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A CONSTRUCTION OF REALIZATIONS OF PERTURBATIONS OF POINCARÉ MAPS

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We present in this paper a construction of a C^r-vector field, the flow of which generates a C^r-perturbation of a given C^r-Poincaré map by the first intersections of its trajectories with a given transversal. This vector field is called the C^r-realization or simply realization of the C^r-perturbation of the Poincaré map. The C^r-realizations are useful for the study of generic properties and generic bifurcations of vector fields (see, e.g., [5, 7]).

D. B. Crespin [2] proved a result of the existence of a smooth realization of a given perturbation of a smooth Poincaré map, which is C^0-close to a given vector field. In the book of J. Palis and W. de Melo [7] there is a result (see Lemma 2.5) of the existence of a smooth realization of a special linear perturbation of a given Poincaré map, which is sufficient for the study of generic properties of vector fields. One of the possible constructions of a C^1-realization C^1-close to a given vector field can be obtained by applying the result of J. Palis and F. Takens (see [8, Appendic, Lemma]). This result is used here in the construction of a suspension of a C^1-diffeomorphism (for the definition of the suspension, see, e.g., [7]). The suspensions are useful for the study of generic properties of vector fields near closed orbits (see, e.g., [7]), but they are not quite convenient for the study of generic bifurcations near closed orbits. A result of the existence of the C^r-realizations of C^r-perturbations of a given parametrized C^r-Poincaré map is given in [5], but unfortunately its proof is not quite correct. To correct this proof, it is necessary to use a special surjective mapping theorem for smooth maps instead of the ordinary one for C^r-maps ($1 \leqq r < \infty$). We apply here a surjective mapping theorem for mappings of Fréchet spaces, which is a corollary of some new version of the Nash-Moser implicit function theorem (see, e.g., [9]) published by R. S. Hamilton [3] in 1982. We shall give a somewhat different formulation of the result from that given in [5, Lemma 6]. This result is also important in the construction of realizations of given perturbations of the Poincaré maps in the space of parametrized second order ordinary differential equations on differentiable manifolds presented in the paper [6].

Let X be a compact C^{r+1}-manifold, P a compact C^r-manifold, $r \geqq 1$ and

$V^r(P, X)$ be the set of all parametrized C^r-vector fields on X depending on a parameter in P. We denote the set of all C^r-vector field on X by $\Gamma^r(X)$. If $F \in V^r(P, X)$ and $p \in P$, then we define $F_p \in \Gamma^r(X)$ by $F_p(x) = F(p, x)$, $x \in X$. We denote by $\varphi^F(x, p, t)$ the parametrized flow of $F \in V^r(P, X)$ and by $\varphi^G(x, t)$ the flow of $G \in \Gamma^r(X)$. We can endow the sets $\Gamma^r(X)$ and $V^r(P, X)$ with the structure of a Banach space with norms induced by the C^r-norms of local representations (see, e.g., [4, Appendix A, III]).

Using the implicit function theorem one can prove the following theorem (see, e.g., [10]).

Theorem 1. *Let X be a compact C^{r+1}-manifold, P a compact C^r-manifold, $1 \leqq r < \infty$ and let $F \in V^r(P, X)$. Suppose that the vector field $F_{p_0} \in \Gamma^r(X)$, $p_0 \in P$, has a periodic trajectory γ through a point x_0 with the prime period τ_0 and let Σ be a local transversal to γ at the point x_0. Then there exists a neighbourhood $B_0 \times U_0 \times V_0 \times W_0$ of the point (F, x_0, p_0, τ_0) in $V^r(P, X) \times \Sigma \times P \times R$ and a unique C^r-function τ: $B_0 \times U_0 \times V_0 \to W_0$ such that $\tau(F, x_0, p_0) = \tau_0$ and $\varphi^G(x, p, t) \in \Sigma$ for $(G, x, p) \in B_0 \times U_0 \times V_0$ if and only if $t = \tau(G, x, p)$.*

Corollary. *Let X be a compact C^{r+1}-manifold and $F \in \Gamma^r(X)$ ($1 \leqq r < \infty$). Suppose that F has a periodic trajectory γ through a point x_0 with the prime period τ_0 and let Σ be a local transversal to γ at x_0. Then there exists a neighbourhood $D_0 \times U_0 \times W_0$ of the point (F, x_0, τ_0) in $\Gamma^r(X) \times \Sigma \times R$ and a unique C^r-function τ: $D_0 \times U_0 \to W_0$ such that $\tau(F, x_0) = \tau_0$ and $\varphi^G(x, t) \in \Sigma$ for $(G, x) \in D_0 \times U_0$ if and only if $t = \tau(G, x)$.*

Definition 1. *Let the assumptions of Theorem 1 and its corollary be satisfied. Then for every $G \in D_0 \subset \Gamma^r(X)$ the mapping π_G: $U_0 \mapsto \Sigma$, $\pi_G(x) = \varphi^G(x, \tau(G, x))$ is defined. This mapping is called the Poincaré map and we shall also denote it by $\pi_G[F, \gamma, \Sigma, x_0, U_0]$. We shall often write π, or $\pi[F, \gamma, \Sigma, x_0, U_0]$ instead of $\pi_F[F, \gamma, \Sigma, x_0, U_0]$. For every $G \in B_0 \subset V^r(P, X)$ the mapping H_G: $U_0 \times V_0 \mapsto \Sigma$, $H_G(x, p) = \varphi^G(x, p, \tau(G, x, p))$ is defined. This mapping is called the parametrized Poincaré map and we shall denote it by $H_G[F, \gamma, \Sigma, x_0, p_0, U_0, V_0]$. We shall also write H, or $H[F, \gamma, \Sigma, x_0, p_0, U_0, V_0]$ instead of $H_F[F, \gamma, \Sigma, x_0, p_0, U_0, V_0]$.*

The mappings H_G and π_G are C^r-differentiable. Moreover, π_G is a C^r-diffeomorphism onto its image (see [4], or [7]).

Definition 2. *Let X, P be as above, $p_0 \in P$, $G \in V^r(P, X)$, $1 \leqq r \leqq \infty$, γ be a periodic trajectory of the vector field G_{p0} through a point $x_0 \in X$, let $M_{p_0} \subset P$ be an open neighbourhood of p_0 and $N_\gamma \subset X$ be an open neighbourhood of γ. Then we define the set:*

$$V_G^r(P[M_{p_0}], X[N_\gamma]) = \{\tilde{G} \in V^r(P, X): \tilde{G}(x, p) = G(x, p) \text{ for all } (x, p) \in X \times P \setminus (N_\gamma \times M_{p_0})\}.$$

Let Σ be a local transversal to γ at x_0 and $U \subset \Sigma$, $V \subset P$ be open sets such that $x_0 \in U$, $p_0 \in P$ and the parametrized Poincaré map $H = H[G, \gamma, \Sigma, x_0, p_0, U, V]$ is defined. Let $K \subset U$, $I \subset V$ be such open sets that $\bar{K} \subset U$, $\bar{L} \subset V$, $x_0 \in K$, $p_0 \in L$. Then we define the set:

$$Z_J^r(U[K], V[L]) = \{\tilde{H} \in C^r(U \times V, \Sigma) : \tilde{H}(x, p) = H(x, p) \text{ for all } (x, p) \in U \times V \setminus (K \times L)\}.$$

Theorem 2. *Let X be a compact C^{l+1}-manifold, P a compact C^l-manifold, $1 \leqq l \leqq \infty$, $G \in V^r(P, X)$, $1 \leqq r \leqq l$, $r < \infty$, $p_0 \in P$, γ be a periodic trajectory of the vector field G_{p_0} through $x_0 \in X$. Then there exists a local transversal Σ to γ at x_0 such that for the parametrized Poincaré map $H = H[G, \gamma, \Sigma, x_0, p_0, U, V]$ the following holds: There exists an open neighbourhood K of the point x_0 in Σ, $\bar{K} \subset U$, open neighbourhoods M_{p_0}, L of p_0 in P, $\bar{L} \subset \bar{M}_{p_0} \subset V$, an open neighbourhood $N_\gamma \subset X$ of γ, a neighbourhood $\mathcal{V}(H)$ of the mapping H in $Z_H^r(U[K], V[L])$ and a continuous map $\chi: \mathcal{V}(H) \to V_G^r(P[M_{p_0}], X[N_\gamma])$ such that for every $\tilde{H} \in \mathcal{V}(H)$ the parametrized vector field $\tilde{G} = \chi(\tilde{H})$ is such that the parametrized Poincaré map $H_{\tilde{G}} = H_{\tilde{G}}[G, \gamma, \Sigma, x_0, U, V]$ is defined and $H_{\tilde{G}} = \tilde{H}$.*

Definition 3. *Let X be as above, γ be a periodic trajectory of the vector field $F \in \Gamma^r(X)$ through $x_0 \in X$ and let N_γ be a neighbourhood of γ. Then we define the set:*

$$\Gamma_F^r(X[N_\gamma]) = \{\tilde{F} \in \Gamma^r(X) : \tilde{F}(x) = F(x) \text{ for all } x \in X \setminus N_\gamma\}.$$

Let Σ be a local transversal to γ at x_0 and $U \subset \Sigma$ be an open neighbourhood of x_0 in Σ such that the Poincaré map $\pi = \pi[F, \gamma, \Sigma, x_0, U]$ is defined. Let $K \subset U$ be a neighbourhood of x_0 in Σ such that $\bar{K} \subset U$. Then we define the set:

$$Z_\pi^r(U[K]) = \{\tilde{\pi} \in C^r(U, \Sigma) : \tilde{\pi}(x) = \pi(x) \text{ for all } x \in U \setminus K\}.$$

As a consequence of Theorem 2 we have the following theorem.

Theorem 3. *Let X be a compact C^{l+1}-manifold, $1 \leqq l \leqq \infty$ and γ be a periodic trajectory of the vector field $F \in \Gamma^r(X)$, $1 \leqq r \leqq l$, $r < \infty$, through $x_0 \in X$. Then there exists a local transversal Σ to γ at x_0 such that for the Poincaré map $\pi = \pi[F, \gamma, \Sigma, x_0, U]$ the following holds: There exists an open neighbourhood $K \subset U$ of the point x_0 in Σ, $\bar{K} \subset U$, an open neighbourhood $N_\gamma \subset X$ of γ, a neighbourhood $\mathcal{V}(\pi)$ of the mapping π in $Z_\pi^r(X[N_\gamma])$ and a continuous map $\chi: \mathcal{V}(\pi) \to \Gamma_F^r(X[N_\gamma])$* such that for *each $\tilde{\pi} \in \mathcal{V}(\pi)$ the vector field $\tilde{F} = \chi(\tilde{\pi})$* is such that the Poincaré map $\pi_{\tilde{F}} = \pi_{\tilde{F}}[F, \gamma, \Sigma, x_0, U]$ *is defined and $\pi_{\tilde{F}} = \tilde{\pi}$.*

Now we recall some definitions from Hamilton's paper which are necessary for the formulation of his surjective mapping theorem.

Similarly to the definition of the Gateaux derivative of mappings between Banach spaces, it is possible to define the derivative $f'(x): F_1 \mapsto F_2$ of a mapping $f: U \to F_2$ at $x \in U$, where F_1, F_2 are Fréchet spaces U is an open set in F_1 (see e.g. [9], or [3]). For the Gateaux derivative we shall use the same notation.

Definition 4. *We say that a Fréchet space F is graded if its topology is defined by a countable collection of seminorms $\{\|.\|_n\}_{n=0}^{\infty}$ satisfying $\|x\|_n \leqq \|x\|_{n+1}$ for each $n \geqq 0$, $x \in F$.*

Definition 5. *Let F, G be graded Fréchet spaces. We say that a linear map $L: F \mapsto G$ is a tame map if there exists a natural number r and a number b such that $\|Lf\|_n \leqq C\|f\|_{n+r}$ for all $f \in F$, $n \geqq b$, where C is a positive constant which may depend on n.*

Let B be a Banach space with norm $\|.\|_B$ and let $\Sigma(B)$ denote the space of all sequences $\{x_k\}_{k=0}^{\infty}$ of elements in the B such that $\|\{x_k\}_{k=0}^{\infty}\|_n \overset{\text{def}}{=} \sum_{k=0}^{\infty} e^{nk} \|x_k\|_B < \infty$ for all $n \geqq 0$. The space $\Sigma(B)$ with the topology defined by the system of seminorms $\{\|.\|_n\}_{n=0}^{\infty}$ is a graded Fréchet space.

Definition 6. *We say that a graded Fréchet space F is tame if there exists a Banach space B and linear tame mappings $L: F \to \Sigma(B)$, $M: \Sigma(B) \to F$ such that $M \circ L: F \to F$ is the identity.*

From [3, Lemma 1.3.4, Lemma 1.3.6] we have

Proposition 1. *If $U \subset R^n$ is an open set with compact closure, then $C^\infty(\bar{U}, R^m)$ is a tame Fréchet space.*

Definition 7. *Let F, G be graded Fréchet spaces, $U \subset F$ an open set and $P: U \to G$ a nonlinear map. We say that the map P is tame if the following conditions are satisfied:*

(1) *P is continuous*

(2) *For each $f_0 \in U$ there exists its neighbourhood $V \subset U$, a natural number r and a number b such that $\|P(f)\|_n \leqq C(1 + \|f\|_{n+r})$ for all $f \in V$ and all $n \geqq b$, where C is a constant which may depend on n.*

Proposition 2 ([3, Theorem 2.1.6]). *The composition of two tame maps is a tame map.*

Theorem 4 (the Hamilton surj. mapping th.; [3, Th. 1.1.3]). *Let F, G be tame spaces, $U \subset F$ an open set, $P: U \to G$ a smooth tame map and let $P': U \times F \to G$, $P'(g, f) = P'(g)f$ be a smooth tame map. Suppose that for each $g \in U$ the map $P'(g): F \to G$ is surjective and there exists a smooth tame map $RP: U \times G \to F$ such that $P'(g) \circ RP(g, f) = f$ for all $g \in U$ and $f \in F$. Then for any $g_0 \in U$ there exists its neighbourhood $W \subset U$ such that the set $P(W)$ is open in G. Moreover, there exists a smooth tame map $Q: P(W) \to W$ such that $P \circ Q(w) = w$ for all $w \in P(W)$.*

Lemma 1. *Let E, F be Banach spaces, $U \subset E$ a convex compact set and $\Phi \in C^r(U, F)$, $r \geqq 2$. Then there exists a mapping $H \in C^{r-1}(U \times U, L(E, F))$ such that*

$$\Phi(y) - \Phi(x) = (d_x\Phi + H(x, y))(y - x)$$

for all x, $y \in U$, where $\|H(x, y)\|_1 \leqq \|\Phi\|_2 \|y - x\|$, $\|.\|_k$ is the norm on $L^k(E, F)$

($L^k(E, F)$ *denotes the set of continuous* k*-multilinear maps from* E *into* F), $\|\Phi\|_2 = \max_{x \in U} (\|d_x\Phi\|_1 + \|d_x^2\Phi\|_2$, $d_x^i\Phi$ *is the* i*-the Frechet derivative of* Φ *at* x.

Proof. Let $\Phi'(z) \in L(E, F)$ be the Gateaux derivative of Φ at z and $g(t) = \Phi(x + t(y-x))$ for $x, y \in U$, $0 \leqq t \leqq 1$. Then

$$\Phi(y) - \Phi(x) = g(1) - g(0) = \left(\int_0^1 \Phi'(x + t(y-x))\, dt\right)(y-x)$$

$$= (\Phi'(x) + \left(\int_0^1 [\Phi'(x + t(y-x)) - \Phi'(x)]\, dt\right)(y-x).$$

Let us define

$$H\colon U \times U \to L(E, F),\ H(x, y) = \int_0^1 [\Phi'(x + t(y-x)) - \Phi'(x))]\Big)\, dt.$$

Using the mean value theorem we obtain

$$\|H(x, y)\|_1 \leqq \int_0^1 \|\Phi'(x + t(y-x)) - \Phi'(x)\|_1\, dt \leqq \|\Phi\|_2 \|y - x\|.$$

Since $\Phi'(x) = d_x\Phi$, our lemma is proved.

Lemma 2. *Let* X *be a* C^{r+1}*-manifold,* $r \geqq 1$, dim $X = n$ *and* $x_0 \in X$ *be a regular point of the vector field* $F \in \Gamma^r(X)$. *Then there exists a chart* (U, α) *on* X *such that*

(1) $x_0 \in U$, $\alpha(x_0) = 0 \in R^n$, $\alpha(U) = \{(t, z_1, z_2, \ldots, z_{n-1}) \in R \times R^{n-1}: |t| < 1, |z_i| < 1, i = 1, 2, \ldots, n-1\}$.
(2) *If* f_α *is the main part of the local representative of* F *with respect to the local chart* (U, α), *then* $f_\alpha(t, z) = (1, 0, \ldots, 0) \in R^n$ *for all* $(t, z) \in \alpha(U)$.

Using this so called flow box lemma one can easily prove the following lemma.

Lemma 3. *Let* X, P *be as above,* dim $X = n$, dim $P = k$, *let* $G \in V^r(P, X)$, x_0 *be a regular point of the vector field* G_{p_0}, $p_0 \in P$. *Then there exists a chart* (W, h) *on* $X \times P$ *such that*

(1) $(x_0, p_0) \in W$, $h(x_0, p_0) = (0, 0) \in R^n \times R^k$, $W = W_1 \times W_2$, *where* $W_1 \subset X$, $W_2 \subset P$ *are open sets,* $h\colon W_1 \times W_2 \to R^n \times R^k$, $h(x, p) = (h_1(x, p), h_2(p))$, $h_1\colon W_1 \times W_2 \to R^n$, $h_2\colon W_2 \to R^k$.
(2) *If* $p \in W_2$, *then the map* $h_{1p}\colon W_1 \to R^n$, $h_{1p}(x) = h_1(x, p)$, *is a* C^r*-diffeomorphism of* W_1 *onto* I^n, *where* $I = (-1, 1)$.
(3) *The map* h_2 *is a* C^r-diffeomorphism of W_2 *onto* I^k.
(4) *The main part of the local representative of the parametrized vector field* G *with respect to the chart* (W, h) *has the form* $g_h(t, z, \mu) = (1, 0, \ldots, 0) \in R^n$ *for all* $(t, z, \mu) \in I \times I^{n-1} \times I^k$.

Let us consider a parametrized vector field $G \in V^r(P, X)$, where dim $X = n$, dim $P = k$. Suppose that γ is a periodic trajectory of the vector field G_{p_0}, $p_0 \in P$. Let (W, h) be a chart on $X \times P$ having the properties (1)—(4) from Lemma 3. By the property (4) of this lemma the main part g_h of the local representative of G has the form $g_h(y, z, \mu)=(1, 0, \ldots, 0)$ for all $(t, z, \mu)\in I\times I^{n-1}\times I^k$, which defines the system of

$$\frac{dy}{dt}=1, \quad \frac{dz}{dt}=0\in R^{n-1}. \tag{1}$$

Let $B_m(r)=\{v\in R^m: \|v\|<r\}$, where $\|v\|=\max_i |v_i|$. The set $W^\infty=\{f\in C^\infty(R^{n+k}, R^{n-1}): f(v)=0 \text{ for all } v\in R^{n+k}\setminus B_{n+k}(1/2)\}$ is a Fréchet space with the topology defined by the system of seminorms $\{|\cdot|_j\}_{j=1}^\infty$, $|f|_j=\max_{0\leqq i\leqq j}\left(\max_{x\in B_{n+k}(1/2)}\|d_x^i f\|_i\right)$ and the set $W_0^\infty=\{f\in W^\infty: |f|_j<1/2, j=0, 1, \ldots\}$ is its open subset.

Let us consider the following system of differential equations depending on the parameter $\mu\in R^k$:

$$\frac{dy}{dt}=1, \quad \frac{dz}{dt}=g(y, z, \mu), \tag{2}$$

where $g\in W^\infty$, $y\in R$, $z\in R^{n-1}$, $\mu\in R^k$. This system defines a parametrized flow $\varphi^g(y, z, \mu, t)=(Y^g(y, z, \mu, t), Z^g(y, z, \mu, t))$ on R^n, where $Y^g(y, z, \mu, t)=y+t$ and Z^g satisfies the following integral identity:

$$Z^g(y, z, \mu, t)=z+\int_0^t g(y+s, Z^g(y, z, \mu, s), \mu)\, ds. \tag{3}$$

Let $0<T<1$. For $g\in W^\infty$ we define the mapping

$$Q^g: \{0\}\times I^{n-1}\times I^k\to\{T\}\times R^{n-1},$$

$$Q^g(0, z, \mu)=\varphi^g(0, z, \mu, T)=(T, Z^g(0, z, \mu, T)).$$

Let $Z^\infty=\{h\in C^\infty(I^{n+k-1}, R^{n-1}): h(w)=0 \text{ for all } w\in R^{n+k-1}\setminus B_{n+k-1}(1/2)\}$. Define the map

$$\mathscr{F}: W^\infty\to Z^\infty, \quad \mathscr{F}(g)(z, \mu)=Z^g(0, z, \mu, T)-z.$$

Lemma 4.

(1) *$Q^g\in C^\infty$, $\mathscr{F}(g)\in C^\infty$ for each $g\in W^\infty$.*
(2) *If $g\in W_0^\infty$, then $Q^g(\{0\}\times I^{n-1}\times I^k)\subset\{T\}\times I^{n-1}$.*
(3) *$\mathscr{F}(W_0^\infty)\subset C^\infty(I^{n-1}\times I^k, I^{n-1})$.*

Proof. The assertion (1) follows from the smoothness of φ^g. Since $g(v)=0$ for all $v\in R^{n+k}\setminus B_{n+k}(1/2)$ we have $Z^g(0, z, \mu, t)=z$ for all $t\in R$ and $\|(0, z, , \mu)\|\geqq 1/2$. Therefore $\|Z^g(0, z, \mu, T)\|=\|z\|\leqq 1$. If $\|(0, z, \mu)\|<1/2$, then $\|Z^g(0, z, \mu, T)\|\leqq\|z\|+\int_0^T\|g(s, Z^g(0, z, \mu, s), \mu)\|\, ds\leqq 1$ for each $g\in W_0^\infty$ and so the assertion (2) is proved. The assertion (3) is obvious.

Lemma 5.

(1) $\mathcal{F}/W_0^\infty$ *is a smooth tame map.*
(2) *The derivative* $\mathcal{F}'(g)f$ *exists for each* $f, g \in W^\infty$ *and* $\mathcal{F}'(g)f(z, \mu) = c_0(T, z, \mu, g, f)$ *for* $(z, \mu) \in I^{n-1} \times I^k$, *where* $c_0(t, z, \mu, g, f)$ *is a solution of the differential equation*

$$\frac{dw}{dt} = A^g(t, z, \mu)w + b^{g,f}(t, z, \mu), \tag{4}$$

satisfying the initial condition $w(0) = 0$, *where* $A^g(t, z, \mu) = \dfrac{\partial g(v)}{\partial z}$, $v = (t, Z^g(0, z, \mu, t), \mu)$, $b^{g,f}(t, z, \mu) = f(t, Z^g(0, z, \mu, t), \mu)$.

Proof. Let us denote $B(t, v, z, \mu, g, f) = Z^{g+vf}(0, z, \mu, t) - Z^g(0, z, \mu, t)$ for $t \in [0, T]$, $z \in I^{n-1}$, $\mu \in I^k$, $v \in [0, \varepsilon]$, $\varepsilon > 0$ and let $K_1 = |g|_1$ (C^1-norm of g), $K_2 = \max_v \left(\max_{(z, \mu, t)} \int_0^t \|f(s, Z^{g+vf}(0, z, \mu, s), \mu)\| \, ds \right)$. The mean value theorem and the equality (3) imply $\|B(t, v, z, \mu, g, f)\| \leqq \int_0^t \|g(s, Z^{g+vf}(0, z, \mu, s), \mu) - g(s, Z^g(0, z, \mu, s), \mu)\| \, ds + v\int_0^t \|f(s, Z^{g+vf}(0, z, \mu, s), \mu)\| \, ds \leqq K_1 \int_0^t \|B(s, v, z, \mu, g, f)\| \, ds + \gamma K_2$. Applying the Gronwall lemma we obtain

$$\max_t \|B(t, v, z, \mu, g, f)\| \leqq vK, \tag{5}$$

where $K = K_2 \exp K_1$. If $b(v, g, f) = \max_{(z, \mu, t)} \|B(t, v, z, \mu, g, f)\|$, then from (5) we obtain

$$\lim_{v \to 0} b(\gamma, g, f) = 0. \tag{6}$$

By Lemma 1 there exists a map $H \in C^\infty(I^{n+k} \times I^{n+k}, L(R^{n+k}, R^{n-1}))$ such that for any $u, v \in I^{n+k}$

$$g(v) - g(u) = (d_u g + H(u, v))(v - u), \tag{7}$$

$$\|H(u, v)\|_1 \leqq R\|v - u\|, \tag{8}$$

where $R = |g|_2$. For each $f, g \in W^\infty$, $t \in [0, T]$, $v \in [0, \varepsilon]$, $z \in R^{n-1}$ and $\mu \in I^k$ we have $\dfrac{d}{dt} B(t, v, z, \mu, g, f) = g(t, Z^{g+vf}(0, z, \mu, t), \mu) - g(t, Z^g(0, z, \mu, t), \mu) + vf(t, Z^{g+vf}(0, z, \mu, t), \mu) = (M(g, z, \mu, t) + N(t, v, z, \mu, g, f))B(t, v, z, \mu, g, f) + vf(t, Z^{g+vf}(0, z, \mu, t), \mu)$, where $M(g, z, \mu, t) = d_u g$, $N(t, v, z, \mu, g, f)$

$= H(u, v)$, $u = (t, Z^g(0, z, \mu, t), \mu)$, $v = (t, Z^{g+vf}(0, z, \mu, t), \mu)$. Using (8) we obtain the inequality

$$\|N(t, v, z, \mu, g, f)\| \leqq R\|B(t, v, z, \mu, g, f)\|. \tag{9}$$

For $C(t, v, z, \mu, g, f) = v^{-1}B(t, v, z, \mu, g, f)$ we obtain

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} = (M(t) + N(t))C(t) + F(t), \tag{10}$$

where $C(t) = C(t, v, z, \mu, g, f)$, $M(t) = M(g, z, \mu, t)$, $N(t) = N(t, v, z, \mu, g, f)$, $F(t) = f(t, Z^{g+vf}(0, z, \mu, t), \mu)$, $C(0, v, z, \mu, g, f) = 0$. The inequalities (5), (9) and (6) imply

$$\lim_{v \to 0} \sigma(\gamma, g, f) = 0, \tag{11}$$

where $\sigma(v, g, f) = \max_{(t, z, \mu)} \|N(t)C(t)\|$. Let $c_0(t, z, \mu, g, f)$ be the solution of the linear differential equation (4) satisfying the initial condition $c_0(0, z, \mu, g, f) = 0$. From (10) and (4) we obtain $\|C(t, v, z, \mu, g, f) - c_0(t, z, \mu, g, f)\| \leqq \sigma(v, g, f) + M(g)\int_0^t \|C(s, v, z, \mu, g, f) - c_0(s, z, \mu, g, f)\|\, \mathrm{d}s$, where $M(g) = \max_{(z, \mu, t)} \|M(g, z, \mu, t)\|$. Applying the Gronwall lemma we have $\max_{(t, z, \mu)} \|C(t, v, z, \mu, g, f) - c_0(t, z, \mu, g, f)\| \leqq \sigma(v, g, f) \exp M(g)$ and therefore (11) implies

$$\lim_{v \to 0} \max_{(t, z, \mu)} \|C(t, v, z, \mu, g, f) - c_0(t, z, \mu, g, f)\| = 0. \tag{12}$$

Let us define the maps:

$$c_0(g, f);\ I^{n-1} \times I^k \to I^{n-1},\ c_0(g, f)(z, \mu) = c_0(T, z, \mu, g, f),$$

$$c(v, g, f)\colon I^{n-1} \times I^k \to I^{n-1},\ c(v, g, f)(z, \mu) = C(T, v, z, \mu, g, f).$$

These mappings are obviously smooth and for any $(z, \mu) \in I^{n-1} \times I^k$ we have

$$\|v^{-1}[\mathscr{F}(g + vf)(z, \mu) - \mathscr{F}(g)(z, \mu)] - c_0(g, f)(z, \mu)\| =$$

$$= \|v^{-1}[Z^{g+vf}(0, z, \mu, T) - Z^g(0, z, \mu, T)] - c_0(g, f)(z, \mu)\| =$$

$$= \|c(v, g, f)(z, \mu) - c_0(g, f)(z, \mu) \leqq \sigma(v, g, f) \exp M(g)$$

and so the equality (11) yields

$$\lim_{v \to 0} v^{-1}[\mathscr{F}(g + vf) - \mathscr{F}(g)] = c_0(g, f) \tag{13}$$

(the limit with respect to the seminorm $|\,.\,|_0$). It is necessary to prove the existence of the limit (13) with respect to every seminorm $|\,.\,|_m$, $m \geqq 0$. However, this is not difficult to prove now. The partial derivative of $c_0(g, f)(z, \mu)$ with respect to z and μ, respectively, also the higher order one, is a solution of a linear differential equation in a corresponding Banach space, satisfying the zero initial condition, which can be obtained by differentiating the equation (4) with respect to z and μ, respectively. Therefore one can use the same procedure as that we have used in the case of $m=0$. Since Lemma 1 is formulated for mappings of Banach spaces, we may also use it for $m \geqq 1$. Therefore all estimations are analogous to these we have performed in the case of $m=0$ and we omit them. It suffices to prove the asserrtion (1) of our lemma. From the form of the linear differential equation (4) there follows the smoothness of the map $\mathcal{F}/W_0^\infty$. It remains to show that this map is tame. The assertion (3) of Lemma 4 implies that $\mathcal{F}(f)(I^{n-1} \times I^k) \subset I^{n-1}$ for all $f \in W_0^\infty$ and so $|\mathcal{F}(f)|_0 \leqq 1 \leqq 1 + |f|_0$. Applying the operator $\partial/\partial z$ to the integral identity (3) and using the Gronwall inequality one can show that

$$\max_{(z,\mu) \in I^{n-1} \times I^k} \left\| \frac{\partial \mathcal{F}(f)(z, \mu)}{\partial z} \right\| \leqq \exp |f|_1 .$$

Using the same procedure one can prove that for any natural number m and natural numbers i, j, $i+j=m$, there exists a continuous nonnegative function $\Phi^m_{i,j}(u_0, u_1, \ldots, u_m)$ defined for $u_s \geqq 0$, $s=0, 1, \ldots, m$, nondecreasing in each variable and such that

$$\max_{(z,\mu) \in I^{n-1} \times I^k} \left\| \frac{\partial^{i+j} \mathcal{F}(f)(z, \mu)}{\partial z^i \, \partial \mu^j} \right\| \leqq \Phi^m_{i,j}(|f|_0, |f|_1, \ldots, |f|_m)$$

for all $f \in W_0^\infty$. Since $|f|_j < 1/2$ for all $f \in W_0^\infty$, $j=0, 1, \ldots, m$, we obtain from the above inequality that for any natural number m there exists a positive constant C_m (independent of f) such that $|\mathcal{F}(f)|_m \leqq C_m \leqq C_m(1 + |f|_m)$ for all $f \in W_0^\infty$. Thus we have proved that the mapping $\mathcal{F}/W_0^\infty$ is tame.

Remark. We have used the mean value theorem in the proof of the existence of the limit (13) with respect to the seminorm $|\,.\,|_0$ and we needed the C^1-differentiability of g. This means that the C^r-differentiability of g is not sufficient for the existence of the limit (13) with respect to the seminorm $|\,.\,|_r$. Therefore it is necessary to work with the class C^∞.

Lemma 6.

(1) *The mapping* $\mathcal{F}' : W_0^\infty \times W^\infty \to Z^\infty$, $\mathcal{F}'(g, f) = \mathcal{F}'(g)f$ *is a smooth tame map.*
(2) *There exists a smooth tame map* $R\mathcal{F}: Z^\infty \to W_0^\infty \times W^\infty$ *such that* $\mathcal{F}'(g) \circ R\mathcal{F}(g)h = h$ *for all* $g \in W_0^\infty$, $h \in Z^\infty$.

Proof. The smoothness of $\mathcal{F}'$ follows from Lemma 5 (1). From Lemma 4 (2) and the variation of constants formula we obtain

$$\mathcal{F}'(g)f(z, \mu) = \int_0^T \Phi^g(T, s, z, \mu) f(s, Z^g(0, z, \mu, s), \mu) \, ds \tag{14}$$

for all $g \in W_0^\infty$, $f \in W^\infty$, $(z, \mu) \in I^{\,-1} \times I^k$, where $\Phi^g(t, s, z, \mu)$ is the resolvent of the equation

$$\frac{dw}{dt} = A^g(t, z, \mu)w \tag{15}$$

(A^g is defined in Lemma 5). The mapping $s \to \Phi(T, s, z, \mu)$ is a solution of the equation (15) and so by a procedure similar to that we have used in the proof of the inequality $|\mathcal{F}(f)|_m \leqq C_m$ (see the proof of Lemma 5), one can prove that for any natural number m there exists a positive constant L_m such that

$$|\Phi^g|_m \leqq L_m \tag{16}$$

for all $g \in W_0^\infty$, where L_m is independent of g. Let us define the mapping $Z\colon W_0^\infty \to C^\infty(I^{n-1} \times I^k \times I, R^{n-1})$ $Z(g)(z, \mu, t) = Z^g(0, z, \mu, t)$. From the definition of $\mathcal{F}$ and from (3) we have that $Z(g)(z, \mu, T) = \mathcal{F}(g)(z, \mu) - z$ and therefore

$$|Z(g)|_m \leqq C_m + 1 \quad \text{for all} \quad g \in W_0^\infty. \tag{17}$$

Using the formula (14), the inequalities (16), (17), the Leibnitz formula and Proposition 2, one can easily prove that there exists a sequence $\{R_m\}_{m=1}^\infty$ of such constants that $|\mathcal{F}'(g, f)|_m \leqq R_m(1 + |f|_m)$ for all $g \in W_0^\infty$ and $f \in W^\infty$. We leave the details to the reader. Now it suffices to prove the assertion (2). Let us define the mapping $R\mathcal{F}\colon W_0^\infty \times Z^\infty \to W^\infty$,

$$R\mathcal{F}(g, f)(s, z, \mu) = T^{-1}\psi^g(T, s, z, \mu)h(Z^g(0, z, \mu, -s), \mu), \tag{18}$$

where $\psi^g(t, s, z, \mu) = [\Phi^g(t, s, z, \mu)]^{-1}$. From the properties of the resolvent it follows that map $t \to \psi^g(T, t, z, \mu)$ is a solution of the equation

$$\frac{dX}{dt} = -XA^g(t, z, \mu). \tag{19}$$

Therefore the smoothness of the mapping $(g, s, z, \mu) \mapsto \psi^g(s, z, \mu)$ is obvious. The mapping $R\mathcal{F}$ has the same structure as the mapping $\mathcal{F}'$ and therefore using the same procedure as that in the proof of the assertion 1 one can prove that $R\mathcal{F}$ is a tame map. From the formulae (14), (18) we obtain that

$$\mathcal{F}'(g) \circ R\mathcal{F}(g)h(z, \mu) =$$
$$= \int_0^T \Phi^g(T, s, z, \mu)T^{-1}\psi^g(T, s, z, \mu)h(Z^g(0, Z^g(0, z, \mu, s), -s), \mu)\, ds =$$
$$= \int_0^T T^{-1}h(z, \mu)\, ds = h(z, \mu) \quad \text{for all} \quad (z, \mu) \in I^{n-1} \times I^k,\ h \in Z^\infty$$

and this completes the proof.

As a direct consequence of Lemma 4, Lemma 5 and Theorem 4 we obtain the following lemma.

Lemma 7. *There exists a neighbourhood* $U_0 \subset W_0^\infty$ *of the zero map* $0 \in W_0^\infty$ *such that* $\tilde{\mathcal{F}} = \mathcal{F}/U_0\colon U_0 \to \mathcal{F}(U_0)$ *is surjective, where* $\mathcal{F}(U_0) \subset C^\infty(I^{n-1} \times I^k, I^{n-1})$ *is an open neighbourhood of the mapping* $h_0\colon I^{n-1} \times I^k \to I^{n-1}$, $h_0(z, \mu) = z$. *Moreover, there exists a smooth tame map* $G\colon \mathcal{F}(U_0) \to U_0$ *such that* $\tilde{\mathcal{F}} \circ G(h) = h$ *for all* $h \in \mathcal{F}(U_0)$.

Proof of Theorem 2. Let (W, h) be a chart on $X \times P$ having the properties (1)—(4) from Lemma 3. Let W^∞, W_0^∞, Z^∞ be the Fréchet spaces defined as above. Let us recall that $\varphi^g = (Y^g, Z^g)$ is the parametrized flow defined by the system (2) for $g \in W^\infty$ and $\mathcal{F}\colon W^\infty \to Z^\infty$, $\mathcal{F}(g)(z, \mu) = Z^g(0, z, \mu, T) - z$, where $0 < T < 1$. The set $\Sigma = h_{p_0}^{-1}(\{T\} \times I^{n-1})$ is the global transversal of the vector field G_{p_0}/W_1 and $h_{p_0}(\gamma \cap W_1) = I \times \{0\}$. If $H = H[G, \gamma, \lambda, x_0, p_0, U, V]$ is the parametrized Poincaré map, then Lemma 3 (4) implies that for each $(x, p) \in U \times V$ there exists $(z, \mu) \in I^{n-1} \times I^k$ such that

$$h_{1p} \circ H(x, p) = \varphi^0(0, z, \mu, T) = (T, z). \tag{20}$$

The flow φ^g defines in the coordinates of the chart (W, h) a parametrized vector field $\hat{G}^g$ on W, where $\hat{G}^g = G$ on $W \setminus h^{-1}(B_{n+k}(1/2))$. Let us define the parametrized vector field F^g on $X \times P$ as follows: $F^g = \tilde{G}^g$ on W and $F^g = G$ on $X \times P \setminus W$. This is obviously C^r. From Theorem 1 it follows that there exists an open neighbourhood U_0 of the zero map $0 \in W_0^\infty$ such that if $U \times V$ is a sufficiently small neighbourhood of the point (x_0, p_0) in $\Sigma \times P$, then for each $g \in U_0$ the mapping $H^g = H_{F^g}[G, \gamma, \Sigma, x_0, p_0, U, V]$ is defined and for such $(x, p) \in U \times V$ for which the equality (20) is valid we have

$$h_{1p} \circ H^g(x, p) = \varphi^g(0, z, \mu, T) = (T, \sigma_0(z, \mu) + \mathcal{F}(g)(z, \mu)), \tag{21}$$

where $\sigma_0 \in C^\infty(I^{n-1} \times I^k, I^{n-1})$, $\sigma_0(z, \mu) = z$ for all (z, μ). If U_0 is sufficiently small, then by Lemma 7 there exists an open neighbourhood $V_0 \subset Z^\infty$ of the zero map $0 \in Z^\infty$ such that $\mathcal{F}/U_0\colon U_0 \to V_0$ is surjective. Moreover, there exists a smooth map $Q\colon U_0 \to V_0$ such that $\mathcal{F} \circ Q(\sigma) = \sigma$ for all $\sigma \in V_0$. This means that for arbitrary $\sigma \in V_0$ there exists $g \in U_0$ such that $\mathcal{F}(g) = \sigma$. Therefore for arbitrary $\sigma \in U(\sigma_0) = \{\sigma_0 + \sigma\colon \sigma \in U_0\}$ there exists $g \in U_0$ such that

$$h_{1p} \circ H^g(x, p) = \varphi^g(0, z, \mu, T) = (T, \tilde{\sigma}(z, \mu)) \tag{22}$$

for all $(x, p) \in U \times V$. Let $K = h_1^{-1}(B_{n+k}(1/2)) \cap U$, $L = h_2^{-1}(B_k(1/2)) \cap V$, $M_{p_0} = V$ and let N_γ be an open neighbourhood of γ such that $\bigcup_{p \in V} h_{1p}^{-1}(I \times I^{n-1}) \subset N_\gamma$. Let $\vartheta(H) = \{\tilde{H} \in Z_H^\infty(U[K], V[L])$: there exists $\tilde{\sigma} \in U(\sigma_0)$ such that $h_{1p} \circ \tilde{H}(x, p) = (T, \tilde{\sigma}(z, \mu))$ for all $(x, p) \in U \times V\}$. Let us define the map $\chi\colon \mathcal{V}(H) \mapsto V^r(P[M_{p_0}], X[N_\gamma])$ as follows: If $\tilde{H} \in \mathcal{V}(H)$ and $h_{1p} \circ \tilde{H}(x, p) = (T, \tilde{\sigma}(z, \mu))$, where $\tilde{\sigma} \in U(\sigma_0)$, then $\chi(\tilde{H}) = F^g$, where $g \in U_0$ is the map for which the equality

(22) is valid. From the construction of F^g it follows that χ is continuous and from the equality (22) we have that $H_{F^g}[G, \gamma, \Sigma, x_0, p_0, U, V] = \tilde{H}$.

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КОНСТРУКЦИЯ РЕАЛИЗАЦИЙ ВОЗМУЩЕНИЙ ОТОБРАЖЕНИЙ ПУАНКАРЕ

Milan Medveď

Резюме

В статье дана одна конструкция векторного поля класса C^r, поток которого порождает C^r-возмущение данного отображения Паункере класса C^r.