Štefan Černák

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ON THE MAXIMAL DEDEKIND COMPLETION OF A HALF PARTIALLY ORDERED GROUP

ŠTEFAN ČERNÁK

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ABSTRACT. The notion of the maximal Dedekind completion is extended for the case of a half partially ordered group. The main result is formulated in 2.16.

For the maximal Dedekind completion $M(G)$ of a partially ordered group G, cf. L. Fuchs [3; Chapter V, §10]. C. J. Everett [2] has proved that $M(G)$ is a lattice ordered group whenever G is a commutative lattice ordered group. The same result was obtained in [1] for an arbitrary lattice ordered group.

M. Giraudet and F. Lucas [4] have defined and systematically studied the notion of a half partially ordered group as a generalization of the notion of a partially ordered group.

In this paper, the maximal Dedekind completion of a half partially ordered group is studied.

1. Preliminaries

We shall summarize the essentials of the MacNeille completion of a partially ordered set (see [6] and [3]).

Let G be a partially ordered set, and let X be a subset of G. Denote

$$U(X) = \{g \in G :\ g \geq x \ \text{for each}\ x \in X\}\,,$$
$$L(X) = \{g \in G :\ g \leq x \ \text{for each}\ x \in X\}\,.$$

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If $U(X) \neq \emptyset$ $(L(X) \neq \emptyset)$, then X is called an *upper* (*lower*) *bounded subset* of G. Let us denote by $G^{\#}$ the system of all subsets of G of the form $L\big(U(X)\big)$, where X is a nonvoid upper bounded subset of G. Each element of $G^{\#}$ is said to be the *Dedekind cut* of G. The system $G^{\#}$ is a conditionally complete conditional lattice under set-inclusion ([3; p. 92]). By a *conditional lattice*, is meant a partially ordered set in which every two elements having an upper (lower) bound have the least upper bound (greatest lower bound).

Let $Z_i \in G^{\#}$ $(i \in I)$, and let there exist an element $Z_0 \in G^{\#}$ with $Z_i \subseteq Z_0$ for each $i \in I$. Then for the least upper bound of Z_i $(i \in I)$ we have $\bigvee_{i\in I} Z_i = L\Big(U\Big(\bigcup_{i\in I} Z_i\Big)\Big)$. Analogously, if the system Z_i $(i \in I)$ has a lower bound in $G^{\#}$, then for the greatest lower bound of Z_i $(i \in I)$ we get $\bigwedge_{i\in I} Z_i = \bigcap_{i\in I} Z_i$.

Define the mapping $\varphi\colon G \to G^{\#}$ by the rule $\varphi(g) = L\big(U(\{g\})\big)$ for each $g \in G$. Then φ is an injection, and it preserves all greatest lower bounds and least upper bounds existing in G. In what follows, we shall identify g and $\varphi(g)$. In this sense, G is a subset of $G^{\#}$, and the following conditions are satisfied:

(a) If X is a nonempty and upper (lower) bounded subset of G, then X has the least upper bound (greatest lower bound) in $G^{\#}$.

(b) If $z \in G^{\#}$, then there exist nonempty subsets X and Y of G such that X is upper bounded in G, Y is lower bounded in G, and $z = \sup X = \inf Y$ in $G^{\#}$.

Remark 1.1. If we suppose that G is a lattice (linearly ordered set), then $G^{\#}$ is a conditionally complete lattice (linearly ordered set). When identifying g and $\varphi(g)$, G is a sublattice of $G^{\#}$.

Now, we recall the notion of a half partially ordered group (cf.[4]).

Let G be a group with the group operation $+$, and let $\leq$ be a partial order on G. The relation $\leq$ is called *compatible from the right* if $x, y, z \in G$ and $x \leq y$ imply $x + z \leq y + z$. An element $z \in G$ is said to be *increasing* (*decreasing*) if $x, y \in G$ and $x \leq y$ imply $z + x \leq z + y$ $(z + x \geq z + y)$. The set of all increasing (decreasing) elements of G will be denoted by $G\!\uparrow$ $(G\!\downarrow)$.

G is said to be a *half partially ordered group* if the following conditions are fulfilled:

(I) $\leq$ is a non-trivial partial order on G.

(II) $\leq$ is compatible from the right.

(III) $G = G\!\uparrow \cup\, G\!\downarrow$.

If $G\!\uparrow$ is a lattice (linearly ordered set), then G will be called a *half lattice ordered group* (*half linearly ordered group*).

Let G be a half partially ordered group. From the definition of G, it immediately follows:

(1) If $x \in G\downarrow$, then $-x \in G\downarrow$.
(2) If $x, y \in G\downarrow$, then $x + y \in G\uparrow$,
if $x \in G\uparrow$, $y \in G\downarrow$, then $x + y \in G\downarrow$, $y + x \in G\downarrow$.
(3) If $x, y \in G\downarrow$, $x \leq y$, then $-x \leq -y$.

We shall apply the following result [4; Proposition I.1.3]).

PROPOSITION 1.2. *Let G be a half partially ordered group such that $G\downarrow \neq \emptyset$. Then*

(i) *$G\uparrow$ is a subgroup of G, and G is a disjoint union of $G\uparrow$ and $G\downarrow$.*
(ii) *$G\uparrow$ and $G\downarrow$ are isomorphic and also antiisomorphic partially ordered sets.*
(iii) *If $x \in G\uparrow$ and $y \in G\downarrow$, then x and y are incomparable.*

2. The maximal Dedekind completion of a half partially ordered group

In the whole section, G is assumed to be a half partially ordered group. The maximal Dedekind completion of G will be constructed. The method from [3] for partially ordered groups will be applied for G.

Let us denote $H = G\uparrow$, $K = G\downarrow$.

From 1.2 (iii) it immediately follows:

LEMMA 2.1. *Let $X \subseteq G$, $X \neq \emptyset$, $U(X) \neq \emptyset$. Then:*

(i) *Either $X \subseteq H$ (and then $U(X) \subseteq H$) or $X \subseteq K$ (and then $U(X) \subseteq K$).*
(ii) *If there exists $g \in G$, $g = \sup X$ in G, then $g \in H$ ($g \in K$) if and only if $X \subseteq H$ ($X \subseteq K$).*
(iii) *If $X \subseteq H$ ($X \subseteq K$), then $\sup X$ exists in $H(K)$ if and only if $\sup X$ exists in G, and $\sup X$ in G is equal to $\sup X$ in $H(K)$.*

Analogous assertions are valid for $L(X)$ and $\inf X$.

Let $X \subseteq G^{\#}$. Denote

$$U_{G^{\#}}(X) = \{z \in G^{\#} : z \geq x \text{ for each } x \in X\},$$
$$L_{G^{\#}}(X) = \{z \in G^{\#} : z \leq x \text{ for each } x \in X\}.$$

Remark 2.2. In 2.1, G, H, K and $U(X)$ can be replaced by $G^{\#}$, $H^{\#}$, $K^{\#}$ and $U_{G^{\#}}(X)$, respectively.

From 2.1 (i), we infer that $L(U(X)) \subseteq H(K)$ whenever $X \subseteq H(K)$. Hence, in view of 1.2 (i), we get the following result.

Lemma 2.3. *$G^{\#}$ is a disjoint union of $H^{\#}$ and $K^{\#}$.*

Lemma 2.4. *Let X and Y be nonempty subsets of G, and let $V = \{x + y : x \in X,\ y \in Y\}$.*

(i) *Assume that one of the following conditions is satisfied:*

(a_1) *X and Y are upper bounded subsets of H.*

(a_2) *X is an upper bounded subset of K, and Y is a lower bounded subset of K.*

Then V is a nonempty and upper bounded subset of H.

(ii) *Assume that one of the following conditions is satisfied:*

(a_3) *X is an upper bounded subset of H, and Y is an upper bounded subset of K.*

(a_4) *X is an upper bounded subset of K, and Y is a lower bounded subset of H.*

Then V is a nonempty and upper bounded subset of K.

P r o o f . Since X and Y are nonempty, V is nonempty as well.

Suppose that (a_2) is satisfied. Then there exist elements $x', y' \in G$ with $x \leq x'$, $y' \leq y$ for all $x \in X$, $y \in Y$. According to 2.1 (i), we get $x', y' \in K$. By (II), we have $x + y \leq x' + y$. Since x' is decreasing, $x' + y \leq x' + y'$. Hence, $x + y \leq x' + y'$. With respect to (2), we have $x + y \in H$ and $x' + y' \in H$.

Assume that (a_3) is fulfilled. Then there exist elements $x', y' \in G$ such that $x \leq x'$, $y \leq y'$. From 2.1 (i), we infer that $x' \in H$ and $y' \in K$. By using of (II), we obtain $x + y \leq x' + y$. As for x' is increasing, we get $x' + y \leq x' + y'$. Hence, $x + y \leq x' + y'$. Applying (2), we have $x + y \in K$ and $x' + y' \in K$.

The remaining assertions can be verified similarly. □

Remark 2.5. The dual lemma to 2.4 also holds true.

For an element $z \in H^{\#}$ we denote

$$U_H(z) = \{h \in H : h \geq z\}, \qquad L_H(z) = \{h \in H : h \leq z\}.$$

Symbols $U_K(z)$, $L_K(z)$ have an analogous meaning for $z \in K^{\#}$. In view of (b), the sets $U_H(z)$, $L_H(z)$ are nonempty subsets of H. Hence, $U_H(z)$ is lower bounded, and $L_H(z)$ is upper bounded in H. We get an analogous result for subsets $U_K(z)$ and $L_K(z)$ of K.

Therefore

$$z = \sup L_H(z) = \inf U_H(z) \qquad \text{in} \quad H^{\#} \tag{4}$$

if $z \in H^{\#}$,

$$z = \sup L_K(z) = \inf U_K(z) \qquad \text{in} \quad K^{\#} \tag{5}$$

if $z \in K^{\#}$.

We intend to define a binary operation $z_1 + z_2$ in $G^{\#}$. The following four possibilities can occur:

(a'_1) $z_1, z_2 \in H^{\#}$;

by (4), we have $z_1 = \sup L_H(z_1)$, $z_2 = \sup L_H(z_2)$ in $H^{\#}$. Hence, 2.4 (i) yields that the set $Z = \{h_1 + h_2 : h_1 \in L_H(z_1),\ h_2 \in L_H(z_2)\}$ is a nonempty and upper bounded subset of H.

(a'_2) $z_1, z_2 \in K^{\#}$;

then (5) implies that $z_1 = \sup L_K(z_1)$, $z_2 = \inf U_K(z_2)$ in $K^{\#}$. Similarly as in (a'_1), we get that $Z = \{k_1 + k_2 : k_1 \in L_K(z_1),\ k_2 \in U_K(z_2)\}$ is a nonempty and upper bounded subset of H.

(a'_3) $z_1 \in H^{\#}$, $z_2 \in K^{\#}$;

from (4) and (5), it follows that $z_1 = \sup L_H(z_1)$ in $H^{\#}$, $z_2 = \sup L_K(z_2)$ in $K^{\#}$. By using of 2.4 (ii), we obtain that the set $Z = \{h_1 + k_2 : h_1 \in L_H(z_1),\ k_2 \in L_K(z_2)\}$ is a nonempty and upper bounded subset of K.

(a'_4) $z_1 \in K^{\#}$, $z_2 \in H^{\#}$;

according to (5) and (4), we get $z_1 = \sup L_K(z_1)$ in $K^{\#}$, $z_2 = \inf U_H(z_2)$ in $H^{\#}$. Analogously as in (a'_3), we get that $Z = \{k_1 + h_2 : k_1 \in L_K(z_1),\ h_2 \in U_H(z_2)\}$ is a nonempty and upper bounded subset of K.

With respect to (a), we can conclude that, in all four cases, there exists $\sup Z$ in $G^{\#}$. In the cases (a'_1) and (a'_2) ((a'_3) and (a'_4)), there exists also $\sup Z$ in $H^{\#}$ ($K^{\#}$). But 2.2 yields that $\sup Z$ in $G^{\#}$ coincides with $\sup Z$ in $H^{\#}$ ($K^{\#}$).

The operation $+$ in $G^{\#}$ is defined as follows. We put $z_1 + z_2 = \sup Z$ in $G^{\#}$ for each $z_1, z_2 \in G^{\#}$.

From the definition, we immediately obtain:

(2′) If $z_1, z_2 \in H^{\#}$, then $z_1 + z_2 \in H^{\#}$,
if $z_1, z_2 \in K^{\#}$, then $z_1 + z_2 \in H^{\#}$,
if $z_1 \in H^{\#}$, $z_2 \in K^{\#}$, then $z_1 + z_2 \in K^{\#}$, $z_2 + z_1 \in K^{\#}$.

Remark 2.6. The operation $+$ in $G^{\#}$ need not be associative, in general. Thus $G^{\#}$ fails to be a semigroup, in general (see 3.5 (A)). Hence, in this point, the situation essentially differs from that concerning partially ordered groups. Namely, if G is a partially ordered group, then $G^{\#}$ is a semigroup ([3; p. 94]).

In the following lemma, we show that the operation $z_1 + z_2$ in $G^{\#}$ does not depend on a choice of subsets of G having supremum equal to z_1 and supremum or infimum equal to z_2 in $G^{\#}$.

LEMMA 2.7. *Let $z_1, z_2 \in G^{\#}$, and let X_1, X_2 be nonempty subsets of G. Assume that some of the following conditions is satisfied:*

(b$_1$) $X_1 \subseteq H$, $X_2 \subseteq H$, $z_1 = \sup X_1$, $z_2 = \sup X_2$ *in* $G^{\#}$,
(b$_2$) $X_1 \subseteq K$, $X_2 \subseteq K$, $z_1 = \sup X_1$, $z_2 = \inf X_2$ *in* $G^{\#}$,
(b$_3$) $X_1 \subseteq H$, $X_2 \subseteq K$, $z_1 = \sup X_1$, $z_2 = \sup X_2$ *in* $G^{\#}$,
(b$_4$) $X_1 \subseteq K$, $X_2 \subseteq H$, $z_1 = \sup X_1$, $z_2 = \inf X_2$ *in* $G^{\#}$.

Then $z_1 + z_2 = \sup\{x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\}$ *in* $G^{\#}$.

Proof. Suppose that the condition (b$_4$) is satisfied. Then $z_1 \in K^{\#}$, $z_2 \in H^{\#}$. By the definition of the operation $+$ in $G^{\#}$, we have $z_1 + z_2 = \sup\{k_1 + h_2 : k_1 \in L_K(z_1), h_2 \in U_H(z_2)\}$ in $K^{\#}$. According to (2′), $z_1 + z_2 \in K^{\#}$ holds. Let us form the set $V = \{x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\}$. From (2), we infer that $V \subseteq K$. Since X_1 is a nonempty upper bounded subset of K, and X_2 is a nonempty lower bounded subset of H, 2.4 (ii) yields that V is a nonempty upper bounded subset of K. Whence, there exists an element $v \in K^{\#}$, $v = \sup V$ in $K^{\#}$. We have to show that $z_1 + z_2 = v$. From $X_1 \subseteq L_K(z_1)$, $X_2 \subseteq U_H(z_2)$, it follows that $V \subseteq Z$, and so $v \leq z_1 + z_2$. Now, we prove that $z_1 + z_2 \leq v$, i.e., $U_K(v) \subseteq U_K(z_1 + z_2)$. Let $g \in U_K(v)$. Then $g \geq v$, and thus $g \geq x_1 + x_2$ for each $x_1 \in X_1$, $x_2 \in X_2$. Since $x_1 \in K$, by (1), $-x_1 \in K$, and we get $-x_1 + g \leq x_2$, $-x_1 + g \leq z_2 \leq h_2$ for each $h_2 \in U_H(z_2)$ and $g \geq x_1 + h_2$. Then (II) yields $g - h_2 \geq x_1$, $g - h_2 \geq z_1 \geq k_1$, $g \geq k_1 + h_2$ for each $k_1 \in L_K(z_1)$, $h_2 \in U_H(z_2)$. Therefore $g \geq z_1 + z_2$, and so $g \in U_K(z_1 + z_2)$.

If (b$_1$)–(b$_3$) are fulfilled, the proofs are similar. □

LEMMA 2.8. *Let $z_1, z_2 \in G^{\#}$, $z_1 \leq z_2$. Then*

(i) $z_1 + z \leq z_2 + z$ *for each* $z \in G^{\#}$,
(ii) $z + z_1 \leq z + z_2$ *for each* $z \in H^{\#}$,
(iii) $z + z_1 \geq z + z_2$ *for each* $z \in K^{\#}$.

Proof. We shall prove only (iii). Inequalities (i) and (ii) can be verified in a similar manner.

According to 2.3 and 2.2, both elements z_1 and z_2 belong either to $H^{\#}$ or to $K^{\#}$. Consider the case $z_1, z_2 \in H^{\#}$. Assume that $z \in K^{\#}$. From (2′), it follows that $z + z_1 \in K^{\#}$, $z + z_2 \in K^{\#}$. We have $z + z_1 = \sup\{k + h_1 : k \in L_K(z), h_1 \in U_H(z_1)\}$, $z + z_2 = \sup\{k + h_2 : k \in L_K(z), h_2 \in U_H(z_2)\}$ in $K^{\#}$. We have to prove that $z + z_1 \geq z + z_2$, i.e., that $U_K(z + z_1) \subseteq U_K(z + z_2)$. Let $g \in U_K(z + z_1)$. Hence $g \in K$, $g \geq z + z_1$, $g \geq k + h_1$. Since $k \in K$, we get $-k + g \leq h_1$ for each $h_1 \in U_H(z_1)$. Whence, $-k + g \leq z_1$. The hypothesis $z_1 \leq z_2$ implies that $-k + g \leq z_2$. Therefore $-k + g \leq h_2$. Because of $k \in K$, we get $g \geq k + h_2$ for each $k \in L_K(z)$, $h_2 \in U_H(z_2)$. We conclude that $g \geq z + z_2$, and so $g \in U_K(z + z_2)$. □

Assume that there exists an inverse $z' \in G^{\#}$ to $z \in G^{\#}$. Since $0 \in H$, it is easy to see that the following results hold true:

(1′) If $z \in H^{\#}$, then $z' \in H^{\#}$,
if $z \in K^{\#}$, then $z' \in K^{\#}$.

Remark 2.9. If $z \in G^{\#}$, then, in general, z need not have an inverse in $G^{\#}$ (see 3.5 (C)).

Let $M_h(G)$ $(I(K^{\#}))$ be the set of all elements of $G^{\#}$ $(K^{\#})$ possessing an inverse in $G^{\#}$. The set of all elements of $H^{\#}$ having an inverse in $G^{\#}$ (that is in $H^{\#}$) is the maximal Dedekind completion $M(H)$ of a partially ordered group H (cf. [3]).

The following lemma is an immediate consequence of 2.3.

Lemma 2.10. *$M_h(G)$ is a disjoint union of $M(H)$ and $I(K^{\#})$.*

By interchanging U_H (U_K) and L_H (L_K) in (a_1')–(a_4'), we get a set W instead of the set Z. With respect to 2.5, W is a nonempty and lower bounded subset of G. Then there exists $w \in G^{\#}$, $w = \inf W$.

Lemma 2.11. *Let $z_1, z_2 \in M_h(G)$. Then $z_1 + z_2 = w$.*

Proof. Let $z_1, z_2 \in M_h(G)$. From 2.10, we infer that z_1 (z_2) belongs either to $M(H)$ or to $I(K^{\#})$. Assume that $z_1 \in I(K^{\#})$, $z_2 \in M(H)$. Since $I(K^{\#}) \subseteq K^{\#}$ and $M(H) \subseteq H^{\#}$, we have $z_1 + z_2 = \sup\{k_1 + h_2 : k_1 \in L_K(z_1),\ h_2 \in U_H(z_2)\}$, $w = \inf\{k_1' + h_2' : k_1' \in U_K(z_1),\ h_2' \in L_H(z_2)\}$. Since $k_1 \le k_1'$ and $h_2' \le h_2$, we get $k_1 + h_2 \le k_1' + h_2'$. Hence $z_1 + z_2 \le w$. We have to verify that $w \le z_1 + z_2$, i.e., $L_K(w) \subseteq L_K(z_1 + z_2)$. Let $g \in L_K(w)$. Then $g \in K$, $g \le w$. Hence, $g \le k_1' + h_2'$ for each $k_1' \in U_K(z_1)$, $h_2' \in L_H(z_2)$. From (II), we infer that $g - h_2' \le k_1'$, and so $g - h_2' \le z_1$. According to 2.8 (i), we have $g \le z_1 + h_2'$. The assumption $z_1 \in M_h(G)$ implies that there exists an inverse to z_1 in $G^{\#}$. Then according to 2.8 (iii), $-z_1 + g \ge h_2'$ for each $h_2' \in L_H(z_2)$. Therefore $-z_1 + g \ge z_2$. Applying 2.8 (iii) again we obtain $g \le z_1 + z_2$, and so $g \in L_K(z_1 + z_2)$.

Proofs of the remaining cases are similar. □

Remark 2.12. If $z_1, z_2 \in G^{\#}$, then, in general, the elements $z_1 + z_2$ and w need not be equal (see 3.5 (B)).

Remark 2.13. Let $z_1, z_2 \in M_h(G)$. Then the dual lemma to 2.7 is also valid.

Lemma 2.14. *$(M_h(G), +)$ is a group.*

Proof. At first, we prove that the operation $+$ is associative, i.e., $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for each $z_1, z_2, z_3 \in M_h(G)$.

Let $z_1, z_2, z_3 \in M_h(G)$. According to 2.10, each of the elements z_1, z_2, z_3 belongs either to $M(H)$ or to $I(K^\#)$. Only two cases will be investigated. Proofs of the remaining cases are analogous.

Let $z_1 \in M(H)$, $z_2, z_3 \in I(K^\#)$. Since $M(H) \subseteq H^\#$, $I(K^\#) \subseteq K^\#$, with respect to 2.7(b$_2$), we obtain $(z_1 + z_2) + z_3 = \sup\{h_1 + k_2 : h_1 \in L_H(z_1),\ k_2 \in L_K(z_2)\} + \inf U_K(z_3) = \sup\{(h_1 + k_2) + k_3 : h_1 \in L_H(z_1),\ k_2 \in L_K(z_2),\ k_3 \in U_K(z_3)\} = \sup\{h_1 + (k_2 + k_3) : h_1 \in L_H(z_1),\ k_2 \in L_K(z_2),\ k_3 \in U_K(z_3)\}$. On the other hand, according to 2.7(b$_1$), we have $z_1 + (z_2 + z_3) = \sup L_H(z_1) + \sup\{k_2 + k_3 : k_2 \in L_K(z_2),\ k_3 \in U_K(z_3)\} = \sup\{h_1 + (k_2 + k_3) : h_1 \in L_H(z_1),\ k_2 \in L_K(z_2),\ k_3 \in U_K(z_3)\}$.

Now, let $z_1, z_2, z_3 \in I(K^\#)$. Then 2.7 (b$_3$) implies that $(z_1 + z_2) + z_3 = \sup\{k_1 + k_2 : k_1 \in L_K(z_1),\ k_2 \in U_K(z_2)\} + \sup L_K(z_3) = \sup\{(k_1 + k_2) + k_3 : k_1 \in L_K(z_1),\ k_2 \in U_K(z_2),\ k_3 \in L_K(z_3)\} = \sup\{k_1 + (k_2 + k_3) : k_1 \in L_K(z_1),\ k_2 \in U_K(z_2),\ k_3 \in L_K(z_3)\}$. In view of 2.11 and 2.7(b$_4$), we get $z_1 + (z_2 + z_3) = \sup L_K(z_1) + \inf\{k_2 + k_3 : k_2 \in U_K(z_2),\ k_3 \in L_K(z_3)\} = \sup\{k_1 + (k_2 + k_3) : k_1 \in L_K(z_1),\ k_2 \in U_K(z_2),\ k_3 \in L_K(z_3)\}$.

It remains to verify that $z_1 + z_2 \in M_h(G)$ whenever $z_1, z_2 \in M_h(G)$. Let $z_1, z_2 \in M_h(G)$. There are elements $z_1', z_2' \in M_h(G)$ with $z_1 + z_1' = z_1' + z_1 = 0$, $z_2 + z_2' = z_2' + z_2 = 0$. By using of associativity, we get $(z_1 + z_2) + (z_2' + z_1') = z_1 + (z_2 + z_2') + z_1' = 0$, $(z_2' + z_1') + (z_1 + z_2) = z_2' + (z_1' + z_1) + z_2 = 0$. Hence $z_2' + z_1'$ is an inverse to $z_1 + z_2$ in $G^\#$, and thus $z_1 + z_2 \in M_h(G)$. □

The partial order $\le$ is non-trivial on $M_h(G)$ because of $\le$ is a non-trivial partial order on G. From 2.8 (i), it follows that $\le$ is compatible from the right. From 2.8 (ii) and 2.8 (iii), we infer that $M_h(G)\uparrow = M(H)$ and $M_h(G)\downarrow = I(K^\#)$.

By using of 2.10, we have obtained the following result.

THEOREM 2.15. *Let G be a half partially ordered group. Then $M_h(G)$ is a half partially ordered group, and $M_h(G)\uparrow = M(H)$, $M_h(G)\downarrow = I(K^\#)$.*

A half partially ordered group $M_h(G)$ is said to be the *maximal Dedekind completion of G*.

In [1] (in [5; p. 162]), it was proved that the maximal Dedekind completion of a lattice ordered group (linearly ordered group) is a lattice ordered group (linearly ordered group). From this fact and from 2.15, it follows:

THEOREM 2.16. *Let G be a half lattice ordered group (half linearly ordered group). Then the maximal Dedekind completion $M_h(G)$ of G is a half lattice ordered group (half linearly ordered group).*

3. Inverse elements in $G^{\#}$

Elements of $G^{\#}$ having an inverse in $G^{\#}$ will be characterized in this section.

We shall use the notation $X_1 = L_H(z)$, $Y_1 = U_H(z)$ if $z \in H^{\#}$, and $X = L_K(z)$, $Y = U_K(z)$ if $z \in K^{\#}$. Further denote $-X = \{-x \in G : x \in X\}$. Symbols $-Y$, $-X_1$, $-Y_1$ have an analogous meaning.

LEMMA 3.1.

(i) *Assume that $z \in H^{\#}$. Then there exists $z' \in H^{\#}$ such that $z' = \sup(-Y_1) = \inf(-X_1)$.*

(ii) *Assume that $z \in K^{\#}$. Then there exists $z'' \in K^{\#}$ such that $z'' = \sup(-X) = \inf(-Y)$.*

Proof.

(ii) Let $z \in K^{\#}$. According to (5), we have $z = \sup X = \inf Y$. By using of (3), from $x \le y$, we get $-x \le -y$ for each $x \in X$, $y \in Y$. Hence there exist $z'', z^* \in K^{\#}$, $z'' = \sup(-X)$, $z^* = \inf(-Y)$. Since $z'' \le z^*$, we have to show that $z^* \le z''$, i.e., $U_K(z'') \subseteq U_K(z^*)$. Let $g \in U_K(z'')$. Then $g \ge z''$. Thus $g \ge -x$ and $-g \ge x$ for each $x \in X$. Hence $-g \ge z$, and so $-g \in Y$ and $g \in -Y$. From this, we infer that $g \ge z^*$ and $g \in U_K(z^*)$.

The proof of (i) is analogous. □

LEMMA 3.2. *Assume that the following conditions are fulfilled:*

(i) *If $z \in H^{\#}$, then $\bigwedge\{y_1 - x_1 : x_1 \in X_1,\ y_1 \in Y_1\} = 0$ in G.*
(ii) *If $z \in K^{\#}$, then $\bigvee\{x - y : x \in X,\ y \in Y\} = 0$ in G.*

Then z has a right inverse in $G^{\#}$.

Proof. Assume that $z \in K^{\#}$, and let z'' be as in 3.1 (ii). We want to show that z'' is a right inverse to z in $G^{\#}$. With respect to (2′), we obtain $z + z'' \in H^{\#}$, $z + z'' = \sup\{x + y : x \in X,\ y \in -Y\} = \sup\{x - y : x \in X,\ y \in Y\}$ in $G^{\#}$. The assumption implies that $\sup\{x - y : x \in X,\ y \in Y\} = 0$ in G. Hence, $\sup\{x - y : x \in X,\ y \in Y\} = 0$ in $G^{\#}$ as well. Therefore $z + z'' = 0$, and thus z'' is a right inverse to z in $G^{\#}$.

Assume that (i) is satisfied. In a similar manner, can be verified (cf. [1]) that z' is a right inverse to z in $G^{\#}$. □

Remark 3.3. In an analogical way, we obtain that z' (z'') is a left inverse to z in $G^{\#}$ whenever $\bigwedge\{-x_1 + y_1 : x_1 \in X_1,\ y_1 \in Y_1\} = 0$ ($\bigvee\{-x + y : x \in X,\ y \in Y\} = 0$) in G.

THEOREM 3.4.

(i) *Assume that* $z \in H^{\#}$. *Then* $z \in M_h(G)$ *if and only if the following conditions are satisfied in* G:

(c_1) $\bigwedge\{y_1 - x_1 :\ x_1 \in X_1,\ y_1 \in Y_1\} = 0$,

(c_1') $\bigwedge\{-x_1 + y_1 :\ x_1 \in X_1,\ y_1 \in Y_1\} = 0$.

(ii) *Assume that* $z \in K^{\#}$. *Then* $z \in M_h(G)$ *if and only if the following conditions are satisfied in* G:

(c_2) $\bigvee\{x - y :\ x \in X,\ y \in Y\} = 0$,

(c_2') $\bigvee\{-x + y :\ x \in X,\ y \in Y\} = 0$.

P r o o f.

(ii) Let $z \in K^{\#}$, and let both conditions (c_2) and (c_2') be satisfied. Then 3.2 and 3.3 yield that the element $z'' = \inf(-Y)$ is an inverse to z in $G^{\#}$. Hence $z \in M_h(G)$. Conversely, let $z \in M_h(G)$. Since $z \in K^{\#}$, $X \subseteq K$ and $Y \subseteq K$. In view of (3), from $x \leq y$ we infer that $-x \leq -y$ for each $x \in X$, $y \in Y$. Since x is decreasing, $x - y \leq 0$ for each $x \in X$, $y \in Y$. Let $g \in G$, $x - y \leq g$ for each $x \in X$, $y \in Y$. By (II), we get $x \leq g + y$ for each $x \in X$, and thus $z \leq g + y$. As for $g \in H$, by using of 2.8 (ii), $-g + z \leq y$ holds for each $y \in Y$, and so $-g + z \leq z$. The assumption $z \in M_h(G)$ implies that there is an inverse to z in $G^{\#}$. According to 2.8 (i), we get $-g \leq 0$ and $g \geq 0$. We conclude that $\bigvee\{x - y :\ x \in X,\ y \in Y\} = 0$ in G, and (c_2) is valid. The proof of (c_2') is analogous.

(i) can be proved in a similar manner (cf. [1]). □

The question of the independence of the conditions (c_1) and (c_1') ((c_2) and (c_2')) remains open.

EXAMPLE 3.5. Let C be the additive group of all integers with the natural linear order, and let H be the lexicographic product $H = C \circ C$. If $h, h' \in H$, $h = (c_1, c_2)$, $h' = (c_1', c_2')$, $c_i, c_i' \in C$ ($i = 1, 2$), then $h \leq h'$ if and only if $c_1 < c_1'$ or $c_1 = c_1'$ and $c_2 \leq c_2'$. The operation $+$ in H is defined componentwise. H is a linearly ordered group.

We apply the idea of the proof of Lemma III.3 from [4] to construct a half linearly ordered group G with $G\uparrow = H$ that is not a linearly ordered group.

Let a be a symbol, and let $a + H$ be the set of symbols $a + H = \{a + h : h \in H\}$. Denote by G a (disjoint) union of G and $a + H$. The operation $+$ and the order $\leq$ on H will be extended on the whole G in the following way. For each $h, h' \in H$ we put $(a + h) + (a + h') = -h + h'$, $h + (a + h') = a + (-h + h')$, $(a + h) + h' = a + (h + h')$. Further we put $a + h \leq a + h'$ if and only if $h' \leq h$, $a + h$ and h' incomparable. Then G turns into a half linearly ordered group such that $G\uparrow = H$, $G\downarrow = a + H$. Since $G\uparrow \neq \emptyset$, G fails to be a linearly ordered group.

Form the sets:

$$\begin{aligned} X_1 &= \{(b_1, c) \in H : b_1 \in C, \ b_1 \le 0, \ c \in C\}, \\ Y_1 &= \{(c_1, c) \in H : c_1 \in C, \ c_1 \ge 1, \ c \in C\}, \\ X_2 &= \{(b_2, c) : b_2 \in C, \ b_2 \le 1, \ c \in C\}, \\ Y_2 &= \{(c_2, c) : c_2 \in C, \ c_2 \ge 2, \ c \in C\}, \\ X_3 &= \{(b_3, c) \in H : b_3 \in C, \ b_3 \le 2, \ c \in C\}, \\ Y_3 &= \{(c_3, c) \in H : c_3 \in C, \ c_3 \ge 3, \ c \in C\}. \end{aligned}$$

We have $x_i \le y_i$ for each $x_i \in X_i$, $y_i \in Y_i$, $(i = 1,2,3)$. Therefore there exist elements $v_1, v_2, v_3 \in H^{\#}$ such that $v_i = \sup X_i = \inf Y_i$ $(i = 1,2,3)$ in $H^{\#}$, and $X_i = L_H(v_i)$, $Y_i = U_H(v_i)$ $(i = 1,2,3)$. From $a+x_i, a+y_i \in a + H$, $a + y_i \le a + x_i$ for each $x_i \in X_i$, $y_i \in Y_i$ $(i = 1,2,3)$ it follows that there are elements $z_1, z_2, z_3 \in (a+H)^{\#}$ such that $z_i = \sup\{a+y_i : y_i \in Y_i\} = \inf\{a+x_i : x_i \in X_i\}$ $(i = 1,2,3)$ in $(a + H)^{\#}$, and $\{a + y_i : y_i \in Y_i\} = L_{a+H}(z_i)$, $\{a + x_i : x_i \in X_i\} = U_{a+H}(z_i)$ $(i = 1,2,3)$.

(A) We get $z_1 + z_2 = \sup\{(a + y_1) + (a + x_2) : y_1 \in Y_1, \ x_2 \in X_2\} = \sup\{-y_1 + x_2 : y_1 \in Y_1, \ x_2 \in X_2\} = \sup X_1 = v_1$ in $H^{\#}$; $(z_1 + z_2) + z_3 = v_1 + z_3 = \sup\{x_1 + (a + y_3) : x_1 \in X_1, \ y_3 \in Y_3\} = \sup\{a + (-x_1 + y_3) : x_1 \in X_1, \ y_3 \in Y_3\} = \sup\{a + y_3 : y_3 \in Y_3\} = z_3$. On the other hand, $z_2 + z_3 = \sup\{(a+y_2)+(a+x_3) : y_2 \in Y_2, \ x_3 \in X_3\} = \sup\{-y_2+x_3 : y_2 \in Y_2, \ x_3 \in X_3\} = \sup X_1 = v_1$; $z_1 + (z_2 + z_3) = z_1 + v_1 = \sup\{(a + y_{1i}) + y_{1j} : y_{1i}, y_{1j} \in Y_1\} = \sup\{a+(y_{1i}+y_{1j}) : y_{1i}, y_{1j} \in Y_1\} = \sup\{a+y_2 : y_2 \in Y_2\} = z_2$. Hence, $(z_1 + z_2) + z_3 \neq z_1 + (z_2 + z_3)$.

(B) We have seen in (A) that $z_1 + z_2 = v_1$. But $w = \inf W = \inf\{(a + x_1) + (a+y_2) : x_1 \in X_1, \ y_2 \in Y_2\} = \inf\{-x_1+y_2 : x_1 \in X_1, \ y_2 \in Y_2\} = \inf Y_2 = v_2$. Therefore $z_1 + z_2 \neq w$.

(C) There does not exist $\bigwedge\{y_1 - x_1 : x_1 \in X_1, \ y_1 \in Y_1\}$ in G. With respect to 3.4 (i) the element $v_1 \in G^{\#}$ has no inverse in $G^{\#}$.

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Department of Mathematics
Faculty of Civil Engineering
Technical University
Vysokoškolská 4
SK-042 02 Košice
SLOVAKIA