} + \ldots + \frac{1}{d_k} < 1$$

Denote the least common multiple of $d_1, \ldots, d_k$ by d. It is easy to see that every arithmetic sequence $a_i + \langle d_i \rangle$, $i = 1, 2, \ldots, k$, can be expressed in the form

$$a_i + \langle d_i \rangle = a_i + \langle d \rangle \cup a_i + d_i + \langle d \rangle \cup \ldots \cup a_i + r_i \,.\, d_i + \langle d \rangle$$

where $r_i = \dfrac{d}{d_i} - 1$, $i = 1, 2, \ldots, k$. The decomposition on the right-hand side is disjoint and contains exactly $\dfrac{d}{d_i}$ arithmetic sequences. From (7) it follows that

$$A \dot{\subset} \bigcup_{j=1}^{r} b_j + \langle d \rangle, \qquad b_j \in \mathbb{N}, j = 1, \ldots, r \tag{8}$$

and

$$\frac{r}{d} = \frac{1}{d_1} + \ldots + \frac{1}{d_k} < 1$$

Then $r < d$, and therefore $b_1, \ldots, b_r$ is not the complete residue system modulo d. By virtue of (8), there exists such an arithmetic sequence $b + \langle d \rangle$ that at most a finite number of elements of A belong to $b + \langle d \rangle$.

But it is trivial that for $n = 1, 2, \ldots$ there hold

$$b + nd + (b + nd)! \in b + \langle d \rangle,$$

whence the sequence $b + \langle d \rangle$ has infinitely many common elements with A — a contradiction. This proves (6).

As a consequence of (6) we obtain that $\mu(B) = 1$ for every set $B \supseteq A$. It can be easily proved that if $\mu(C) = 0$ then C has zero asymptotic density. This implies that if we consider the measure density μ, then μ is a compact submeasure, and $Z(\mu)$ is a proper subset of the system of all sets with asymptotic density zero.

To conclude with, let us remark that (6) is also valid in the case ewhen

$$A = \{n + (n!)^{k_n}, \qquad n = 0, 1, 2, \ldots\}$$

where $\{k_n\}$ is an arbitrary sequence of positive integers.

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Katedra matematiky
Stavebná fakulta SVŠT
Radlinského 11
863 68 Bratislava