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Tadeusz Wesołowski

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SUBDIRECTLY IRREDUCIBLE DECOMPOSITION OF SOME ALGEBRAS HAVING THE SEMILATTICE STRUCTURE

TADEUSZ WESOŁOWSKI

0. In this paper we consider algebras of type $\tau\colon \{+, \cdot\} \to \mathbb{N}$, where $\tau(+) = \tau(\cdot) = 2$. Denote by $\boldsymbol{D}$ the variety of all distributive lattices of type τ and by $\boldsymbol{S}_0$ the variety of all algebras of type τ satisfying the following identities:

(1) $x \cdot y = z \cdot t$;

(2) $x + (x \cdot y) = x$;

(3) identities which define $+$ — semilattices.

In [5] algebras from the join $\boldsymbol{D} \vee \boldsymbol{S}_0$ of varieties $\boldsymbol{D}$ and $\boldsymbol{S}_0$ were studied. In particular, the following facts were proved there:

(i) identities (2), (3) and the following identities (4) — (7):

(4) $x \cdot y = y \cdot x$;

(5) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

(6) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$;

(7) $(x \cdot x) \cdot y = x \cdot y$,

form an equational base of $\boldsymbol{D} \vee \boldsymbol{S}_0$;

(ii) if $\mathscr{A} = (A; +, \cdot) \in \boldsymbol{D} \vee \boldsymbol{S}_0$, then the mapping $h\colon A \to A$ defined by the formula:

$$h(x) = x \cdot x \qquad \text{for} \quad x \in A$$

is a retraction of $\mathscr{A}$ such that $(h(A); +, \cdot)$ is a distributive lattice, $h(x) \leq x$ and $x \cdot y = h(x) \cdot h(y)$ for all $x, y \in A$.

In this paper we describe all subdirectly irreducible algebras from $\boldsymbol{D} \vee \boldsymbol{S}_0$. In order to attain this we shall use the notion of a disjunctive lattice, which was introduced in [4] as an utilization of the notion of a disjunctive poset for lattices (cf in [1], [3]).

Let us recall that a lattice $\mathscr{L} = (L; +, \cdot)$ with the least element $0 \in L$ is called disjunctive if for all $a, b \in L$ the following condition holds:

(iii) if $a < b$, then there exists $c \in L \setminus \{0\}$ such that $c \leq b$ and $a \cdot c = 0$.

Lemma 1. *Let $\mathscr{L} = (L; +, \cdot)$ be a distributive lattice with the least element $0 \in L$. Then $\mathscr{L}$ is disjunctive iff for each nontrivial congruence Θ of $\mathscr{L}$ there exists $c \in L \setminus \{0\}$ such that $c \equiv 0(\Theta)$.*

Proof. ($\Rightarrow$). It was proved in [4].
($\Leftarrow$). Let $a < b$ for $a, b \in L$. Then the principial congruence $\Theta(a, b)$ of $\mathscr{L}$ is not trivial, so $c \equiv 0(\Theta(a, b))$ for some $c \in L \setminus \{0\}$. Using the G. Gratzer—E. T. Schmidt theorem (cf [2], p. 74) we have $a \cdot c = a \cdot 0 = 0$ and $b + c = b + 0 = b$.

1. It is known that each nondegenerated subdirectly irreducible member of $\boldsymbol{D}$ is isomorphic to the two-element lattice $\mathbf{2} = (\{0, 1\};\ +,\ \cdot)$, where $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ for $a, b \in \{0, 1\}$. Similarly, each nondegenerated subdirectly irreducible member of $\boldsymbol{S}_0$ is isomorphic to the algebra $\bar{\mathbf{2}} = (\{0, 1\};\ +,\ \cdot)$, in which $a + b = \max\{a, b\}$ and $a \cdot b = 0$ for a, $b \in \{0, 1\}$. In fact, if $\mathscr{A} = (A;\ +,\ \cdot) \in \boldsymbol{S}_0$, then the reduct $(A;\ +)$ of $\mathscr{A}$ is a semilattice and congruences of $(A;\ +)$ and $\mathscr{A}$ coincide.

Of course, algebras $\mathbf{2}$ and $\bar{\mathbf{2}}$ are examples of subdirectly irreducible members of $\boldsymbol{D} \vee \boldsymbol{S}_0$. For another example let us consider a distributive disjunctive lattice $\mathscr{L} = (L;\ \oplus,\ \odot)$ with the least element $0 \in L$ and let us put $L_e = L \cup \{e\}$, where $e \notin L$. Now we define on L_e two binary operations $+$ and $\cdot$ as follows. If $a, b \in L$, then $a + b = a \oplus b$ and $a \cdot b = a \odot b$. If $a \in L_e \setminus \{0\}$, then $a + e = e + a = a$. Finally we put $0 + e = e + 0 = e$ and $a \cdot e = e \cdot a = 0$ for each $a \in L_e$. It is easy to check that the algebra $\mathscr{L}_e = (L_e;\ +,\ \cdot)$ satisfies identities (2) – (7), so by (i), $\mathscr{L}_e \in \boldsymbol{D} \vee \boldsymbol{S}_0$. Observe that L is a subalgebra of $\mathscr{L}_e$ and $L = h(L_e)$, where h is a retraction of $\mathscr{L}_e$ defined in (ii). Indeed, for $x \in L_e$ we have $h(x) = x$ for $x \in L$ and $h(e) = 0$. Below, the operations $\oplus$ and $\odot$ will be denoted by $+$ and $\cdot$, respectively.

Theorem 1. *If $\mathscr{L} = (L;\ +,\ \cdot)$ is a distributive disjunctive lattice and $e \notin L$, then the algebra $\mathscr{L}_e$ is subdirectly irreducible.*

Proof. Let $\sim$ be the kernel of h. We have $[0]_\sim = \{0, e\}$ and $[a]_\sim = \{a\}$ for each $a \in L \setminus \{0, e\}$. It means that $\sim$ is an atom in the lattice of all congruences of $\mathscr{L}_e$. If $\mathscr{L}_e$ is subdirectly reducible, then there exists a nontrivial congruence Θ of $\mathscr{L}_e$ such that $\sim \cap\, \Theta = \omega_{L_e}$. Hence $0 \not\equiv e(\Theta)$ and the restriction Θ_1 of Θ to the subalgebra L of $\mathscr{L}_e$ is a nontrivial congruence of $\mathscr{L}$. Therefore, by Lemma 1 there exists $c \in L \setminus \{0\}$ such that $c \equiv 0(\Theta_1)$. Then $c \equiv 0(\Theta)$ and consequently $c = c + e \equiv 0 + e = e(\Theta)$. Thus $e \equiv 0(\Theta)$ – a contradiction.

Note that the algebra $\bar{\mathbf{2}}$ is of the form $\mathbf{1}_e$, where $\mathbf{1} = (\{0\};\ +,\ \cdot)$ is the one-element disjunctive lattice and $e = 1$.

2. For an algebra $\mathscr{A} = (A;\ +,\ \cdot) \in \boldsymbol{D} \vee \boldsymbol{S}_0$ denote by h the retraction of $\mathscr{A}$ defined in (ii). Let $\mathscr{L}^h$ denote the distributive lattice $(h(A);\ +,\ \cdot)$ and let $\sim$ be the kernel of h. Assume that 0 is the least element of $\mathscr{A}$.

Lemma 2. (a). *If $u \in A$, then $[u]_\sim$ is a subalgebra of $\mathscr{A}$ and $([u]_\sim;\ +,\ \cdot) \in \boldsymbol{S}_0$;*
(b) *Each congruence Θ of $([0]_\sim;\ +,\ \cdot)$ can be extended to some congruence Θ^* of $\mathscr{A}$;*

(c). *If* $x \in h(A)$, *then the relation* $\varrho_x \subseteq A \times A$ *defined as follows*:

$$a\varrho_x b \quad \textit{iff} \quad a + x = b + x$$

is a congruence of $\mathscr{A}$. *Moreover,* ϱ_x *is trivial iff* $x = 0$.

Proof. (a). Since $(A;\ +)$ is a semilattice, $[u]_\sim$ is closed under the operation $+$. Further, if $x, y \in [u]_\sim$, then $h(x \cdot y) = h(x) \cdot h(y) = h(u) \cdot h(u) = h(u)$ and $x \cdot y = h(x) \cdot h(y) = h(u)$. Thus $x \cdot y \in [u]_\sim$ and the algebra $([u]_\sim;\ +,\ \cdot)$ satisfies (1).

(b). For a congruence Θ of $([0]_\sim;\ +,\ \cdot)$ we define a relation $\Theta^* \subseteq A \times A$ putting

$$x \equiv y(\Theta^*) \quad \text{iff} \quad x \sim y \quad \text{and} \quad x \equiv y(\Theta) \quad \text{if} \quad x, y \in [0]_\sim.$$

We see that Θ^* is an equivalence on A. Let $a \equiv b(\Theta^*)$ and $c \equiv d(\Theta^*)$. Then $a \cdot c = h(a) \cdot h(c) = h(b) \cdot h(d) = b \cdot d$, so $a \cdot c \equiv b \cdot d(\Theta^*)$. Now observe that if $h(x + y) = 0$, then $h(x) = 0$ and $h(y) = 0$. Therefore, if $a, b \notin [0]_\sim$ or $c, d \notin [0]_\sim$, then $a + c \notin [0]_\sim$ and $b + d \notin [0]_\sim$. Hence $a + c \equiv b + d(\Theta^*)$. If $a, b, c, d \in [0]_\sim$, then $a \equiv b(\Theta)$ and $c \equiv d(\Theta)$, so $a + c \equiv b + d(\Theta)$ and consequently $a + c \equiv \equiv b + d(\Theta^*)$.

(c). Let $x \in h(A)$. Obviously ϱ_x is an equivalence on A. Let $a \equiv b(\varrho_x)$ and $c \equiv d(\varrho_x)$. Then $a + c \equiv b + d(\varrho_x)$ and $(a \cdot c) + x = (h(a) \cdot h(c)) + h(x) = = (h(a) + h(x)) \cdot (h(c) + h(x)) = h((a + x) \cdot (c + x)) = h((b + x) \cdot (d + x)) = = (b \cdot d) + x$. Thus $a \cdot c \equiv b \cdot d(\varrho_x)$. If $x = 0$, then $\varrho_x = \omega_A$. On the other hand we have $a + x \equiv a(\varrho_x)$ for each $a \in A$. Therefore, if $\varrho_x = \omega_A$, then $a + x = a$, so $x = 0$.

Lemma 3. *If an algebra* $\mathscr{A} = (A;\ +,\ \cdot) \in \boldsymbol{D} \vee \boldsymbol{S}_0$ *is subdirectly irreducible and* $\sim\ \neq \omega_A$, *then the lattice* $\mathscr{L}^h$ *is disjunctive and there exists* $e \in A \setminus h(A)$ *such that* $\mathscr{A} = \mathscr{L}^h_e$.

Proof. Observe that the lattice $\mathscr{L}^h$ has the least element $0 \in A$. Indeed, otherwise all relations ϱ_x, $x \in h(A)$ from Lemma 2(c) are nontrivial congruences of $\mathscr{A}$. If $a \equiv b(\bigcap \{\varrho_x\colon x \in h(A)\})$ for $a, b \in A$, then $a \equiv b(\varrho_{a \cdot b})$ since $a \circ b \in h(A)$. Hence $a = a + (a \cdot b) = b + (a \cdot b) = b$ – a contradiction.

Put $B = \{x \in A \setminus [0]_\sim : |[x]_\sim| > 1\}$ and $\mathscr{F} = \{\varrho_{h(x)} : x \in B\}$. For each congruence Θ of the algebra $([0]_\sim;\ +,\ \cdot)$ denote by Θ^* the extension of Θ from Lemma 2(b). Let $\mathscr{D}^* = \{\Theta^* : \Theta \in \mathscr{D}\}$, where $\mathscr{D}$ is the family of all congruences of $([0]_\sim;\ +,\ \cdot)$.

We see that if $B \neq \emptyset$, then families $\mathscr{F}$ and $\mathscr{D}^*$ are not empty and $\sim\ \in \mathscr{D}^*$, since $\sim$ is the extension of $[0]_\sim \times [0]_\sim \in \mathscr{D}$. Further, all congruences from the family $\mathscr{H} = \mathscr{F} \cup \mathscr{D}^*$ are not trivial. Let $a \equiv b(\bigcap \mathscr{H})$ and $a \neq b$ for $a, b \in A$. Then $a \sim b$, i.e. $h(a) = h(b)$. If $h(a) = h(b) \neq 0$, then $a, b \in B$ and $a \equiv b(\varrho_{h(a)})$. Hence $a = = a + h(a) = b + h(a) = b + h(b) = b$ — a contradiction. If $h(a) = h(b) = 0$, then $a, b \in [0]_\sim$ and $a \equiv b(\Theta^*)$ for each $\Theta \in \mathscr{D}$. In particular, $a \equiv b(\omega^*_{[0]_\sim})$, so $a \equiv b(\omega_{[0]_\sim})$ — a contradiction.

We have proved $B = \emptyset$. It can be easily verified that the algebra $([0]_\sim; +, \cdot) \in \boldsymbol{S}_0$ is subdirectly irreducible. Therefore, $|[0]_\sim| = 2$. Hence $A \setminus h(A) = \{e\}$ for some $e \in A$. It means that the set $\{0, e\}$ is the only one nondegenerated congruence class of $\sim$, so $\sim$ is the atom in the lattice of all congruences of $\mathscr{A}$. We have:

(iv) $$a \cdot e = 0 \quad \text{for all} \quad a \in A$$

since $a \cdot e = h(a) \cdot h(e) = h(a) \cdot 0 = 0$. Further,

(v) $$a + e = a \quad \text{for all} \quad a \in A \setminus \{0\}.$$

In fact, $e + e = e$ and $a \in h(A)$ for $a \in A \setminus \{0, e\}$. Hence the congruence ϱ_a of $\mathscr{A}$ is not trivial, so $\sim \subseteq \varrho_a$. Thus $0 \equiv e(\varrho_a)$, which gives (v).

It follows from (iv) and (v) that $\mathscr{A} = \mathscr{L}^h_e$. To prove that the lattice $\mathscr{L}^h$ is disjunctive we use Lemma 1. Of course, if $\mathscr{L}^h$ has exactly one element, then it is disjunctive. Let $|h(A)| > 1$ and Θ be a nontrivial congruence of $\mathscr{L}^h$. Let us assume that $[0]_\Theta = \{0\}$. Then the relation $\Theta_e = \Theta \cup \{\langle e, e \rangle\}$ is a congruence of $\mathscr{A}$. Indeed, let $a \equiv b(\Theta_e)$ and $c \equiv d(\Theta_e)$ for a, b, c, $d \in A$. If $\langle a, b \rangle \in \Theta$ and $\langle c, d \rangle \in \Theta$ or $\langle a, b \rangle = \langle c, d \rangle = \langle e, e \rangle$, then obviously $a \cdot c \equiv b \cdot d(\Theta_e)$ and $a + c \equiv b + d(\Theta_e)$. If $\langle a, b \rangle \in \Theta$ and $c = d = e$, then by (iv) we have: $a \cdot c = a \circ e = 0 = b \cdot e = b \circ d$, so $a \circ c \equiv b \circ d(\Theta_e)$. If $a = 0$, then also $b = 0$ and $a + c \equiv b + d(\Theta_e)$. For $a \neq 0$ we have $b \neq 0$ and by (v), $a + c = a + e = a$ and $b + d = b + e = b$. Hence $a + c \equiv b + d(\Theta_e)$. Then congruence Θ_e is not trivial, so $\sim \subseteq \Theta_e$. Thus $0 \equiv e(\Theta_e)$ — a contradiction. Therefore $|[0]_\Theta| > 1$, which ends the proof of the Lemma.

Theorem 2. *If an algebra $\mathscr{A} = (A; +, \cdot) \in \boldsymbol{D} \vee \boldsymbol{S}_0$ is subdirectly irreducible and $|A| > 1$, then $\mathscr{A} \cong \mathbf{2}$ or there exists a distributive disjunctive lattice $\mathscr{L} = (L; +, \cdot)$ and an element $e \notin L$ such that $\mathscr{A} = \mathscr{L}_e$.*

Proof. If $\sim = \omega_A$, then $h(A) = A$. Hence $\mathscr{A} \in \boldsymbol{D}$ and $\mathscr{A} \cong \mathbf{2}$. If $\sim \neq \omega_A$, we use Lemma 3.

It was proved in [5] that the varieties $\boldsymbol{D} \vee \boldsymbol{S}_0$, $\boldsymbol{D}$, $\boldsymbol{S}_0$ and the trivial variety $\boldsymbol{T}$ of type τ are the only subvarieties of $\boldsymbol{D} \vee \boldsymbol{S}_0$. Therefore we have

Corollary. If e is not a member of $\mathbf{2}$, then the algebra $\mathbf{2}_e$ generates the variety $\boldsymbol{D} \vee \boldsymbol{S}_0$.

Proof. Obviously, the lattice $\mathbf{2}$ is disjunctive, so $\mathbf{2}_e$ is a subdirectly irreducible member of $\boldsymbol{D} \vee \boldsymbol{S}_0$. Let $\boldsymbol{K} = \mathrm{HSP}(\mathbf{2}_e)$. Then $\boldsymbol{K} \subseteq \boldsymbol{D} \vee \boldsymbol{S}_0$. But $\boldsymbol{K} \neq \boldsymbol{D}$ since $\mathbf{2}_e \notin \boldsymbol{D}$ and $\boldsymbol{K} \neq \boldsymbol{S}_0$ since $\mathbf{2}_e \notin \boldsymbol{S}_0$. Thus $\boldsymbol{K} = \boldsymbol{D} \vee \boldsymbol{S}_0$.

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Wyższa Szkoła Pedagogiczna
im. Powstańców Śląskich
skrytka pocztowa nr 313
PL-45-951 OPOLE 1
POLAND

ПОДПРЯМО НЕРАЗЛОЖИМОЕ РАЗБИТИЕ НЕКОТОРЫХ АЛГЕБР С ПОЛУРЕШЁТОЧНОЙ СТРУКТУРОЙ

Tadeusz Wesołowski

Резюме

В работе исследуется объединение двух многообразий алгебр с полурешёточной структурой. Получено описание всех подпрямо неразложимых алгебр из рассматриваевого класса и доказано, что он порождается трёхэлементной алгеброй.