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Dedicated to Professor Tibor Katriňák

TORSION CLASSES AND SUBDIRECT PRODUCTS OF CARATHÉODORY VECTOR LATTICES

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ABSTRACT. In this paper we prove that there exists a one-to-one correspondence between torsion classes of Carathéodory vector lattices and torsion classes of generalized Boolean algebras. Further, we deal with the relations between completely subdirect product decompositions of a Carathéodory vector lattice V and completely subdirect product decompositions of the generalized Boolean algebra B which generates V.

1. Introduction

The notion of torsion class of lattice ordered groups has been defined and investigated by Martinez [11]; it was dealt with in several papers (cf., e.g., Martinez [12], Conrad [3] and the author [7]). For the torsion classes of generalized Boolean algebras, cf. [8]. Analogously we define torsion classes of vector lattices.

Carathéodory vector lattices were investigated by Gofman [5] and by the author [9], [10].

Let $\mathcal{C}$ be the class of all Carathéodory vector lattices. We show that $\mathcal{C}$ is a torsion class of vector lattices. We denote by K_1 the collection of all torsion classes of Carathéodory vector lattices. Further, let K_2 be the collection of all torsion classes of generalized Boolean algebras. We prove that there exists a bijection $\varphi\colon K_2 \to K_1$ such that for $X_1, X_2 \in K_2$ we have

$$X_1 \subseteq X_2 \iff \varphi(X_1) \subseteq \varphi(X_2)\,.$$

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For the analogous result concerning Specker lattice ordered groups cf. the author's paper [8]. Earlier, Conrad and Darnel [4] proved that the system $\mathcal{S}$ of all Specker lattice ordered groups is a torsion class.

The notion of a completely subdirect product decomposition of a partially ordered group is due to Šik [13]. Analogously we can define this notion for vector lattices and for generalized Boolean algebras.

Let B be a generalized Boolean algebra and let V be a Carathéodory vector lattice which is generated by B. We show that there is a one-to-one correspondence between internal completely subdirect product decompositions of B and those of V. (For direct product decompositions, the situation is different; cf. [10].)

2. Preliminaries

For lattices, lattice ordered groups and vector lattices we apply the terminology and the notation as in Birkhoff [1] and Conrad [2].

A *generalized Boolean algebra* is defined to be a lattice B having the least element 0 such that for each $b \in B$, the interval $[0, b]$ of B is a Boolean algebra.

A vector lattice V is called a *Carathéodory vector lattice* (cf. [9], [10]) if it satisfies the following conditions:

(i) there exists a generalized Boolean algebra B such that B is a sublattice of the underlying lattice $\ell(V)$ of V and the least element of B coincides with the neutral element of V;
(ii) for each $x \in V$, there exist elements $b_1, \dots, b_n$ of B and reals $a_1, \dots, a_n$ such that $x = a_1 b_1 + \dots + a_n b_n$.

Under the conditions as above we say that the Carathéodory vector lattice V *is generated by the generalized Boolean algebra* B and we express this situation by writing $V = f(B)$.

According to [5] and [9], for each generalized Boolean algebra B there exists a Carathéodory vector lattice V such that $V = f(B)$.

We remark that we use the same symbol 0 for denoting the zero real, the neutral element of a vector lattice and the least element of a generalized Boolean algebra. From the context it will be always clear which is the meaning of this symbol. $\mathbb{R}$ denotes the set of all reals.

Let B be a generalized Boolean algebra. We denote by $c(B)$ the system of all subsets X of B such that X is a convex sublattice of B and $0 \in X$.

For a vector lattice V we denote by $c(V)$ the set of all nonempty subsets Y of V such that

(i) Y is a convex sublattice of $\ell(V)$;
(ii) if $y_1, y_2 \in Y$ and $r \in \mathbb{R}$, then $y_1 + y_2 \in Y$ and $ry_1 \in Y$.

Both the systems $c(B)$ and $c(V)$ are partially ordered by the set-theoretical inclusion; then they are complete lattices.

Let $\mathcal{B}$ and $\mathcal{V}$ be the class of all generalized Boolean algebras, and the class of all vector lattices, respectively.

Assume that Z is a nonempty subclass of $\mathcal{B}$ such that

(i_1) Z is closed with respect to homomorphisms;
(ii_1) if $B_1 \in Z$, $B_2 \in c(B_1)$, then $B_2 \in Z$;
(iii_1) if $B \in \mathcal{B}$ and $B_i \in Z \cap c(B)$, then $\bigvee\limits_{i \in I} B_i \in Z$.

Under these assumptions, Z is called a *torsion class* of generalized Boolean algebras; we also say that Z is a torsion class in $\mathcal{B}$.

By using analogous conditions (applying $\mathcal{V}$ and $c(V)$ instead of $\mathcal{B}$ and $c(B)$) we define the notion of torsion class in $\mathcal{V}$.

Further, let $\mathcal{G}$ be the class of all lattice ordered groups. For $G \in \mathcal{G}$ we denote by $c(G)$ the system of all convex ℓ-subgroups of $\mathcal{G}$. By the conditions analogous to (i_1), (ii_1) and (iii_1) we define the notion of torsion class of lattice ordered groups.

For $V \in \mathcal{V}$, let $g(V)$ be the underlying lattice ordered group of V. If $Y \in c(V)$, then clearly $Y \in c\big(g(V)\big)$. If, conversely, $Y_1 \in c\big(g(V)\big)$, then it is easy to verify that $Y_1 \in c(V)$. Thus $c(V) = c\big(g(V)\big)$.

The notion of internal completely subdirect product decomposition of a generalized Boolean algebra (or of a vector lattice, respectively), is defined in Section 4 below.

3. The class $\mathcal{C}$

In this section we prove that $\mathcal{C}$ is a torsion class of vector lattices. We need some lemmas.

Let L be a distributive lattice with the least element 0. Assume that I is a nonempty set of indices and that for each $i \in I$, B_i is a generalized Boolean algebra such that B_i is an ideal of L. We denote by B the set of all $x \in L$ which can be expressed in the form $x = b_1 \vee \dots \vee b_n$, where $b_1, \dots, b_n \in \bigcup\limits_{i \in I} B_i$.

LEMMA 3.1.1. *Let x and $b_1,\dots,b_n$ be as above. Then there are elements $b_1^*,\dots,b_n^*$ of $\bigcup_{i\in I} B_i$ such that $x = b_1^* \vee \dots \vee b_n^*$, $b_1^* \leqq b_1$, ..., $b_n^* \leqq b_n$ and $b_{i(1)}^* \wedge b_{i(2)}^* = 0$ whenever $i(1), i(2) \in \{1,2,\dots,n\}$, $i(1) \neq i(2)$.*

Proof. We proceed by induction with respect to n. For $n = 1$, the assertion is valid. Assume that $n > 1$ and that the assertion holds for $n-1$. Put $b_1^* = b_1$, $b_{20} = b_1 \wedge b_2$, ..., $b_{n0} = b_1 \wedge b_n$. Further, let b_{21} be the complement of b_{20} in the interval $[0, b_2]$ of B, and let $b_{31},\dots,b_{n1}$ have an analogous meaning. Then

$$\begin{aligned} x &= b_1^* \vee (b_1 \vee b_2) \vee \dots \vee (b_1 \vee b_n) \\ &= b_1^* \vee (b_1 \vee b_{20} \vee b_{21}) \vee \dots \vee (b_1 \vee b_{n0} \vee b_{n1}) \\ &= b_1^* \vee (b_1 \vee b_{21}) \vee \dots \vee (b_1 \vee b_{n1}) \\ &= b_1^* \vee b_{21} \vee \dots \vee b_{n1} . \end{aligned}$$

We have $b_1^* \wedge (b_{21} \vee \dots \vee b_{n1}) = 0$. Now it suffices to apply the induction assumption for the element $b_{21} \vee \dots \vee b_{n1}$. □

LEMMA 3.1.2.

(i) *B is an ideal of L;*
(ii) *B is a generalized Boolean algebra.*

Proof. Put $D = \bigcup_{i\in I} B_i$. Let $x, x' \in B$, $x = b_1 \vee \dots \vee b_n$, $x' = b_1' \vee \dots \vee b_m'$, where $b_1,\dots,b_n,b_1',\dots,b_m' \in D$. Then, clearly, $x \vee x' \in B$. In view of the distributivity of L, from $z \in L$, $z \leqq x$ we obtain $z = z \wedge x = (z\wedge b_1) \vee \dots \vee (z \wedge b_n)$. Since $z \wedge b_1, \dots, z \wedge b_n \in D$, we get $z \in B$. Hence (i) is valid.

Again, let x and z be as above. There are $i(1),\dots,i(n) \in I$ such that $b_1 \in B_{i(1)}$, ..., $b_n \in B_{i(n)}$. In view of 3.1.1 we can assume that $b_{k(1)} \wedge b_{k(2)} = 0$ whenever $k(1), k(2) \in \{1,2,\dots,n\}$, $k(1) \neq k(2)$. Then we have also $z \wedge b_1 \in B_{i(1)}$, ..., $z \wedge b_n \in B_{i(n)}$. Hence there exist elements $c_1 \in B_{i(1)}$, ..., $c_n \in B_{i(n)}$ such that c_1 is the complement of $z \wedge b_1$ in the interval $[0, b_1]$ of B_1, and analogously for $c_2,\dots,c_n$. Put $c = c_1 \vee \dots \vee c_n$. Then c is a complement of the element z in the interval $[0, x]$ of B. Thus B is a generalized Boolean algebra. □

Let $V \in \mathcal{V}$. Assume that I is a nonempty set and that for each $i \in I$, $X_i \in c(V)$. We denote by Y the set of all $y \in V$ such that there exist $u_1,\dots,u_n \in \bigcup_{i\in I} X_i = D_1$ with $y = u_1 + \dots + u_n$.

LEMMA 3.2. *We have* $\bigvee_{i\in I} X_i = Y$.

Proof. For each $i \in I$, X_i is an element of $c(g(V))$. It is well known that Y is the supremum of the system $\{X_i\}_{i\in I}$ in the lattice $c(g(V))$. Since $c(V)$ and $c(g(V))$ coincide, we conclude that Y is the supremum of the system $\{X_i\}_{i\in I}$ also in the lattice $c(V)$. □

LEMMA 3.3. *Let V, X_i and Y be as above. Assume that $X_i \in \mathcal{C}$ for each $i \in I$. Then $Y \in \mathcal{C}$.*

Proof. For each $i \in I$ there exists $B_i \in \mathcal{B}$ such that $X_i = f(B_i)$. Put $L = V^+$. Then B_i is an ideal of L. Let B be as in 3.1.

Let $b \in B$. There exist $i(1),\dots,i(n) \in I$ and $b_1 \in B_{i(1)}, \dots, b_n \in B_{i(n)}$ with $b = b_1 \vee \dots \vee b_n$. Then $0 \leqq b_1 \in X_{i(1)}, \dots, 0 \leqq b_n \in X_{i(n)}$, hence $0 \leqq b \leqq b_1 + \dots + b_n \in Y$. Thus $B \subseteq Y$. By applying 3.1.2 we obtain that B is a convex sublattice of Y. Clearly $0 \in B$.

Let $y \in Y$. Then y is a linear combination of some elements of D_1. Further, if $i \in I$ and $x_i \in X_i$, then x_i is a linear combination of some elements of B_i. Therefore y is a linear combination of some elements of B. Thus we have $Y = f(B)$ and hence $Y \in \mathcal{C}$. □

For each $X \in K_2$ we put

$$\varphi(X) = \{f(B) :\ B \in X\}.$$

Hence we have $\varphi(X) \in \mathcal{C}$.

LEMMA 3.4. *Let $X \in K_2$. Then $\varphi(X)$ is a torsion class of vector lattices.*

Proof. We consider the conditions (i_1), (ii_1) and (iii_1) from the definition of the torsion class.

(i_1) Let $V \in \varphi(X)$, $V' \in \mathcal{V}$ and let φ_1 be a homomorphism of V onto V'. There exists $B \in X$ such that $V = f(B)$. Put $B' = \varphi_1(B)$. Then B' is a generalized Boolean algebra and $V' = f(B')$. Hence $V' \in \varphi(X)$.

(ii_1) Let $V_1 \in \varphi(X)$, $V_2 \in c(V_1)$. There exists $B_1 \in X$ with $V_1 = f(B_1)$. Put $B_2 = V_2 \cap B_1$. Then $B_2 \in c(B_1)$, hence $B_2 \in X$. Moreover, $V_2 = f(B_2)$. Thus $V_2 \in \varphi(X)$.

(iii_1) Let $V \in \mathcal{V}$, $X_i \in c(V) \cap \varphi(X)$ for $i \in I$, where I is a nonempty set of indices. For each $i \in I$ there exists $B_i \in X$ with $X_i = f(B_i)$. Let B be as in the proof of 3.3; then we have $V = f(B)$. From the definition of B (cf. also 3.1.2) we conclude that in the lattice $c(B)$ we have $B = \bigvee_{i\in I} B_i$. Thus $B \in X$ and hence $V \in \varphi(X)$. □

We have $\mathcal{B} \in K_2$ and $\varphi(\mathcal{B}) = \mathcal{C}$. Thus from 3.4 we obtain as a corollary:

PROPOSITION 3.5. *$\mathcal{C}$ is a torsion class of vector lattices.*

If T is a nonempty collection of torsion classes of vector lattices, then the intersection of all torsion classes belonging to T is a torsion class again. This yields:

LEMMA 3.6. *Let C_0 be a nonempty subclass of $\mathcal{V}$. Then there exists a torsion class Y of vector lattices such that*

(i) *$C_0 \subseteq Y$;*
(ii) *if Y_1 is a torsion class of vector lattices with $C_0 \subseteq Y_1$, then $Y \subseteq Y_1$.*

If C_0 and Y are as in 3.6, then we say that the torsion class Y is generated by C_0.

As a consequence of 3.5 and 3.6 we obtain:

COROLLARY 3.7. *Let $\emptyset \neq C_0 \subseteq \mathcal{C}$. Then the torsion class of vector lattices generated by C_0 is a subclass of $\mathcal{C}$.*

A torsion class Y of vector lattices will be called *Carathéodory torsion class* if $Y \subseteq \mathcal{C}$.

LEMMA 3.8. (Cf. [10].) *Let $B_1, B_2 \in \mathcal{B}$. Then we have*

$$B_1 \simeq B_2 \iff f(B_1) \simeq f(B_2).$$

LEMMA 3.9. *Let B_i $(i \in I)$ and B be elements of $\mathcal{B}$ such that $B_i \in c(B)$ for each $i \in I$, and $\bigvee\limits_{i\in I} B_i = B$. Then $\bigvee\limits_{i\in I} f(B_i) = f(B)$.*

P r o o f. Let us remark that, given $B \in \mathcal{B}$, the vector lattice $f(B)$ is determined up to isomorphisms leaving the elements of B fixed. From $B_i \in c(B)$ we conclude that there exists $f(B_i)$ with $f(B_i) \in c\bigl(f(B)\bigr)$, hence, for these $f(B_i)$ we have $\bigvee\limits_{i\in I} f(B_i) \in c\bigl(f(B)\bigr)$. Let $x \in f(B)$. Thus x can be written in the form $x = a_1 b_1 + \dots + a_n b_n$, where $a_1, \dots, a_n \in \mathbb{R}$ and $\{b_1, \dots, b_n\}$ is an orthogonal subset of B (cf. [9]). Further, each b_k $(k = 1, 2, \dots, n)$ is a join of a finite number of elements belonging to $\bigcup\limits_{i\in I} B_i$. Since the interval $[0, b_k]$ of B is a Boolean algebra and $[0, b_k] \cap B_i$ are ideals of $[0, b_k]$, we conclude (cf. 3.1.1) that without loss of generality we can suppose that this join consists of an orthogonal system of elements. Then x is a linear combination of some elements of the set $\bigcup\limits_{i\in I} B_i$. This yields that $x \in \bigvee\limits_{i\in I} f(B)$. □

For each $Y \in K_1$ we put

$$\psi(Y) = \bigl\{B \in \mathcal{B} :\ f(B) \in Y\bigr\}.$$

LEMMA 3.10. *Let $Y \in K_1$. Then $\psi(Y) \in K_2$.*

Proof. Let us consider the conditions (i_1), (ii_1) and (iii_1) from the definition of torsion class.

Let $B \in \psi(Y)$; thus $f(B) \in Y$. Assume that B' is a homomorphic image of B. Then in view of [10], $f(B')$ is a homomorphic image of $f(B)$. Hence $f(B') \in Y$ and so $B' \in \psi(Y)$.

Let $B_1 \in \psi(Y)$, $B_2 \in c(B_1)$. Then we have $f(B_2) \in c\big(f(B_1)\big)$, whence $f(B_2) \in Y$ and $B_2 \in \psi(Y)$.

Further, assume that $B \in \mathcal{B}$ and $B_i \in \psi(Y) \cap c(B)$ for $i \in I$. Then $f(B_i) \in Y$ and $f(B_i) \in c\big(f(B)\big)$. Since $Y \in K_1$, we obtain $\bigvee_{i\in I} f(B_i) \in Y$. In view of 3.9 we get $\bigvee_{i\in I} f(B_i) = f\Big(\bigvee_{i\in I} B_i\Big)$. Hence $f\Big(\bigvee_{i\in I} B_i\Big) \in Y$ and thus $\bigvee_{i\in I} B_i \in \psi(Y)$. □

From the definitions of φ and ψ, and from 3.9 and 3.10 we conclude that $\psi = \varphi^{-1}$. Further, if $X_1, X_2 \in K_2$ and $Y_1, Y_2 \in K_1$, then

$$X_1 \subseteq X_2 \implies \varphi(X_1) \subseteq \varphi(X_2)\,, \tag{1}$$

$$Y_1 \subseteq Y_2 \implies \psi(Y_1) \subseteq \psi(X_2)\,, \tag{2}$$

Thus we obtain:

PROPOSITION 3.11. *The mapping φ is a bijection of K_2 onto K_1. Moreover, $\psi = \varphi^{-1}$ and the relations* (1), (2) *are valid.*

We recall that in view of [8], K_2 has many elements (in the sense that there exists a monomorphism of the class of all infinite cardinals into K_2); hence K_1 has many elements as well.

4. Completely subdirect products

Assume that $(A_i)_{i\in I}$ is an indexed system of algebras of the same type having a nulary operation 0. Further, let A be an algebra of the same type and suppose that φ is an isomorphism of A onto a subalgebra of the direct product $\prod_{i\in I} A_i$ such that, whenever $i \in I$ and $x^i \in A_i$, then there exists $a \in A$ with $\big(\varphi(a)\big)_i = x^i$. In other words, φ is a *subdirect decomposition* of A. We call φ a *completely subdirect decomposition* if it satisfies the following condition:

(1) Whenever $i \in I$ and $x^i \in A_i$, then there exists $x \in A$ such that $\big(\varphi(x)\big)_i = x^i$ and $\big(\varphi(x)\big)_j = 0$ for each $j \in I$, $j \neq i$.

For the case of partially ordered groups, the notion of completely subdirect decomposition was introduced by Šik [13].

We slightly strengthen the condition (1) as follows:

(2) Whenever $i \in I$, then A_i is a subalgebra of A and for each $x \in A_i$ we have $\big(\varphi(x)\big)_i = x$, $\big(\varphi(x)\big)_j = 0$ if $j \in I, j \neq i$.

If (2) is valid, then φ will be called an *internal completely subdirect decomposition.*

To each completely subdirect decomposition φ there corresponds in a natural way an internal completely subdirect decomposition $\overline{\varphi}$. We construct $\overline{\varphi}$ as follows.

Let $i \in I$. We put

$$\overline{A}_i = \Big\{x \in A : \ \big(\varphi(x)\big)_j = 0 \text{ for each } j \in I, \ j \neq i\Big\}.$$

Then $\overline{A}_i$ is a subalgebra of A. It is easy to verify that $\overline{A}_i$ is isomorphic to A_i.

For each $y \in A$ consider the element x of $\overline{A}_i$ which satisfies the relation

$$\big(\varphi(x)\big)_i = \big(\varphi(y)\big)_i .$$

We put $\overline{\varphi}_i(y) = x$ and

$$\overline{\varphi}(y) = \big(\overline{\varphi}_i(y)\big)_{i \in I} .$$

Then $\overline{\varphi}(y) \in \prod\limits_{i \in I} \overline{A}_i$ and $\overline{\varphi}$ is an internal completely subdirect decomposition of A.

Below, when speaking about a completely subdirect decompostion, we suppose that the condition (2) is satisfied (i.e., completely subdirect decompositions under consideration are assumed to be internal). The algebras A_i are called *completely subdirect factors* of A. We also say that A is a *completely subdirect product* of the system $(A_i)_{i \in I}$.

If, in particular, $\varphi(A) = \prod\limits_{i \in I} A_i$, then φ is a *direct product decomposition* of A; in view of (2), since A_i are subalgebras of A, we speak about an *internal direct product decomposition* and A_i are called *internal direct factors* of A.

EXAMPLE 1. Let A_0 be the Boolean algebra of all subsets of an infinite set M. For $m \in M$ let $A_m = \{\emptyset, m\}$ (we have now $\emptyset$ instead of 0). For $a \in A_0$ and $m \in M$ we put $\big(\varphi(a)\big)_m = m$ if $m \in a$, and $\big(\varphi(a)\big)_m = \emptyset$ otherwise. Then φ is an isomorphism of A_0 onto $\prod\limits_{m \in M} A_m$; in fact, φ is an internal direct decomposition of A_0 and each A_m is an internal direct factor of A_0.

EXAMPLE 2. Under the notation as in Example 1, let A be the subsystem of A_0 consisting of all $a \in A_0$ such that either $a = M$ or a is finite. The system A is a meet-semilattice. For $a \in A_0$ we define $\varphi(a)$ similarly as above. Then φ is a completely subdirect product of A_0. For each $m \in M$, A_m is a completely subdirect factor of A, but it fails to be a direct factor of A.

Now let us consider completely subdirect decompositions of generalized Boolean algebras and of vector lattices.

Assume that B and B_i $(i \in I)$ are generalized Boolean algebras; let $\varphi\colon B \to \prod_{i\in I} B_i$ be a completely subdirect product decomposition of B. For $i \in I$ we put

$$B_i' = \left\{ b \in B : \ \left(\varphi(b)\right)_i = 0 \right\}.$$

It is easy to verify that $B_i' \in c(B)$ and that $B_i' \cap B_i = \{0\}$.

For $b \in B$ we denote $b_i = \left(\varphi(b)\right)_i$. Then $b_i \leqq b$, hence there exists the complement b_i^* of b_i in the interval $[0, b]$ of B. Thus $b_i \wedge b_i^* = 0$ and $b_i \vee b_i^* = b$. Then

$$(b_i)_i \wedge (b_i^*)_i = 0, \qquad (b_i)_i \vee (b_i^*)_i = b_i.$$

Since $(b_i)_i = b_i$, we obtain $(b_i^*)_i = 0$, whence $b_i^* \in B_i'$. From this we get:

LEMMA 4.1. *Under the notation as above, put $\varphi_i(b) = (b_i, b_i^*)$. Then $\varphi_i\colon B \to B_i \times B_i'$ is an internal direct product decomposition of B.*

COROLLARY 4.2. *Each completely subdirect factor of a generalized Boolean algebra of B is an internal direct factor of B.*

LEMMA 4.3. *Under the notation as above we have*

(i) *if $i(1)$, $i(2)$ are distinct elements of I and $b^1 \in B_{i(1)}$, $b^2 \in B_{i(2)}$, then $b^1 \wedge b^2 = 0$;*

(ii) *if $b \in B$, then $b = \bigvee_{i\in I} b_i$.*

Proof. Let $i(1)$, $i(2)$, b^1 and b^2 be as in (i). From $b^1 \in B_{i(1)}$ we obtain $(b^1)_{i(2)} = 0$. By way of contradiction, suppose that $b^1 \wedge b^2 = c > 0$. Then $c \in B_{i(2)}$, whence $c_{i(2)} = c > 0$. At the same time, $(b^1)_{i(2)} \geqq c_{i(2)} > 0$, which is a contradiction. Hence (i) is valid.

Let $b \in B$. We have $b_i \leqq b$ for each $i \in I$. Assume that there is $d \in B$ such that $b_i \leqq d < b$ for each $i \in I$. From $b_i \leqq d$ we obtain $(b_i)_i \leqq d_i$; since $(b_i)_i = b_i$, we get $b_i \leqq d_i$ for each $i \in I$. Therefore $b \leqq d$, which is a contradiction. Thus (ii) is valid. □

Lemma 4.4. *Let B be a generalized Boolean algebra and let J be a nonempty set of indices. Assume that for each $j \in J$ we have an internal direct product decomposition $\varphi_j\colon B \to B_j \times B'_j$. For $b \in B$ let b_j be the component of b in the internal direct factor B_j. Assume that the conditions* (i) *and* (ii) *from* 4.3 *are satisfied (where I, $i(1)$ and $i(2)$ are replaced by J, $j(1)$ and $j(2)$). For $b \in B$ put $\varphi_0(b) = (b_j)_{j\in J}$. Then φ_0 is a completely subdirect decomposition of B.*

P r o o f . In view of the definition of φ_0 we conclude that φ_0 is a homomorphism of B into $\prod\limits_{j\in J} B_j$. If $b, b' \in B$ such that $\varphi_0(b) = \varphi_0(b')$, then in view of (ii) we obtain $b = b'$. Thus φ_0 is a monomorphism. Therefore φ_0 is an isomorphism of B onto a subalgebra of $\prod\limits_{j\in J} B_j$.

Let $j \in J$. In view of φ_j, B_j is a subalgebra of B and for $x \in B_j$ we have $\big(\varphi_0(x)\big)_j = x$. Further, let $j(1) \in J$, $j(1) \neq j$. According to (ii), $\big(\varphi_0(x)\big)_{j(1)} = x_{j(1)} \leqq x$. On the other hand, (i) yields that $x_{j(1)} \wedge x = 0$, whence $x_{j(1)} = 0$. □

Proposition 4.5. *Let B be a generalized Boolean algebra and let B_i $(i \in I)$ be internal direct factors of B. Then B is a completely subdirect product of the system $(B_i)_{i\in I}$ if and only if the conditions* (i) *and* (ii) *from* 4.3 *are satisfied.*

P r o o f . This is a consequence of 4.2, 4.3 and 4.4. □

Let us assume that V is a vector lattice and that $\varphi\colon V \to \prod\limits_{i\in I} V_i$ is a completely subdirect product decomposition of V. For $x \in V$ and $i \in I$ put $x_i = \big(\varphi(x)\big)_i$. Further, we set

$$V'_i = \{y \in V:\ y_i = 0\}, \qquad V^*_i = \{x - x_i:\ x \in V\}.$$

Then we have $V^*_i \subseteq V'_i$. For $y \in V'_i$ we get $y - y_i = y$, whence $y \in V^*_i$ and thus $V'_i = V^*_i$. Also, $V'_i \in c(V)$. If $y^1 \in V_i$ and $y^2 \in V'_i$, then $|y^1| \wedge |y^2| = 0$. Hence $V_i \cap V'_i = \{0\}$. We obtain:

Lemma 4.6. *For each $i \in I$, the mapping $\varphi_i\colon V \to V_i \times V'_i$ defined by $\varphi_i(x) = (x_i, x - x_i)$ is an internal direct product decomposition of V.*

Corollary 4.7. *Each completely subdirect factor of a vector lattice V is an internal direct factor of V.*

The proof of the following lemma is analogous to that of 4.3.

Lemma 4.8. *Under the assumptions as above we have:*

(i) *Let $i(1)$ and $i(2)$ be distinct elements of I and $x^1 \in V^+_{i(1)}$, $x^2 \in V^+_{i(2)}$. Then $x^1 \wedge x^2 = 0$.*

(ii) *Let $0 \leqq x \in V$. Then $x = \bigvee\limits_{i\in I} x_i$.*

LEMMA 4.9. *Let V be a vector lattice and let $(V_j)_{j\in J}$ be an indexed system of elements of $c(V)$ such that each V_j is an internal direct factor of V and the conditions* (i), (ii) *from* 4.8 *are satisfied (with J, $j(1)$ and $j(2)$ instead of I, $i(1)$, $i(2)$; for $j \in J$, $x \in V$, the symbol x_j denotes the component of x in the direct factor V_j). Then V is a completely subdirect product of the system $(V_j)_{j\in J}$.*

Proof. For each $x \in V$ we put $\varphi_0(x) = (x_j)_{j\in J}$. Hence φ_0 is a homomorphism of V into $\prod_{j\in J} V_j$.

Let $x, x' \in V$ and suppose that $\varphi_0(x) = \varphi_0(x')$; i.e., $x_j = x'_j$ for each $j \in J$. Then $(x^+)_j = (x_j)^+$, $(x'^+)_j = (x'_j)^+$, thus in view of (ii),

$$x^+ = \bigvee_{j\in J} (x_j)^+ = \bigvee_{j\in J} (x'_j)^+ = x'^+ .$$

Similarly we obtain $x^- = (x')^-$. This yields the relation $x = x'$. Hence φ_0 is a monomorphism.

Let $j \in J$ and $x \in V_j$. Since V_j is an internal direct factor of V, we get $x_j = x$. Further, let $j(1) \in J$, $j(1) \neq j$. We have $x^+, x^- \in V_j$. Analogously as in the proof of 4.4 we verify that $(x^+)_{j(1)} = 0 = (x^-)_{j(1)}$. Hence $x_{j(1)} = 0$, completing the proof. □

PROPOSITION 4.10. *Let $(V_i)_{i\in I}$ be an indexed system of elements of $c(V)$, where V is a vector lattice. Then the following conditions are equivalent:*

(a) *V is a completely subdirect product of the system $(V_i)_{i\in I}$.*
(b) *Each V_i is an internal direct factor of V and the conditions* (i), (ii) *from* 4.8 *are valid.*

Proof. The assertion follows from 4.7, 4.8 and 4.9. □

Now let us investigate the case when B is a generalized Boolean algebra and $V = f(B)$.

In view of [10], there is a one-to-one correspondence between internal direct factors of V and internal direct factors of B; if B_0 is an internal direct factor of B and V_0 is the corresponding internal direct factor of V, then we have $V_0 = f(B_0)$.

Suppose that $\varphi\colon B \to \prod_{i\in I} B_i$ is a completely subdirect product decomposition of B.

LEMMA 4.11. *V is a completely subdirect product of the system $\big(f(B_i)\big)_{i\in I}$.*

Proof. Let $i \in I$. In view of 4.2, B_i is an internal direct factor of B; hence $V_i = f(B_i)$ is an internal direct factor of V.

Let $i(1)$, $i(2)$ be distinct elements of I. In view of 4.3 we have $b^1 \wedge b^2 = 0$ whenever $b^1 \in B_{i(1)}$, $b^2 \in B_{i(2)}$. Assume that $0 < x^1 \in V^+_{i(1)}$, $0 < x^2 \in V^+_{i(2)}$. Then there exist $b_1, \dots, b_n \in B_1$ with $b_1 > 0, \dots, b_n > 0$ and reals $0 < a_1, \dots, 0 < a_n$ such that the system $\{b_1, \dots, b_n\}$ is orthogonal and $x^1 = a_1 b_1 + \dots + a_n b_n$. Analogously we can express x^2 in the form $x^2 = a'_1 b'_1 + \dots + a'_m b'_m$ with $b'_1, \dots, b'_m \in B_{i(2)}$. Thus the condition (i) from 4.8 is satisfied.

It remains to verify that the condition (ii) from 4.8 is valid. Let x^1 be as above. In view of φ we obtain

$$b_1 = \bigvee_{i\in I} (b_1)_i\,, \quad \dots, \quad b_n = \bigvee_{i\in I} (b_n)_i\,.$$

According to [9], the symbol $\bigvee\limits_{i\in I} (b_1)_i$ is, at the same time, the join of the system $\big((b_1)_i\big)_{i\in I}$ in $f(B)$. Thus in $f(B)$ we have

$$a_1 b_1 = \bigvee_{i\in I} a_1 (b_1)_i\,, \quad \dots, \quad a_n b_n = \bigvee_{i\in I} a_n (b_n)_i\,.$$

Also, the component of b_1 in B_i coincides with the component of b_1 in $f(B_i)$, and similarly for $b_2, \dots, b_n$. We obtain

$$\begin{aligned} x^1 &= a_1 b_1 + \dots + a_n b_n = \bigvee_{i\in I} (a_1 b_1)_i + \dots + \bigvee_{i\in I} (a_n b_n)_i \\ &= \bigvee_{i\in I} \big((a_1 b_1)_i + \dots + (a_n b_n)_i\big) = \bigvee_{i\in I} (a_1 b_1 + \dots + a_n b_n)_i \\ &= \bigvee_{i\in I} (x^1)_i\,. \end{aligned}$$

□

Assume that $V = f(B)$ and that V is a completely subdirect product of a system $(V_i)_{i\in I}$. For each $i \in I$ we put $B_i = V_i \cap B$; then $V_i = f(B_i)$.

LEMMA 4.12. *B is a completely subdirect product of the system $(B_i)_{i\in I}$.*

Proof. Let $i \in I$. Since V_i is an internal direct factor of V, in view of [10] we obtain that B_i is an internal direct factor of B.

Let $i(1)$ and $i(2)$ be distinct elements of I and $b^1 \in B_{i(1)}$, $b^2 \in B_{i(2)}$. Then $b^1 \in V_{i(1)}$ and $b^2 \in V_{i(2)}$, hence in view of 4.8 we get $b^1 \wedge b^2 = 0$. Thus the condition (i) from 4.3 is satisfied.

Let $b \in B$, $i \in I$ and let b_i be the component of b in B_i. According to 4.8 we have $b = \bigvee_{i \in I} b_i$. Further, in view of [9], the last relation is valid also in B. Hence (ii) from 4.3 is valid. Therefore according to 4.5, B is a completely subdirect product of the system $(B_i)_{i \in I}$. □

Summarizing, 4.11 and 4.12 yield:

PROPOSITION 4.13. *Let B be a generalized Boolean algebra and $V = f(B)$. There is a one-to-one correspondence between completely subdirect decompositions of V and completely subdirect decompositions of B.*

For a generalized Boolean algebra B we denote by $S(B)$ the Specker lattice ordered group which is generated by B (cf. [4], [8]). The proof of the following result will be omitted; it can be performed by the same method as in the case of 4.13.

PROPOSITION 4.14. *Let B be a generalized Boolean algebra. There is a one-to-one correspondence between completely subdirect decompositions of $S(B)$ and completely subdirect decompositions of B.*

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