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ASYMPTOTIC BEHAVIOUR OF A CLASS OF NONOSCILLATORY SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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0. Introduction

The present paper is concerned with n-th order $(n>1)$ differential equations with deviating arguments, which involve the r-derivatives $D_r^{(i)}x$ $(i=0, 1, \ldots, n)$ of the unknown function x defined by

$$D_r^{(0)}x=x, \quad D_r^{(i)}x=r_i(D_r^{(i-1)}x)'$$
$$(i=1, \ldots, n-1) \quad \text{and} \quad D_r^{(n)}x=(D_r^{(n-1)}x)',$$

where r_i $(i=1, \ldots, n-1)$ are positive continuous functions on an interval $[t_0, \infty)$. A real-valued function h is said to be n-times r-differentiable on an interval $[T, \infty)$, $T \geqq t_0$ if $D_r^{(n)}h$ is defined on $[T, \infty)$, and h is said to be n-times continuously r-differentiable on $[T, \infty)$ if $D_r^{(n)}h$ is continuous on $[T, \infty)$. Note that in the special case where $r_1=\ldots=r_{n-1}=1$ the above notion of the r-differentiability specializes from the one of the usual differentiability. Recently, there has been an increasing interest in studying the oscillatory and asymptotic behaviour of differential equations involving the r-derivatives of the unknown function in place of its usual derivatives.

More precisely, the paper deals with the asymptotic behaviour of nonoscillatory solutions of differential equations with deviating arguments of the form

$$\text{(E)} \qquad (D_r^{(n)}x)(t)+F(t; x\langle g_0(t)\rangle,$$
$$(D_r^{(1)}x)\langle g_1(t)\rangle, \ldots, (D_r^{(l)}x)\langle g_l(t)\rangle)=b(t), \quad t \geqq t_0,$$

where: $r_0=1$; l is an integer with $0 \leqq l \leqq n-1$;

$$(D_r^{(i)}x)\langle g_i(t)\rangle=((D_r^{(i)}x)[g_{i1}(t)], \ldots, (D_r^{(i)}x)[g_{iN_i}(t)]),$$
$$g_i=(g_{i1}, \ldots, g_{iN_i}) \quad (i=0, 1, \ldots, l);$$

g_{ik} $(k=1, \dots, N_i;\ i=0, 1, \dots, l)$ are continuous real-valued functions on the interval $[t_0, \infty)$ with

$$\lim_{t\to\infty} g_{ik}(t)=\infty \quad (k=1, \dots, N_i;\ i=0, 1, \dots, l);$$

b is a continuous real-valued function on $[t_0, \infty)$ and F is a continuous real-valued function defined at least on $[t_0, \infty)\times(R_+^N \cup R_-^N)$, where $N=N_0+N_1+\dots+N_l$ and $R_+=(0, \infty)$, $R_-=(-\infty, 0)$. Without any further mention, we suppose that: For every $t\geqq t_0$, the function $|F(t;\ \cdot)|$ is increasing on R_+^N and decreasing on R_-^N. For real-valued functions defined on subsets of R^N monotonicity is considered with respect to the order in R^N defined by the positive cone $\{\mathbf{Y}=(y_1, \dots, y_N)\in R^N: y_1\geqq 0, \dots, y_N\geqq 0\}$. Sufficient smoothness for the existence of solutions of (E) on an infinite subinterval of $[t_0, \infty)$ will be assumed. In what follows the term "solution" is always used only for such solutions $x(t)$ of (E) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T, \infty)$ is called *oscillatory* if the set of its zeros is unbounded above, and otherwise it is called *nonoscillatory*.

For our purposes, for any integers i and λ with $0\leqq i\leqq\lambda\leqq n-1$, we introduce the function $R_{i\lambda}$ which is defined on $[t_0, \infty)$ by

$$R_{i\lambda}(t)=\begin{cases} 1, \text{ if } i=\lambda \\ \displaystyle\int_{t_0}^{t=s_i}\frac{1}{r_{i+1}(s_{i+1})}\int_{t_0}^{s_{i+1}}\frac{1}{r_{i+2}(s_{i+2})}\cdots\int_{t_0}^{s_{\lambda-1}}\frac{1}{r_\lambda(s_\lambda)}\,ds_\lambda\dots ds_{i+2}\,ds_{i+1}, \text{ if } i<\lambda. \end{cases}$$

In particular, for any integer λ with $0\leqq\lambda\leqq n-1$, we put

$$R_\lambda(t)=R_{0\lambda}(t), \quad t\geqq t_0.$$

To obtain our results we need the following lemma from [1].

Lemma 0.1. *Let λ, $0<\lambda\leqq n-1$, be an integer such that*

$$\int^\infty\frac{dt}{r_i(t)}=\infty \quad (i=1, \dots, \lambda).$$

Moreover, let h be a function whose λ-th r-derivative $D_r^{(\lambda)}h$ exists on an interval $[T, \infty)$, $T\geqq t_0$.

If $\lim_{t\to\infty}(D_r^{(\lambda)}h)(t)$ exists in $R^*-\{0\}$, where $R^*=R\cup\{-\infty, \infty\}$ is the extended *real line, then so does* $\lim_{t\to\infty}[h(t)/R_\lambda(t)]$ *and, moreover,*

$$\lim_{t\to\infty}[h(t)/R_\lambda(t)] = \lim_{t\to\infty}(D_r^{(\lambda)}h)(t).$$

The subject of this paper is the following: Let m be an integer with $l \leqq m \leqq n-1$. Provided that

$$(\Sigma[\mathrm{m}]) \qquad \int^{\infty} \frac{dt}{r_i(t)} = \infty \quad (i = 1, \dots, m)$$

for the case $m > 0$ and that the function $|b|$ satisfies a smallness condition depending on m (which holds by itself if $b = 0$), we shall find a condition (depending on m) for the function F, which ensures the existence of at least one (nonoscillatory) solution x of the differential equation (E) with

$$(*) \qquad \lim_{t\to\infty}(D_r^{(m)}x)(t) = \lim_{t\to\infty}\frac{x(t)}{R_m(t)} = L \in R - \{0\}.$$

Next, we shall consider the differential equation (E) with $b = 0$, i.e. the equation

$$(\mathrm{E}_0) \qquad (D_r^{(n)}x)(t) + F(t; x\langle g_0(t)\rangle, (D_r^{(1)}x)\langle g_1(t)\rangle, \dots, (D_r^{(l)}x)\langle g_l(t)\rangle) = 0,$$

and we will suppose that the functions r_i $(i = 1, \dots, n-1)$ are such that

$$(\Sigma) \qquad \int^{\infty} \frac{dt}{r_i(t)} = \infty \quad (i = 1, \dots, n-1)$$

and that F has one of the sign properties

$$(\mathrm{I}) \qquad \begin{cases} F(t\colon \mathbf{Y}) \geqq 0 & \text{for every } t \geqq t_0 \text{ and } \mathbf{Y} \in R_+^N \\ F(t; \mathbf{Y}) \leqq 0 & \text{for every } t \geqq t_0 \text{ and } \mathbf{Y} \in R_-^N, \end{cases}$$

$$(\mathrm{II}) \qquad \begin{cases} F(t; \mathbf{Y}) \leqq 0 & \text{for every } t \geqq t_0 \text{ and } \mathbf{Y} \in R_+^N \\ F(t; \mathbf{Y}) \geqq 0 & \text{for every } t \geqq t_0 \text{ and } \mathbf{Y} \in R_-^N. \end{cases}$$

Then we shall prove that the condition which is sufficient in order that the differential equation (E_0) have at least one (nonoscillatory) solution x satisfying $(*)$ is also necessary. The results obtained extend previous ones due to the author, Sficas and Staikos [2] and to the author and Staikos [3] concerning the special case where $l = 0$. Notice that the methods used here pattern after that of [2] and [3].

1. Sufficient conditions

To obtain our first result (Theorem 1.1) we shall apply the fixed point technique by using the following Schauder's theorem (Schauder [4]).

The Schauder theorem. *Let E be a Banach space and X a nonempty, convex and closed subset of E. Moreover, let S be a continuous mapping of X into itself. If SX is relatively compact, then the mapping S has at least one fixed point (i.e. there exists an $x \in X$ with $x = Sx$).*

A set $\mathcal{F}$ of real-valued functions defined on the interval $[T, \infty)$ is said to be (cf. [5]) equiconvergent at ∞ if all functions in $\mathcal{F}$ are convergent in R at the point ∞ and, moreover, for every $\varepsilon > 0$ there exists a $T' \geqq T$ such that, for all functions f in $\mathcal{F}$,

$$|f(t) - \lim_{s \to \infty} f(s)| < \varepsilon \quad \text{for every} \quad t \geqq T'.$$

Let now $B([T, \infty))$ be the Banach space of all continuous and bounded real-valued functions on the interval $[T, \infty)$, endowed with the usual sup-norm $\| \ \|$. We need the following compactness criterion for subsets of $B([T, \infty))$, which is a corollary of the Arzelà—Ascoli theorem. For a proof of this criterion we refer to Staikos [5].

Compactness criterion. *Let $\mathcal{F}$ be an equicontinuous and uniformly bounded subset of the Banach space $B([T, \infty))$. If $\mathcal{F}$ is equiconvergent at ∞, it is also relatively compact.*

Theorem 1.1. *Let m, $l \leqq m \leqq n-1$, be an integer such that $(\Sigma[m])$ holds when $m > 0$, and:*

$(C[m])$ *For some nonzero constant c*

$$\begin{cases} \displaystyle\int^{\infty} |F(t; cR_{0,n-1}\langle g_0(t)\rangle, cR_{1,n-1}\langle g_1(t)\rangle, \ldots, cR_{l,n-1}\langle g_l(t)\rangle)| \, dt < \infty \\ \text{if} \quad m = n-1 \\ \displaystyle\int^{\infty} \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |F(s; cR_{0m}\langle g_0(s)\rangle, cR_{1m}\langle g_1(s)\rangle, \ldots \\ \ldots, cR_{lm}\langle g_l(s)\rangle)| \, ds \, ds_{n-1} \ldots ds_{m+1} < \infty \quad \text{if} \quad m < n-1. \end{cases}$$

$(B[m])$ There holds

$$\begin{cases} \displaystyle\int^{\infty} |b(t)| \, dt < \infty \quad \text{if} \quad m = n-1 \\ \displaystyle\int^{\infty} \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)| \, ds \, ds_{n-1} \ldots ds_{m+1} < \infty \quad \text{if} \quad m < n-1. \end{cases}$$

Then for every real number L with $Lc > 0$ and $\frac{|c|}{2} < |L| < |c|$ there exists a (nonoscillatory) solution x of the differential equation (E) with

$$(P_m(x))\begin{cases}\lim\limits_{t\to\infty}(D_r^{(i)}x)(t)=(\operatorname{sgn} L)^{\infty} \quad (i=0,1,\ldots,m-1) \quad \textit{when } m>0\\ \lim\limits_{t\to\infty}(D_r^{(m)}x)(t)=\lim\limits_{t\to\infty}\dfrac{x(t)}{R_m(t)}=L\\ \lim\limits_{t\to\infty}(D_r^{(j)}x)(t)=0 \quad (j=m+1,\ldots,n-1) \quad \textit{when } m<n-1.\end{cases}$$

Proof. The substitution $u=-x$ transforms (E) into the equation

$$(D_r^{(n)}u)(t)+\hat{F}(t;u\langle g_0(t)\rangle,\\ (D_r^{(1)}u)\langle g_1(t)\rangle,\ldots,(D_r^{(l)}u)\langle g_l(t)\rangle)=\hat{b}(t),$$

where $\hat{F}(t;\mathbf{Y})=-F(t;-\mathbf{Y})$ and $\hat{b}(t)=-b(t)$. The transformed equation is subject to the assumptions of the theorem with $-c$ in place of c. Thus we can confine our discussion only to the case of positive c.

Let L be a (positive) number with $\frac{c}{2}<L<c$. By condition ($B[m]$) we choose a $T_0>t_0$ such that $c-L>d$, where

$$d=\begin{cases}\displaystyle\int_{T_0}^{\infty}|b(t)|\,dt \text{ if } m=n-1\\ \displaystyle\int_{T_0}^{\infty}\frac{1}{r_{m+1}(s_{m+1})}\cdots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}|b(s)|\,ds\,ds_{n-1}\ldots ds_{m+1} \text{ if } m<n-1.\end{cases}$$

Next, by taking into account condition ($C[m]$), we consider a $T\geqq T_0$ so that

$$g_{ik}(t)\geqq T_0 \quad \text{for every} \quad t\geqq T \quad (k=1,\ldots,N_i;\ i=0,1,\ldots,l)$$

and

$$\begin{cases}\displaystyle\int_T^{\infty}|F(t;cR_{0,n-1}\langle g_0(t)\rangle,cR_{1,n-1}\langle g_1(t)\rangle,\ldots,cR_{l,n-1}\langle g_l(t)\rangle)|dt\leqq c-L-d\\ \text{if } \quad m=n-1\\ \displaystyle\int_T^{\infty}\frac{1}{r_{m+1}(s_{m+1})}\cdots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}|F(s;cR_{0m}\langle g_0(s)\rangle,cR_{1m}\langle g_1(s)\rangle,\ldots\\ \ldots,cR_{lm}\langle g_l(s)\rangle)|ds\,ds_{n-1}\ldots ds_{m+1}\leqq c-L-d \text{ if } m<n-1.\end{cases}$$

Now, we consider the Banach space E of all real-valued functions with continuous and bounded n-th r-derivative on the interval $[T,\infty)$, endowed with the norm $|||\ \ |||$ defined by

$$|||h|||=\begin{cases}\|h\|, \text{ if } m=0\\ \|D_r^{(m)}h\|+\displaystyle\sum_{i=0}^{m-1}\frac{|(D_r^{(i)}h)(T)|}{R_{im}(T)} \quad \text{if} \quad m>0.\end{cases}$$

We observe that the restriction of the function LR_m on $[T,\infty)$ belongs in E, and so we consider the nonempty, convex and closed subset X of E, which contains all function $x \in E$ with

$$|||x - LR_m||| \leqq c - L,$$

namely

$$\begin{cases} \|x - L\| \leqq c - L & \text{if } m = 0 \\ \|D_r^{(m)}x - L\| + \sum_{i=0}^{m-1} \left| \frac{(D_r^{(i)}x)(T)}{R_{im}(T)} - L \right| \leqq c - L & \text{if } m > 0. \end{cases}$$

Next, we shall prove that for any function x in x these holds

$$\left| \frac{(D_r^{(i)}x)(t)}{R_{im}(t)} - L \right| \leqq c - L \quad \text{for all } t \geqq T \quad (i = 0, 1, \ldots, m).$$

This is obvious for $m = 0$. Let us suppose that $m > 0$ and let us consider an arbitrary function x in X. Then we observe that

$$|(D_r^{(m)}x)(t) - L| \leqq c - L \quad \text{for all } t \geqq T.$$

Hence for every $t \geqq T$ we obtain

$$\frac{(D_r^{(m-1)}x)(t)}{R_{m-1,m}(t)} - L = \frac{1}{R_{m-1,m}(t)} \left[(D_r^{(m-1)}x)(T) + \int_T^t \frac{1}{r_m(s)} (D_r^{(m)}x)(s)\, ds - LR_{m-1,m}(t) \right]$$

$$= \frac{1}{R_{m-1,m}(t)} \left\{ (D_r^{(m-1)}x)(T) + \int_T^t \frac{1}{r_m(s)} [(D_r^{(m)}x)(s) - L]\, ds + L \int_T^t \frac{ds}{r_m(s)} - LR_{m-1,m}(t) \right\}$$

$$= \frac{1}{R_{m-1,m}(t)} \left\{ [(D_r^{(m-1)}x)(T) - LR_{m-1,m}(T)] + \int_T^t \frac{1}{r_m(s)} [(D_r^{(m)}x)(s) - L]\, ds \right\}$$

$$= \frac{R_{m-1,m}(T)}{R_{m-1,m}(t)} \left[\frac{(D_r^{(m-1)}x)(T)}{R_{m-1,m}(T)} - L \right] + \frac{1}{R_{m-1,m}(t)} \int_T^t \frac{1}{r_m(s)} [(D_r^{(m)}x)(s) - L]\, ds$$

and hence

$$\left| \frac{(D_r^{(m-1)}x)(t)}{R_{m-1,m}(t)} - L \right| \leqq \frac{R_{m-1,m}(T)}{R_{m-1,m}(t)} \left| \frac{(D_r^{(m-1)}x)(T)}{R_{m-1,m}(T)} - L \right| +$$

$$+ \frac{1}{R_{m-1,m}(t)} \int_T^t \frac{1}{r_m(s)} |(D_r^{(m)}x)(s) - L|\, ds$$

$$\leqq \frac{R_{m-1,m}(T)}{R_{m-1,m}(t)} \left| \frac{(D_r^{(m-1)}x)(T)}{R_{m-1,m}(T)} - L \right| + \|D_r^{(m)}x - L\| \left[\frac{1}{R_{m-1,m}(t)} \int_T^t \frac{ds}{r_m(s)} \right]$$

$$\leqq \left| \frac{(D_r^{(m-1)}x)(T)}{R_{m-1,m}(T)} - L \right| + \|D_r^{(m)}x - L\|.$$

Thus,

$$\left|\frac{(D_r^{(m-1)}x)(t)}{R_{m-1,m}(t)}-L\right| \leqq c-L \quad \text{for all} \quad t \geqq T.$$

Next, provided that $m>1$, for $t \geqq T$ we get

$$\frac{(D_r^{(m-2)}x)(t)}{R_{m-2,m}(t)}-L=\frac{1}{R_{m-2,m}(t)}\left[(D_r^{(m-2)}x)(T)+\int_T^t \frac{1}{r_{m-1}(s)}(D_r^{(m-1)}x)(s)\,\mathrm{d}s-LR_{m-2,m}(t)\right]$$

$$=\frac{1}{R_{m-2,m}(t)}\left\{(D_r^{(m-2)}x)(T)+\int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)}\left[\frac{(D_r^{(m-1)}x)(s)}{R_{m-1,m}(s)}-L\right]\mathrm{d}s+\right.$$
$$\left.+L\int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)}\,\mathrm{d}s-LR_{m-2,m}(t)\right\}$$

$$=\frac{1}{R_{m-2,m}(t)}\left\{[(D_r^{(m-2)}x)(T)-LR_{m-2,m}(T)]+\int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)}\left[\frac{(D_r^{(m-1)}x)(s)}{R_{m-1,m}(s)}-L\right]\mathrm{d}s\right\}$$

$$=\frac{R_{m-2,m}(T)}{R_{m-2,m}(t)}\left[\frac{(D_r^{(m-2)}x)(T)}{R_{m-2,m}(T)}-L\right]+\frac{1}{R_{m-2,m}(t)}\int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)}\left[\frac{(D_r^{(m-1)}x)(s)}{R_{m-1,m}(s)}-L\right]\mathrm{d}s,$$

and consequently

$$\left|\frac{(D_r^{(m-2)}x)(t)}{R_{m-2,m}(t)}-L\right| \leqq \frac{R_{m-2,m}(T)}{R_{m-2,m}(t)}\left|\frac{(D_r^{(m-2)}x)(T)}{R_{m-2,m}(T)}-L\right|+$$
$$+\frac{1}{R_{m-2,m}(t)}\int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)}\left|\frac{(D_r^{(m-1)}x)(s)}{R_{m-1,m}(s)}-L\right|\mathrm{d}s$$

$$\leqq \frac{R_{m-2,m}(T)}{R_{m-2,m}(t)}\left|\frac{(D_r^{(m-2)}x)(T)}{R_{m-2,m}(T)}-L\right|+\left\|\frac{D_r^{(m-1)}x}{R_{m-1,m}}-L\right\|\left[\frac{1}{R_{m-2,m}(t)}\int_T^t \frac{R_{m-1,m}(s)}{r_{m-1}(s)}\,\mathrm{d}s\right]$$

$$\leqq \left|\frac{(D_r^{(m-2)}x)(T)}{R_{m-2,m}(T)}-L\right|+\left\|\frac{D_r^{(m-1)}x}{R_{m-1,m}}-L\right\|$$

$$\leqq \left|\frac{(D_r^{(m-2)}x)(T)}{R_{m-2,m}(T)}-L\right|+\left|\frac{(D_r^{(m-1)}x)(T)}{R_{m-1,m}(T)}-L\right|+\|D_r^{(m)}x-L\|.$$

Hence

$$\left|\frac{(D_r^{(m-2)}x)(t)}{R_{m-2,m}(t)}-L\right| \leqq c-L \quad \text{for all} \quad t \geqq T.$$

If $m>2$, then, following the same procedure, we conclude that for every $t\geqq T$

$$\left|\frac{x(t)}{R_{0m}(t)}-L\right|\leqq\sum_{i=0}^{m-1}\left|\frac{(D_r^{(i)}x)(T)}{R_{im}(T)}-L\right|+\|D_r^{(m)}x-L\|\leqq c-L.$$

We have thus proved our assertion.

Now, let x be an arbitrary function in X. Then for $i=0,1,\ldots,l$ we have

$$0<(D_r^{(i)}x)(t)\leqq cR_{im}(t)\quad \text{for every}\quad t\geqq T$$

and consequently

$$0<(D_r^{(i)}x)^*(t)\leqq cR_{im}(t)\quad \text{for all}\quad t\geqq T_0,$$

where

$$(D_r^{(i)}x)^*(t)=\begin{cases}(D_r^{(i)}x)(t) & \text{if}\ \ t\geqq T\\ (D_r^{(i)}x)(T)\dfrac{R_{im}(t)}{R_{im}(T)} & \text{if}\ \ T_0\leqq t\leqq T.\end{cases}$$

Thus for all $t\geqq T$

$$0<(D_r^{(i)}x)^*[g_{ik}(t)]\leqq cR_{im}[g_{ik}(t)]\quad (k=1,\ldots,N_i\,;\ i=0,1,\ldots,l)$$

and hence, since for any $t\geqq t_0$ the function $|F(t;\,\cdot)|$ is increasing on R_+^N, we have

$$\begin{aligned}&|F(t;\,x^*\langle g_0(t)\rangle,(D_r^{(1)}x)^*\langle g_1(t)\rangle,\ldots,(D_r^{(l)}x)^*\langle g_l(t)\rangle)|\leqq\\ &\leqq|F(t;\,cR_{0m}\langle g_0(t)\rangle,cR_{1m}\langle g_1(t)\rangle,\ldots,cR_{lm}\langle g_l(t)\rangle)|\ \text{ for every }\ t\geqq T.\end{aligned}$$

Thus, because of $(C[m])$ and $(B[m])$ for any $t\geqq T$ there holds

$$\begin{cases}\left|\displaystyle\int_t^\infty [F(s\,;x^*\langle g_0(s)\rangle,(D_r^{(1)}x)^*\langle g_1(s)\rangle,\ldots,(D_r^{(l)}x)^*\langle g_l(s)\rangle)-b(s)]\,\mathrm{d}s\right|<\infty\ \text{ if }\ m=n-1\\ \left|\displaystyle\int_t^\infty\frac{1}{r_{m+1}(s_{m+1})}\cdots\int_{s_{n-2}}^\infty\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty[F(s\,;x^*\langle g_0(s)\rangle,(D_r^{(1)}x)^*\langle g_1(s)\rangle,\ldots\right.\\ \left.\ldots,(D_r^{(l)}x)^*\langle g_l(s)\rangle)-b(s)]\,\mathrm{d}s\,\mathrm{d}s_{n-1}\ldots\mathrm{d}s_{m+1}\right|<\infty\ \text{ if }\ m<n-1.\end{cases}$$

Next, we define the mapping S as follows:

$$(Sx)(t)=LR_m(t)+\int_T^{t=s_0}\frac{1}{r_1(s_1)}\int_T^{s_1}\frac{1}{r_2(s_2)}\cdots\int_T^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty$$

$$\cdot[F(s\,;x^*\langle g_0(s)\rangle,\ldots,(D_r^{(l)}x)^*\langle g_l(s)\rangle)-b(s)]\,\mathrm{d}s\,\mathrm{d}s_{n-1}\ldots\mathrm{d}s_2\,\mathrm{d}s_1$$

if $m=n-1$,

$$(Sx)(t)=LR_m(t)+(-1)^{n-1-m}\int_T^{t-s_0}\frac{1}{r_1(s_1)}\int_T^{s_1}\frac{1}{r_2(s_2)}\cdots\int_T^{s_{m-1}}\frac{1}{r_m(s_m)}\int_{s_m}^\infty\cdot$$

$$\cdot\frac{1}{r_{m+1}(s_{m+1})}\cdots\int_{s_{n-2}}^\infty\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty[F(s\,;x^*\langle g_0(s)\rangle,\ldots$$

$$\ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle) - b(s)]\, ds\, ds_{n-1} \ldots ds_{m+1}\, ds_m \ldots ds_2\, ds_1$$

if $0 < m < n-1$,

$$(Sx)(t) = LR_m(t) + (-1)^{n-1}\int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots$$

$$\cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty [F(s\,;x^*\langle g_0(s)\rangle) - b(s)]\, ds\, ds_{n-1}\ldots ds_2\, ds_1$$

if $m = 0$.

In order to apply the Schauder theorem it must be verified that S is a mapping of X into itself, SX is relatively compact and S is continuous.

α) $SX \subseteq X$.

Indeed, for any function $x \in X$ and every $t \geqq T$ we obtain

$$|(D_r^{(m)}Sx)(t) - L| =$$

$$= \begin{cases} \left|\int_t^\infty [F(s\,;x^*\langle g_0(s)\rangle, \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle) - b(s)]\, ds\right| & \text{if } m = n-1 \\ \Big|\int_t^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty [F(s\,;x^*\langle g_0(s)\rangle, \ldots & \\ \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle) - b(s)]\, ds\, ds_{n-1}\ldots ds_{m+1}\Big| & \text{if } m < n-1 \end{cases}$$

$$\leqq \begin{cases} \int_t^\infty [|F(s\,;x^*\langle g_0(s)\rangle, \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds & \text{if } m = n-1 \\ \int_t^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty [|F(s\,;x^*\langle g_0(s)\rangle, \ldots & \\ \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds\, ds_{n-1}\ldots ds_{m+1} & \text{if } m < n-1 \end{cases}$$

$$\leqq \begin{cases} \int_t^\infty [|F(s\,;cR_{0,n-1}\bigcup g_0(s)\rangle, \ldots, cR_{l,n-1}\langle g_l(s)\rangle)| + |b(s)|]\, ds & \text{if } m = n-1 \\ \int_t^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty [|F(s\,;cR_{0m}\langle g_0(s)\rangle, \ldots & \\ \ldots, cR_{lm}\langle g_l(s)\rangle)| + |b(s)|]\, ds\, ds_{n-1}\ldots ds_{m+1} & \text{if } m < n-1 \end{cases}$$

$$\leqq (c - L - d) + d$$
$$= c - L.$$

Also, if $m > 0$, we have

$$\frac{(D_r^{(i)}Sx)(T)}{R_{im}(T)} - L = 0 \quad (i = 0, 1, \ldots, m-1).$$

And for any $x \in X$, $|||Sx - LR_m||| \leqq c - L$ and consequently $Sx \in X$.

β) *SX* is relatively compact.

Obviously, it suffices to prove that the set

$$\mathscr{F} = \{D_r^{(m)}Sx : x \in X\}$$

is a relatively compact subset of the space $B([T, \infty))$. Furthermore, by the compactness criterion, $\mathscr{F}$ is relatively compact if it is uniformly bounded, equiconvergent at ∞ and equicontinuous. Now, by the definition of X and the fact that $SX \subseteq X$, there holds

$$\|D_r^{(m)}Sx\| \leqq c \quad \text{for every} \quad x \in X,$$

which means that $\mathscr{F}$ is a uniformly bounded subset of $B([T, \infty))$. Also, for any function $x \in X$ and every $t \geqq T$ we obtain

$$|(D_r^{(m)}Sx)(t) - L| \leqq$$

$$\leqq \begin{cases} \displaystyle\int_t^\infty [|F(s; x^*\langle g_0(s)\rangle, \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds & \text{if } m = n-1 \\ \displaystyle\int_t^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty [|F(s; x^*\langle g_0(s)\rangle, \ldots & \\ \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds\, ds_{n-1} \ldots ds_{m+1} & \text{if } m < n-1 \end{cases}$$

and hence

$$\lim_{t\to\infty} (D_r^{(m)}Sx)(t) = L \quad \text{for every} \quad x \in X,$$

i.e. $\mathscr{F}$ is equiconvergent at ∞. Finally, in order to prove that $\mathscr{F}$ is equicontinuous, for any function $x \in X$ and every t_1, t_2 with $T \leqq t_1 \leqq t_2$ we get

$$|(D_r^{(m)}Sx)(t_2) - (D_r^{(m)}Sx)(t_1)| =$$

$$= \begin{cases} \left|\displaystyle\int_{t_1}^{t_2} [F(s; x^*\langle g_0(s)\rangle, \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle) - b(s)]\, ds\right| & \text{if } m = n-1 \\ \left|\displaystyle\int_{t_1}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty [F(s; x^*\langle g_0(s)\rangle, \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle) - b(s)]\, ds\, ds_{n-1}\right| & \\ \text{if } m = n-2 & \\ \left|\displaystyle\int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \int_{s_{m+1}}^\infty \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty [F(s; x^*\langle g_0(s)\rangle, \ldots\right. & \\ \left.\ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle) - b(s)]\, ds\, ds_{n-1} \ldots ds_{m+2}\, ds_{m+1}\right| & \text{if } n > 2 \text{ and } m < n-2 \end{cases}$$

$$\leqq \begin{cases} \int_{t_1}^{t_2} [|F(s; x^*\langle g_0(s)\rangle, \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds & \text{if } m = n-1 \\ \int_{t_1}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} [|F(s; x^*\langle g_0(s)\rangle, \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds\, ds_{n-1} \\ \text{if } m = n-2 \\ \int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \int_{s_{m+1}}^{\infty} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} [|F(s; x^*\langle g_0(s)\rangle, \ldots \\ \ldots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds\, ds_{n-1} \ldots ds_{m+2}\, ds_{m+1} & \text{if } n > 2 \text{ and } m < n-2 \end{cases}$$

$$\leqq \begin{cases} \int_{t_1}^{t_2} [|F(t; cR_{0,n-1}\langle g_0(s)\rangle, \ldots, cR_{l,n-1}\langle g_l(s)\rangle)| + |b(s)|]\, ds & \text{if } m = n-1 \\ \int_{t_1}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} [|F(t; cR_{0,n-2}\langle g_0(s)\rangle, \ldots, cR_{l,n-2}\langle g_l(s)\rangle)| + |b(s)|]\, ds\, ds_{n-1} \\ \text{if } m = n-2 \\ \int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \int_{s_{m+1}}^{\infty} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} [|F(s; cR_{0m}\langle g_0(s)\rangle, \ldots \\ \ldots, cR_{lm}\langle g_l(s)\rangle)| + |b(s)|]\, ds\, ds_{n-1} \ldots ds_{m+2}\, ds_{m+1} & \text{if } n > 2 \text{ and } m < n-2. \end{cases}$$

γ) The mapping S is continuous.

Let $x \in X$ and $(x_\nu)_{\nu \in N}$ be an arbitrary sequence in X with

$$|||\ ||| - \lim_{\nu \to \infty} x_\nu = x.$$

Then it is easy to verify that for every $t \geqq T_0$

$$\lim_{\nu \to \infty} (D_r^{(i)} x_\nu)^*(t) = (D_r^{(i)} x)^*(t) \quad (i = 0, 1, \ldots, l).$$

On the other hand, for any $\nu \in N$ and every $t \geqq T$ we have

$$|F(t; x_\nu^*\langle g_0(t), \ldots, (D_r^{(l)} x_\nu)^*\langle g_l(t)\rangle)| \leqq$$
$$\leqq |F(t; cR_{0m}\langle g_0(t)\rangle, \ldots, cR_{lm}\langle g_l(t)\rangle)|.$$

Thus, because of condition $(C[m])$, we can apply the Lebesgue dominated convergence theorem to obtain the pointwise convergence

$$\lim_{\nu \to \infty} (Sx_\nu)(t) = (Sx)(t), \quad t \geqq T.$$

In order to prove that

$$|||\ ||| - \lim_{\nu \to \infty} Sx_\nu = Sx,$$

we consider any subsequence $(\xi_{\mu_\nu})_{\nu \in N}$ of $(Sx_\nu)_{\nu \in N}$. Then, because of the relative compactness of SX, there exist a subsequence $(v^{\mu_{\varrho_\nu}})_{\nu \in N}$ of $(\xi_{\mu_\nu})_{\nu \in N}$ and a $w \in E$ so that

$$||| \quad ||| - \lim_{\nu\to\infty} v^{\mu_{\varrho_\nu}} = w.$$

Since $||| \quad |||$-convergence implies the pointwise one to the same limit function, we always have $w = Sx$, which proves the continuity of S.

Finally, the Schauder theorem ensures the existence of at least one fixed point x of the mapping S. Then $x = Sx$ and consequently

$$(D_r^{(n)}x)(t) = -F(t; x^*\langle g_0(t)\rangle, \dots, (D_r^{(l)}x)^*\langle g_l(t)\rangle) + b(t), \quad t \geqq T,$$

namely the fixed point x of the mapping S is a solution on $[T, \infty)$ of the differential equation (E). Moreover, x satisfies $P_m(x)$). Indeed, for every $t \geqq T$ we obtain

$$|(D_r^{(m)}x)(t) - L| \leqq$$

$$\leqq \begin{cases} \displaystyle\int_t^\infty [|F(t; x^*\langle g_0(s)\rangle, \dots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds \text{ if } m = n-1 \\ \displaystyle\int_t^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty [|F(s; x^*\langle g_0(s)\rangle, \dots \\ \dots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds\, ds_{n-1} \dots ds_{m+1} \text{ if } m < n-1 \end{cases}$$

and consequently

$$\lim_{t\to\infty} (D_r^{(m)}x)(t) = L.$$

Also, when $m < n-1$, for $j = m+1, \dots, n-1$ and every $t \geqq T$ we have

$$|(D_r^{(j)}x)(t)| \leqq$$

$$\leqq \begin{cases} \displaystyle\int_t^\infty [|F(t; x^*\langle g_0(s)\rangle, \dots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds, \text{ if } j = n-1 \\ \displaystyle\int_t^\infty \frac{1}{r_{j+1}(s_{j+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty [|f(s; x^*\langle g_0(s)\rangle, \dots \\ \dots, (D_r^{(l)}x)^*\langle g_l(s)\rangle)| + |b(s)|]\, ds\, ds_{n-1} \dots ds_{j+1}, \text{ if } j < n-1. \end{cases}$$

Hence, if $m < n-1$, there holds

$$\lim_{t\to\infty} (D_r^{(j)}x)(t) = 0 \quad (j = m+1, \dots, n-1).$$

In addition, from Lemma 0.1 it follows that

$$\lim_{t\to\infty} \frac{x(t)}{R_m(t)} = L.$$

Moreover, if $m > 0$, then it is easy to verify that

$$\lim_{t\to\infty} (D_r^{(i)}x)(t) = \infty \quad (i = 0, 1, \dots, m-1).$$

Corollary 1.1. *Let m, $l \leqq m \leqq n-1$, be an integer such that ($\Sigma[m]$) holds when*

$m>0$. Under condition ($C[m]$), for every real number L with $Lc>0$ and $\frac{|c|}{2}<|L|<|c|$ there exists a (nonoscillatory) solution x of the differential equation (E_0) *satisfying* ($P_m(x)$).

Next, let us consider the case of more general differential equations with deviating arguments of the form

$$(\mathrm{E}^*) \qquad (D_r^{(n)}x)(t)+\sum_{\varrho=1}^{\mu} F_\varrho(t;x\langle\tau_{\varrho 0}(t)\rangle, (D_r^{(1)}x)\langle\tau_{\varrho 1}(t)\rangle,\dots,(D_r^{(l_\varrho)}x)\langle\tau_{\varrho l_\varrho}(t)\rangle)=b(t),$$

where for any $\varrho=1,\dots,\mu$ we have: $0\leqq l_\varrho\leqq n-1$;

$$\begin{cases}(D_r^{(i)}x)\langle\tau_{\varrho i}(t)\rangle=((D_r^{(i)}x)[\tau_{\varrho i1}(t)],\dots,(D_r^{(i)}x)[\tau_{\varrho iN_{\varrho i}}(t)]) \quad (i=0,1,\dots,l_\varrho);\\ \tau_{\varrho i}=(\tau_{\varrho i1},\dots,\tau_{\varrho iN_{\varrho i}})\end{cases}$$

$\tau_{\varrho ik}$ ($k=1,\dots,N_{\varrho i}$; $i=0,1,\dots,l_\varrho$) are continuous real-valued functions on the interval $[t_0,\infty)$ such that

$$\lim_{t\to\infty}\tau_{\varrho ik}(t)=\infty \quad (k=1,\dots,N_{\varrho i};\ i=0,1,\dots,l_\varrho)$$

and F_ϱ is a continuous real-valued function defined at least on the set $[t_0,\infty)\times(R_+^{\nu_\varrho}\cup R_-^{\nu_\varrho})$, where $\nu_\varrho=N_{\varrho 0}+N_{\varrho 1}+\dots+N_{\varrho l_\varrho}$. It is supposed that: *For any $\varrho=1,\dots,\mu$ and every $t\geqq t_0$, the function $|F_\varrho(t;\cdot)|$ is increasing on $R_+^{\nu_\varrho}$ and decreasing on $R_-^{\nu_\varrho}$.* Following the same technique as in the proof of Theorem 1.1, we can obtain the following more general theorem.

Theorem 1.1′. *Let m be an integer with $\max_{\varrho=1,\dots,\mu} l_\varrho\leqq m\leqq n-1$ and such that ($\Sigma[m]$) holds when $m>0$ and the condition ($B[m]$) is satisfied. Moreover, suppose that:*

($C^*[m]$) For some nonzero constant c and every $\varrho=1,\dots,\mu$

$$\begin{cases}\displaystyle\int^\infty |F_\varrho(t;cR_{0,n-1}\langle\tau_{\varrho 0}(t)\rangle,cR_{1,n-1}\langle\tau_{\varrho 1}(t)\rangle,\dots,cR_{l_\varrho,n-1}\langle\tau_{\varrho l_\varrho}(t)\rangle)|\,\mathrm{d}t<\infty \ \text{ if } m=n-1\\ \displaystyle\int^\infty\frac{1}{r_{m+1}(s_{m+1})}\cdots\int_{s_{n-2}}^\infty\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty |F_\varrho(s;cR_{0m}\langle\tau_{\varrho 0}(s)\rangle,cR_{1m}\langle\tau_{\varrho 1}(s)\rangle,\dots\\ \quad\dots cR_{l_\varrho m}\langle\tau_{\varrho l_\varrho}(s)\rangle)|\,\mathrm{d}s\,\mathrm{d}s_{n-1}\dots\mathrm{d}s_{m+1}<\infty \ \text{ if } m<n-1.\end{cases}$$

Then for every real number L with $Lc>0$ and $\frac{|c|}{2}<|L|<|c|$ there exists a (nonoscillatory) solution x of (E^*) satisfying ($P_m(x)$).

Now, let us consider the linear differential equation with deviating arguments

$$(D_l) \qquad (D_r^{(n)}x)(t) + a_l(t)(D_r^{(l)}x)[\sigma_l(t)] + \ldots + \\ + a_1(t)(D_r^{(1)}x)[\sigma_1(t)] + a_0(t)x[\sigma_0(t)] = b(t),$$

where a_i $(i = 0, 1, \ldots, l)$ are continuous real-valued functions on the interval $[t_0, \infty)$ without any restriction on their sign and σ_i $(i = 0, 1, \ldots, l)$ are also continuous real-valued functions on $[t_0, \infty)$ with

$$\lim_{t\to\infty} \sigma_i(t) = \infty \quad (i = 0, 1, \ldots, l).$$

From Theorem 1.1′ we obtain Corollary 1.2 below concerning the linear equation (D_l). In particular, for $l = n-1$ we have Corollary 1.3 below.

Corollary 1.2. *Let m, $l \leqq m \leqq n-1$, be an integer such that $(\Sigma[m])$ holds when $m > 0$ and the condition $(B[m])$ is satisfied. Moreover, suppose that:*

$(Y[m])$ For every $i = 0, 1, \ldots, l$

$$\begin{cases} \displaystyle\int^{\infty} |a_i(t)| R_{i,n-1}[\sigma_i(t)] \, dt < \infty \quad \textit{if} \quad m = n-1 \\ \displaystyle\int^{\infty} \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |a_i(s)| R_{im}[\sigma_i(s)] \, ds \, ds_{n-1} \ldots ds_{m+1} < \infty \\ \textit{if} \quad m < n-1. \end{cases}$$

Then for every real number L with $L \neq 0$ there exists a (nonoscillatory) solution x of the linear equation (D_l) satisfying $(P_m(x))$.

Corollary 1.3. *Let (Σ) be satisfied and suppose that the conditions $(B[n-1])$ and $(Y[n-1])$, i.e. the conditions*

$$(B[n-1]) \qquad \int^{\infty} |b(t)| \, dt < \infty$$

and

$$(Y[n-1]) \qquad \int^{\infty} |a_i(t)| R_{i,n-1}[\sigma_i(t)] \, dt < \infty \quad (i = 0, 1, \ldots, n-1),$$

hold. Then for every $L \neq 0$ there exists a (nonoscillatory) solution x of the linear equation (D_{n-1}), which satisfies $(P_{n-1}(x))$, i.e.

$$(P_{n-1}(x)) \begin{cases} \displaystyle\lim_{t\to\infty} (D_r^{(i)}x)(t) = (\operatorname{sgn} L)\infty \quad (i = 0, 1, \ldots, n-2) \\ \displaystyle\lim_{t\to\infty} (D_r^{(n-1)}x)(t) = \lim_{t\to\infty} \frac{x(t)}{R_{n-1}(t)} = L. \end{cases}$$

2. Necessary conditions

Our basic purpose in this section is to prove that, for any integer m with $l \leqq m \leqq n-1$, $(C[m])$ is also a necessary condition in order that the differential equation (E_0) have at least one (nonoscillatory) solution x satisfying $(P_m(x))$, provided that (Σ) holds and the function F has the sign property (I) or (II).

Theorem 2.1. *Suppose that (Σ) holds and the function F has the sign property (I) or (II). Moreover, let m be an integer with $l \leqq m < n-1$.*

Then the condition $(C[m])$ is a necessary condition in order that the differential equation (E_0) have at least one (nonoscillatory) solution x such that $\lim_{t\to\infty} (D_r^{(m)}x)(t)$ exists in $R-\{0\}$.

Proof. Let x be a solution on an interval $[T_0, \infty)$, $T_0 > t_0$, of the differential equation (E_0) with $\lim_{t\to\infty} (D_r^{(m)}x)(t) = L$ for some $L \in R-\{0\}$. Since the substitution $u=-x$ transforms (E_0) into an equation of the same form satisfying the assumptions of the theorem, we can restrict ourselves to the case where L is positive.

If $m>0$, then, in view of condition (Σ), we can easily derive that

$$\lim_{t\to\infty} (D_r^{(i)}x)(t) = \infty \quad (i=0, 1, \dots, m-1).$$

Thus the r-derivatives $D_r^{(i)}x$ $(i=0, 1, \dots, l)$ are positive on an interval $[T_0^*, \infty)$, $T_0^* \geqq T_0$. Hence, if we choose a $T \geqq T_0^*$ such that

$$g_{ik}(t) \geqq T_0^* \quad \text{for every} \quad t \geqq T \quad (k=1, \dots, N_i;\ i=0, 1, \dots, l),$$

then from equation (E_0) it follows that for all $t \geqq T$

$$\begin{cases} (D_r^{(n)}x)(t) \leqq 0 & \text{if (I) holds} \\ (D_r^{(n)}x)(t) \geqq 0 & \text{if (II) holds.} \end{cases}$$

Namely, $D_r^{(n)}x$ is of constant sign on $[T, \infty)$. Thus the functions $D_r^{(j)}x$ $(j=m, \dots, n-1)$ are also eventually of constant sign. Without loss of generality, we suppose that $D_r^{(j)}x$ $(j=m, \dots, n-1)$ are of constant sign on the whole interval $[T, \infty)$. Next, provided that $m<n-1$ we can use the assumption (Σ) to obtain

$$\lim_{t\to\infty} (D_r^{(j)}x)(t) = 0 \quad (j=m+1, \dots, n-1).$$

Furthermore, we get

$$|(D_r^{(m)}x)(t) - (D_r^{(m)}x)(T)| = \int_T^t \frac{1}{r_{m+1}(s)} |D_r^{(m+1)}x)(s)|\, ds, \quad t \geqq T,$$

where $r_n = 1$, and hence

$$(1) \qquad \int_T^\infty \frac{1}{r_{m+1}(t)} |(D_r^{(m+1)}x)(t)| \, \mathrm{d}t < \infty.$$

Also, if $m < n-1$, then

$$(2) \qquad |(D_r^{(j)}x)(t)| \geqq \int_t^\infty \frac{1}{r_{j+1}(s)} |(D_r^{(j+1)}x)(s)| \, \mathrm{d}s$$

$$\text{for } t \geqq T \quad (j = m+1, \ldots, n-1).$$

Indeed, for every $w \geqq t$

$$|(D_r^{(j)}x)(t)| + |(D_r^{(j)}x)(w)| \geqq \int_t^w \frac{1}{r_{j+1}(s)} |(D_r^{(j+1)}x)(s)| \, \mathrm{d}s,$$

which gives (2), since $\lim_{w\to\infty} (D_r^{(j)}x)(w) = 0$. Combining (1) and (2), we obtain

$$(3) \quad \begin{cases} \int_T^\infty |(D_r^{(n)}x)(t)| \, \mathrm{d}t < \infty \quad \text{if} \quad m = n-1 \\ \int_T^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |(D_r^{(n)}x)(s)| \, \mathrm{d}s \, \mathrm{d}s_{n-1} \ldots \mathrm{d}s_{m+1} < \infty \\ \text{if} \quad m < n-1. \end{cases}$$

Now, by the L'Hospital rule, we can derive for $i = 0, 1, \ldots, l$

$$\lim_{t\to\infty} \frac{(D_r^{(i)}x)(t)}{R_{im}(t)} = \lim_{t\to\infty} (D_r^{(m)}x)(t) = L > 0$$

and hence there exists a positive constant c so that

$$(D_r^{(i)}x)(t) \geqq cR_{im}(t) \quad \text{for all} \quad t \geqq T_0^* \quad (i = 0, 1, \ldots, l).$$

Thus

$$(D_r^{(i)}x)[g_{ik}(t)] \geqq cR_{im}[g_{ik}(t)] \quad \text{for every} \quad t \geqq T$$
$$(k = 1, \ldots, N_i; \; i = 0, 1, \ldots, l)$$

and consequently, for all $t \geqq T$, there holds

$$\begin{aligned} |(D_r^{(n)}x)(t)| &= |F(t; x\langle g_0(t)\rangle, (D_r^{(1)}x)\langle g_1(t)\rangle, \ldots, (D_r^{(l)}x)\langle g_l(t)\rangle)| \\ &\geqq |F(t; cR_{0m}\langle g_0(t)\rangle, cR_{1m}\langle g_1(t)\rangle, \ldots, cR_{lm}\langle g_l(t)\rangle)|. \end{aligned}$$

And so, because of (3), we have

$$\begin{cases} \int_T^\infty |F(t; cR_{0,n-1}\langle g_0(t)\rangle, \ldots, cR_{l,n-1}\langle g_l(t)\rangle)| \, \mathrm{d}t < \infty \quad \text{if} \quad m = n-1 \\ \int_T^\infty \frac{1}{r_{m+1}(s_{m+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |F(t; cR_{0m}\langle g_0(s)\rangle, \ldots \\ \ldots, cR_{lm}\langle g_l(s)\rangle)| \, \mathrm{d}s \, \mathrm{d}s_{n-1} \ldots \mathrm{d}s_{m+1} < \infty \quad \text{if} \quad m < n-1, \end{cases}$$

which means that $(C[m])$ is satisfied.

Now we turn our attention to the case of the linear differential equation with deviating arguments

$$(D_l)_0 \qquad \begin{aligned}&(D_r^{(n)}x)(t)+a_l(t)(D_r^{(l)}x)[\sigma_l(t)]+\ldots+\\&+a_1(t)(D_r^{(1)}x)[\sigma_1(t)]+a_0(t)x[\sigma_0(t)]=0.\end{aligned}$$

For this equation we have the following corrolaries of Theorem 2.1.

Corollary 2.1. *Let (Σ) be satisfied and suppose that:*

$$(\mathrm{A}_1)_l \qquad a_i \geqq 0 \quad \text{on} \quad [t_0, \infty) \quad (i=0, 1, \ldots, l)$$

or

$$(\mathrm{A}_2)_l \qquad a_i \geqq 0 \quad \text{on} \quad [t_0, \infty) \quad (i=0, 1, \ldots, l).$$

Moreover, let m be an integer with $l \leqq m \leqq n-1$.

Then $(Y[m])$ is a necessary condition in order that the linear equation $(D_l)_0$ have at least one (nonoscillatory) solution x so that $\lim_{t\to\infty}(D_r^{(m)}x)(t)$ exists in $R-\{0\}$.

Corollary 2.2. *Let (Σ) be satisfied and suppose that $(\mathrm{A}_1)_{n-1}$ or $(\mathrm{A}_2)_{n-1}$ holds, i.e.*

$$(\mathrm{A}_1)_{n-1} \qquad a_i \geqq 0 \quad \textit{on} \quad [t_0, \infty) \quad (i=0, 1, \ldots, n-1)$$

or

$$(\mathrm{A}_2)_{n-1} \qquad a_i \leqq 0 \quad \textit{on} \quad [t_0, \infty) \quad (i=0, 1, \ldots, n-1).$$

Then $(Y[n-1])$ is a necessary condition in order that the linear equation $(D_{n-1})_0$ have at least one (nonoscillatory) solution x so that $\lim_{t\to\infty}(D_r^{(n-1)}x)(t)$ exists in $R-\{0\}$.

3. A special case

Here we shall confine our discussion to the special case where

$$r_1=\ldots=r_{n-1}=1.$$

In this case the differential equations (E), (E_0), (D_l) and $(D_l)_0$ become

$$(\tilde{\mathrm{E}}) \qquad x^{(n)}(t)+F(t;\, x\langle g_0(t)\rangle, x'\langle g_1(t)\rangle, \ldots, x^{(l)}\langle g_l(t)\rangle)=b(t),$$

$$(\tilde{\mathrm{E}}_0) \qquad x^{(n)}(t)+F(t;\, x\langle g_0(t)\rangle, x'\langle g_1(t)\rangle, \ldots, x^{(l)}\langle g_l(t)\rangle)=0,$$

$$(\tilde{D}_l) \qquad x^{(n)}(t)+a_l(t)x^{(l)}[\sigma_l(t)]+\ldots+a_1(t)x'[\sigma_1(t)]+a_0(t)x[\sigma_0(t)]=b(t),$$

and

$$(\tilde{D}_l)_0 \qquad x^{(n)}(t)+a_l(t)x^{(l)}[\sigma_l(t)]+\ldots+a_1(t)x'[\sigma_1(t)]+a_0(t)x[\sigma_0(t)]=0,$$

respectively. Moreover, (Σ) is always satisfied and for every integer m with $0<m\leqq n-1$ the assumption $(\Sigma[m])$ holds by itself. Also, for any integers i and λ with $0\leqq i\leqq \lambda\leqq n-1$ we have

$$R_{i\lambda}(t)=\frac{1}{(\lambda-i)!}(t-t_0)^{\lambda-i}, \quad t\geqq t_0.$$

In addition, for a nonnegative continuous function p on an interval of the form $[T,\infty)$, $T\geqq t_0$, and for any integer m with $0\leqq m<n-1$ there holds

$$\int^{\infty}\frac{1}{r_{m+1}(s_{m+1})}\ldots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}p(s)\,ds\,ds_{n-1}\ldots ds_{m+1}<\infty\Leftrightarrow$$

$$\Leftrightarrow\int^{\infty}t^{n-1-m}p(t)\,dt<\infty.$$

Hence, applying Theorems 1.1 and 2.1 and Corollaries 1.2—1.3 and 2.1—2.2 to the special case considered, we derive the following results.

Corollary 3.1. *Let m be an integer with $l\leqq m\leqq n-1$ and suppose that:* $(\tilde{C}[m])$ *for some nonzero constant c*

$$\int^{\infty}t^{n-1-m}\left|F\left(t;\frac{c}{m!}\langle g_0(t)\rangle^m,\frac{c}{(m-1)!}\langle g_1(t)\rangle^{m-1},\ldots,\frac{c}{(m-l)!}\langle g_l(t)\rangle^{m-l}\right)\right|dt<\infty,$$

where

$$\langle g_i(t)\rangle^{m-i}=([g_{i1}(t)]^{m-i},\ldots,[g_{iN_i}(t)]^{m-i})\quad(i=0,1,\ldots,l).$$

$$(\tilde{B}[m]) \qquad \int^{\infty}t^{n-1-m}|b(t)|\,dt<\infty.$$

Then for every real number L with $Lc>0$ and $\frac{|c|}{2}<|L|<|c|$ there exists a (nonoscillatory) solution x of the differential equation $(\tilde{E})$ with

$$(\tilde{P}_m(x))\begin{cases}\lim\limits_{t\to\infty}x^{(i)}(t)=(\operatorname{sgn}L)\infty\quad(i=0,1,\ldots,m-1)\quad\text{when } m>0\\ \lim\limits_{t\to\infty}x^{(m)}(t)=m!\lim\limits_{t\to\infty}\dfrac{x(t)}{t^m}\times L\\ \lim\limits_{t\to\infty}x^{(j)}(t)=0\quad(j=m+1,\ldots,n-1)\quad\text{when } m<n-1.\end{cases}$$

Corollary 3.2. *Suppose that the function F has the sign property* (I) *or* (II) *and let m be an integer with $l\leqq m\leqq n-1$.*

Then $(\tilde{C}[m])$ is a necessary condition in order that the differential equation $(\tilde{E}_0)$ has at least one (nonoscillatory) solution x so that $\lim_{t\to\infty} x^{(m)}(t)$ exists in $R-\{0\}$.

Corollary 3.3. *Let m be an integer with $l \leqq m \leqq n-1$ and suppose that $(\tilde{B}[m])$ holds. Then we have:*

α) *If*

$$(\tilde{Y}[m]) \qquad \int^{\infty} t^{n-1-m}[\sigma_i(t)]^{m-i}|a_i(t)|\,\mathrm{d}t<\infty \quad (i=0, 1, \ldots, l),$$

then for every number L with $L \neq 0$ there exists a (nonoscillatory) solution x of the linear equation $(\tilde{D}_l)$ satisfying $(\tilde{P}_m(x))$.

β) *If $a_i \geqq 0$ on $[t_0, \infty)$ $(i=0, 1, \ldots, l)$ or $a_i \leqq 0$ on $[t_0, \infty)$ $(i=0, 1, \ldots, l)$, then $(\tilde{Y}[m])$ is a necessary condition in order that the linear equation $(\tilde{D}_l)_0$ has at least one (nonoscillatory) solution x so that $\lim_{t\to\infty} x^{(m)}(t)$ exists in $R-\{0\}$.*

Corollary 3.4. *Suppose that the condition $(\tilde{B}[n-1])$, i.e.*

$$(\tilde{B}[n-1]) \qquad \int^{\infty} |b(t)|\,\mathrm{d}t<\infty,$$

holds. Then we have:

α) *If $(\tilde{Y}[n-1])$, i.e. the condition*

$$(\tilde{Y}[n-1]) \qquad \int^{\infty} [\sigma_i(t)^{n-1-i}|a_i(t)\,\mathrm{d}t<\infty \quad (i=0, 1, \ldots, n-1),$$

is satisfied, then for every $L \neq 0$ there exists a (nonoscillatory) solution x of the linear equation $(\tilde{D}_{n-1})$ which satisfies $(\tilde{P}_{n-1}(x))$, i.e.

$$(\tilde{P}_{n-1}(x)) \begin{cases} \lim_{t\to\infty} x^{(i)}(t)=(\operatorname{sgn} L)\infty \quad (i=0, 1, \ldots, n-2) \\ \lim_{t\to\infty} x^{(n-1)}(t)=(n-1)!\lim_{t\to\infty} \dfrac{x(t)}{t^{n-1}}=L. \end{cases}$$

β) *If $a_i \geqq 0$ on $[t_0, \infty)$ $(i=0, 1, \ldots, n-1)$ or $a_i \leqq 0$ on $[t_0, \infty)$ $(i=0, 1, \ldots, n-1)$, then $(\tilde{Y}[n-1])$ is a necessary condition in order that the linear equation $(\tilde{D}_{n-1})_0$, i.e. the equation*

$$(\tilde{D}_{n-1})_0 \qquad \begin{aligned} &x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}[\sigma_{n-1}(t)]+\ldots+ \\ &+a_1(t)x'[\sigma_1(t)]+a_0(t)x[\sigma_0(t)]=0, \end{aligned}$$

have at least one (nonoscillatory) solution x so that $\lim_{t\to\infty} {}^{(n-1)}(t)$ exists in $R-\{0\}$.

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АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ ОДНОГО КЛАССА НЕКОЛЕБЛЮЩИХСЯ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ОТКЛОНЯЮЩИМИСЯ АРГУМЕНТАМИ

Ch. G. Philos

Резюме

В статье изучается асимптотическое поведение одного класса неколеблющихся решений дифференциальных уравнений n-го порядка ($n>1$) с отклоняющимися аргументами. Эти уравнения содержат r-производные $D_r^{(i)}x$ ($i=0, 1, \ldots, n$) неизвестной функции x, определяемые следующим образом:

$$D_r^{(0)}x = x, \quad D_r^{(i)}x = r_i(D_r^{(i-1)}x)' \quad (i = 1, 2, \ldots, n-1) \quad \text{и} \quad D_r^{(n)}x = (D_r^{(n-1)}x)'.$$

где r_i ($i = 1, 2, \ldots, n-1$) — положительные непрерывные функции на интервале $[t_0, \infty)$.

Даются достаточные и необходимые и достаточные условия для существования хотя бы одного (неколеблющегося) решения x такого, что

$$\lim_{t\to\infty} (D_r^{(m)}x)(t)$$

существует в $R-\{0\}$, где m — целое, $1 \leqq m \leqq n-1$.

Полученные результаты обобщают результаты, данные в [2] и [3].