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*Mathematica Slovaca*, Vol. 37 (1987), No. 4, 323--340

Persistent URL: http://dml.cz/dmlcz/129507

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# ORDERING OF OBSERVABLES AND CHARACTERIZATION OF CONDITIONAL EXPECTATION

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In the first half of this paper we study various ways of the ordering of observables. We analyse the relationship between two different definitions of the ordering of observables. In the second half we analyse properties of "relative conditional expectations" for partially compatible observables on quantum logics. These "relative conditional expectations" have been introduced in [14]. The main result is a characterization of "relative conditional expectations" in the sense of Shu-Ten Chen Moy [19] in a quantum logic.

## Preliminaries

Let $L$ be a *logic*, (= an *orthomodular $\sigma$-lattice*). The elements $a, b \in L$ are *orthogonal* ($a \perp b$) if $a \leqslant b^{\perp}$. The elements $a, b \in L$ are *compatible* ($a \leftrightarrow b$) if $a = (a \wedge b) \vee (a \wedge b^{\perp})$, $b = (a \wedge b) \vee (a^{\perp} \wedge b)$. A subset $K \subset L$ is compatible if $a \leftrightarrow b$ for any $a, b \in K$ (see [20]).

**Definition 1.1.** *A subset $M \subset L$ is partially compatible with respect to an element $a \in L$ ($M$ is* p.c. $[a]$) *if*

(*i*) $M \leftrightarrow a$ (*i.e. $b \leftrightarrow a$ for all $b \in M$*);

(*ii*) $M \wedge a = \{b \wedge a | b \in M\}$ *is a compatible subset of $L$.*

Let $a \in L$, $a \neq 0$. The set $L_{[0,a]} = \{b \in L | b \leqslant a\}$ is a logic with the orthocomplementation defined by $b^* = b^{\perp} \wedge a$.

A *set $M \wedge a$ is compatible* in $L$ iff it is compatible in $L_{[0,a]}$.

If $F = \{a_1, \dots, a_n\} \subset L$, put

$$com(F) = \bigvee_{d \in D^n} a_1^{d_1} \wedge \dots \wedge a_n^{d_n}$$

where $D = \{0, 1\}$, $d = (d_1, \dots, d_n)$, $a^0 = a^{\perp}$, $a^1 = a$. The set $F$ if p.c. $[com(F)]$. $F$ is compatible iff $com(F) = 1$.

Let $M \subset L$ be any subset. If there is $\wedge\{com(F)|F$ is a finite subset of $M\}$ (briefly $com(M)$), then the element $com(M)$ is called the *commutator* of the set $M$. If $com(M)$ exists, then $M$ is p.c. $[com(M)]$ (see [17], [18]).

A mapping $x: L_1 \to L_2$ between logics $L_1$, $L_2$ is called a *σ-homomorphism* if it satisfies the following conditions: (i) $x(1_{L_1}) = 1_{L_2}$; (ii) if $a$, $b \in L_1$, $a \perp b$, then $x(a) \perp x(b)$; (iii) $\{a_i\}_{i=1}^{\infty} \subset L_1$ are mutually orthogonal; then $x(\vee a_i) = \vee x(a_i)$. A σ-homomorphism $x$: $B(R) \to L$ is called an *observable on* $L$ ($B(R)$ is the σ-algebra of Borel sets on the real line). If $f$ is a Borel measurable function on $R$ and $x$ is an observable on $L$, then $f \circ x$ (for $E \in B(R): f \circ x(E) = x(f^{-1}(E))$ is also an observable on $L$. The *range* $R(x) = \{x(E)|E \in B(R)\}$ is a Boolean-sub-σ-algebra of $L$. The *spectrum* $\sigma(x)$ of an observable $x$ is the smallest closed set $C \subset R$ such that $x(C) = 1$. The observable $x$ is *bounded* if $\sigma(x)$ is compact. For any $a \in L$, there is an observable $x_a$ such that $\sigma(x_a) \subset \{0, 1\}$ and $x_a(\{1\}) = a$. The observable $x_a$ is called a *proposition observable.*

If $x$ is an observable, and $a \in L$, we write $x \leftrightarrow a$ if $x(E) \leftrightarrow a$ for any $E \in B(R)$. If $x$, $y$ are observables, then $x \leftrightarrow y$ iff $x(E) \leftrightarrow y(F)$ for all $E$, $F \in B(R)$. If $x \leftrightarrow a$, then the map $x \wedge a: B(R) \to L_{[0, a]}$ $(x \wedge a(E) = x(E) \wedge a$, $E \in B(R))$ is an observable on a logic $L_{[0, a]}$.

Observables $x$, $y$ are said to be *simultaneous* ($x \leftrightarrow y$) if $R(x) \leftrightarrow R(y)$ (i.e. $a \leftrightarrow b$ for any $a \in R(x)$, $b \in R(y)$). Observables $x$, $y$ *are* p.c. $[a]$ ($a \in L$, $a \neq 0$) if $R(x) \cup \cup R(y)$ is p.c. $[a]$.

The mapping $m: L \to R$ is called a *measure* on $L$ if (i) $m(0) = 0$; (ii) $\{b_i\}_{i=1}^{\infty} \subset \subset L$ are mutually orthogonal elements; then $m(\vee b_i) = \sum_i m(b_i)$. If $m: L \to [0, 1]$ and $m(1) = 1$, then the measure $m$ is called a *state* on $L$. Let $m$, $n$ be measures on $L$. If $m(b) = 0$ implies $n(b) = 0$, then we write $n \ll m$ ($n$ is *absolutely continuous* to $m$).

Let $x$ be an observable on $L$ and let $m$ be a state on $L$. Then $m_x$: $E \mapsto \mapsto m(x(E))$ for $E \in B(R)$ is a *probability measure* on $B(R)$.

*The expectation* of $x$ in a state $m$ is defined by the formula

$$m(x) = \int x \, \mathrm{d}m = \int \lambda m_x(\mathrm{d}\lambda)$$

if the later integral exists. If $f$ is a Borel function, then

$$m(f(x)) = \int f(\lambda) \, m_x(\mathrm{d}\lambda).$$

It is obvious that

$$m(x) = \int_{\sigma(x)} \lambda m_x(\mathrm{d}\lambda).$$

An observable $x$ on $L$ is called *integrable* in a state $m$ if $m(x)$ exists and it is finite.

If $\sigma(x) \subseteq [0, \infty)$, then $x$ is called a *positive observable* (abbr. $x \geqslant 0$). If $m(x([0, \infty))) = 1$, then we write $x \geqslant 0$ $[m]$ ($m$ is a state on $L$).

Let $M$ be a set of states on $L$. The pair $(L, M)$ is called a *full system* (abbr. *f.s.*) if $m(a) \leqslant m(b)$ for any $m \in M$ implies $a \leqslant b$. The pair $(L, M)$ is a *quite full system* (abbr. *q.f.s.*) if $\{m \in M | m(a) = 1\} \subset \{m \in M | m(b) = 1\}$ implies $a \leqslant b$. S. Gudder [6] showed that if $(L, M)$ is a *q.f.s.*, then $(L, M)$ is an *f.s.* with the following property: If $a \neq 0$, $a \in L$, then there is $m \in M$ such that $m(a) = 1$.

Let $(L, M)$ be *q.f.s.* We say that $L$ has the *property U* if $m(x) = m(y)$ for all $m \in M$ implies $x = y$, where $x$, $y$ are bounded observables on $L$. We say that $L$ has the *property E* if for any pair $x$, $y$ of bounded observables there is a unique bounded observable $z$ such that $m(z) = m(x) + m(y)$ for any $m \in M$. The observable $z$ is called the *sum of observables* $x$, $y$ and we write $z = x + y$. For details see [3], [6], [4], [16]. A pair $(L, M)$ is called a *sum logic* if it is *q.f.s.* and $L$ has the properties $U$ and $E$.

For bounded summable observables let us define *the Segal "product"* by putting

$$x \cdot y = \frac{1}{2}((x + y)^2 - (x - y)^2).$$

Note that if $x \leftrightarrow y$, then there are Borel measurable functions $f$, $g$ and an observable $z$ such that $f \circ z = x$, $g \circ z = y$ (see [20]). Then we have

$$x \cdot y = \frac{1}{4}((f + g)^2 - (f - g)^2) \circ z.$$

Hence for any $E \in B(R)$, we obtain the equality $x \cdot y(E) = (f \circ z \cdot g \circ z)(E) = (f \cdot g) \circ z(E)$. Thus we have $x \cdot y = (f \cdot g) \circ z$.

## 2. Order properties of observables

Recall first that the ordering of observables was considered in [1], [21], [10]. D. Catlin [1] gave the definition of *spectral resolution* $e^x$ such that for each $r \in R$, $e^x(r) = x(-\infty, r)$ and $x \leqslant y$ iff $e^x \geqslant e^y$. S. Gudder and J. Zerbe ([10]) introduced an ordering in the following way. One writes $x \leqslant y$ if for each $r \in R$ $x(r, \infty) \leqslant y(r, \infty)$. Moreover, they introduced an *ordering of observables "modulo" a state m* in the following way: $x \leqslant y$ $[m]$ if $m(x(r, \infty)) \leqslant m(y(r, \infty))$ for all $r \in R$. They proved the following theorem.

**Theorem 2.1.** (Lemma 3.5., [10]). *If $x \leqslant y$ $[m]$ and $m(x)$, $m(y)$ exist, then $m(x) \leqslant m(y)$.*

If $x \leqslant y$ $[m]$ in the sense of S. Gudder and J. Zerbe and $x \leftrightarrow y$ and if we put $x = f \circ z$, $y = g \circ z$, the inequality $f \leqslant g$ a.e. $[m_z]$ need not hold as the following example shows.

Example 2.1. Let $\Omega = \{0, 1\}$, $L = 2^{\Omega}$ and $f(\omega) = \omega$ for $\omega \in \Omega$, $g(0) = 1$, $g(1) = 0$. Let $x = f^{-1}$, $y = g^{-1}$. It is clear that $x \leftrightarrow y$. Let $m$ be a state determined by putting $m(\{1\}) = 1/3$.

If $r \geqslant 1$, then $m(x(r, \infty)) = 0 = m(y(r, \infty))$. If $r \in [0, 1)$, then $m(x(r, \infty)) = m(\{1\}) = 1/3$, $m(y(r, \infty)) = m(\{0\}) = 2/3$. If $r < 0$, then $m(x(r, \infty)) = m(y(r, \infty)) = 1$. Hence $x \leqslant y$ $[m]$ in the sense of S. Gudder and J. Zerbe. On the other hand, $x = f \circ x_1$, $y = g \circ x_1$. If $f \leqslant g$ a.e. $[m_{x_1}]$, then $m_{x_1}(\{\omega \in \Omega | f > g\}) = 0$. But $m_{x_1}(\{\omega \in \Omega | f > g) = 1/3$. Hence $f \nleqslant g$ $[m_{x_1}]$.

This example contradicts the remark following Theorem 3.7 in [10], by which $m(x(r, \infty)) \leqslant m(y(r, \infty))$ for all $r \in R$ $(x \leftrightarrow y)$ would imply $y - x \geqslant 0$ $[m]$.

For that reason we define the ordering of observables in the following way:

**Definition 2.1.** *Let $L$ be a logic, $x$, $y$ be some observables on $L$. We define $\leqslant_1$ for observables as follows:*

(*i*) *if $m$ is a state on $L$, then $x \leqslant_1 y$ $[m]$ if for each $r \in R$*
$m(x(-\infty, r) \wedge y(-\infty, r)) = m(y(-\infty, r))$;

(*ii*) *$x \leqslant_1 y$ if $x(-\infty, r) \geqslant y(-\infty, r)$ for all $r \in R$.*

It is easy to see that $x \leqslant_1 y$ iff $x \leqslant y$ as defined by D. Catlin [1].

**Definition 2.2.** *Let $(L, M)$ be a sum logic, $x$, $y$ be summable observables. We define $\leqslant_2$ as follows:*

(*i*) *if $m \in M$, then $x \leqslant_2 y$ $[m]$ if $y - x \geqslant 0$ $[m]$;*

(*ii*) *$x \leqslant_2 y$ if $y - x \geqslant 0$.*

In what follows the indices 1, 2 will be omited if no misunderstanding is likely to arise.

It is easy to see that $x \leqslant_1 y$ implies $x \leqslant_1 y$ $[m]$ for any $m$ and $x \leqslant_2 y$ implies $x \leqslant_2 y$ $[m]$ for all $m \in M$. Conversely, if $(L, M)$ is *f.s.*, then $x \leqslant_1 y$ $[m]$ for all $m \in M$ iff $x \leqslant_1 y$.

N. Zierler ([21]) proved the following theorem (see also [16]).

**Theorem 2.2.** *Let $(L, M)$ be q.f.s. and $x \leftrightarrow y$. Then $m(x) \leqslant m(y)$ for all $m \in M$ iff whenever $f$, $g$ are Borel function and $z$ is an observable such that $x = f \circ z$, $y = g \circ z$, then $f \leqslant g$ a.e. $[m_z]$ for all $m \in M$.*

**Proposition 2.3.** *Let $(L, M)$ be q.f.s. and $x$ be bounded observable on $L$. Then $m(x) \geqslant 0$ for any $m \in M$ iff $x \geqslant 0$.*

Proof. *If* $x \geqslant 0$, then $\sigma(x) \subset [0, \infty)$ and so we have

$$m(x) = \int_{\sigma(x)} \lambda m_x(\mathrm{d}\lambda) = \int_{[0, \infty)} \lambda m_x(\mathrm{d}\lambda) \geqslant 0 \qquad \text{(for all } m \in M).$$

Let $m(x) \geqslant 0$ for any $m \in M$ and $x \ngeqslant 0$. Then there is $A \in B(R)$ such that

$A \subset (-\infty, 0)$ and $x(A) \neq 0$. Therefore is a state $m \in M$ such that $m_x(A) = 1$. We thus obtain

$$m(x) = \int_A \lambda m(x(\mathrm{d}\lambda)) = \int \chi_A(\lambda) \cdot \lambda m(x(\mathrm{d}\lambda)) < 0. \qquad \text{(Q.E.D.)}$$

Let us note that Proposition 2.3 also follows from Lemma 3 in [21].

**Corollary 2.3.1.** *Let* $(L, M)$ *be a sum logic, then*
(*i*) $x \leqslant_2 y$, $y \leqslant_2 x$ *implies* $x = y$;
(*ii*) *if* $x \geqslant 0$, $y \geqslant 0$, *then* $x + y \geqslant 0$.

**Proposition 2.4.** *Let* $(L, M)$ *be q.f.s. and* $x \leftrightarrow y$. *Then*
(*i*) $x \leqslant_1 y$ $[m]$ *iff* $x \leqslant_2 y$ $[m]$, $m \in M$;
(*ii*) $x \leqslant_1 y$ *iff* $x \leqslant_2 y$.

Proof. Since $x \leftrightarrow y$, there is an observable $z$ and Borel functions $f$, $g$ such that $x = f \circ z$, $y = g \circ z$.

(*i*) Let $x \leqslant_1 y$ $[m]$. It means that for each $r \in R$

$$0 = m(y(-\infty, r)) - m(x(-\infty, r) \wedge y(-\infty, r)) =$$
$$= m(y(-\infty, r) \wedge x[r, \infty)) = m_z(\{\omega | g(\omega) < r, f(\omega) \geqslant r\}),$$

(for all $r \in R$). Hence $0 = m_z(\{\omega | g(\omega) < f(\omega)\})$ (i.e. $f \leqslant g$ $[m_z]$). On the other hand, if $x \leqslant_2 y$ $[m]$, we have $y - x \geqslant 0$ $[m]$. And then $0 = m((y - x)(-\infty, 0)) = m_z((g - f)^{-1}(-\infty, 0)) = m_z(\{\omega | f(\omega) > g(\omega)\})$ i.e. $f \leqslant g$ a.e. $[m_z]$.

(*ii*) $x \leqslant_1 y$ $[m]$ for all $m \in M$ iff $x \leqslant_1 y$. Then we have $x \leqslant_1 y$ $[m]$ for all $m \in M$ iff $x \leqslant_2 y$ $[m]$, for all $m \in M$. Using Proposition 2.3 we have $y - x \geqslant 0$. (Q.E.D.)

**Proposition 2.5.** *Let* $(L, M)$ *be q.f.s. The following statements as equivalent*:
(*i*) $m(x_0(r, \infty)) \leqslant m(x(r, \infty))$ *for all* $r \in R$;
(*ii*) $x_0 \leqslant_1 x$ $[m]$;
(*iii*) $x_0 \leqslant_2 x$ $[m]$;
(*iv*) $x \geqslant 0$ $[m]$.

Proof. Since we have $x_0 \leftrightarrow x$, it is obvious that (*ii*) is equivalent to (*iii*). Let $x_0 \leqslant_2 x$ $[m]$. Then $m((x - x_0)[0, \infty)) = 1$. But $x_0 = f \circ x$, where $f(r) = 0$ for all $r \in R$. Let $g$ be the identity function on $R$. Then

$$1 = m((x - x_0)[0, \infty)) = m((g \circ x - f \circ x)[0, \infty)) =$$
$$= m_x((g - f)^{-1}[0, \infty)) = m_x(\{\omega | g(\omega) \in [0, \infty)\}) = m(x[0, \infty)).$$

Then we have $x \geqslant 0$ $[m]$. It means that (*iii*) implies (*iv*).

Let us suppose that $x \geqslant 0$ $[m]$. If $r < 0$, then we have $m(x_0(r, \infty)) = 1$ and $m(x(r, \infty)) = 1$. If $r \geqslant 0$, then $m(x_0(r, \infty)) = 0$. But $m(x(r, \infty)) \geqslant 0$ for all $r \in R$. It means that (*iv*) implies (*i*).

Let $m(x_0(r, \infty)) = m(x(r, \infty))$ for all $r \in R$. Let $r_n \in R$ be taken such that $r_n \in (-1/2^n, -1/2^{n+1})$. We obtain $x_0(r_n, \infty) = 1$ for all $n$ and therefore $m(x(r_n, \infty)) = 1$ for all $n$. Hence for any $n$ $m_x(-\infty, r_n) = 0$. Now we have $\lim_{n \to \infty} m_x(-\infty, r_n) = m_x(-\infty, 0) = 0$ and $m(x[0, \infty)) = 1$. It means $x \geqslant 0$ $[m]$ and so (*i*) implies (*iii*). (Q.E.D.)

Note that $\leqslant_1 [m]$, $\leqslant_1$, $\leqslant_2 [m]$, $\leqslant_2$ are reflexive and $\leqslant_2$, $\leqslant_1$ are transitive. If $x, y$ are observables on $L$ such that $x \leqslant_1 y$ and $y \leqslant_1 x$, then $x = y$. In fact, let $x \leqslant_1 y$, $y \leqslant_1 x$; then for each $r \in R$, $x(-\infty, r) = y(-\infty, r)$. Hence for $r_1, r_2 \in R$: $r_1 < r_2$. we have $x([r_1, r_2)) = y([r_1, r_2))$. But if two observables are equal on all generators for Borel sets, then they are indentical (see e.g. D. Catlin [1]). Since a sum logic has the property $U$, we have $x \leqslant_2 y$, $y \leqslant_2 x$ iff $x = y$.

**Proposition 2.6.** *If $x$, $y$ are observables on $L$, and if there is $r_0 \in R$ such that $\sigma(x) \subset (-\infty, r_0)$, $\sigma(y) \subset [r_0, \infty)$, then $x \leqslant_1 y$.*

The proof is obvious.

From Proposition 2.6 it follows that $x \leqslant_1 y$ does not imply $x \leftrightarrow y$. It is sufficient to take the observables $x \not\leftrightarrow y$, $\sigma(x) \subset (-\infty, t)$, $\sigma(y) \subset [t, \infty)$ $(t \in R)$.

From Theorem 2.1 it is obvious that $x \leqslant_1 y$ implies $x \leqslant_2 y$ on a sum logic. If $(L, M)$ is *f.s.*, then $x \leqslant y$ $[m]$ for all $m \in M$ in the sense of S. Gudder and J. Zerbe iff $x \leqslant_1 y$.

Example 2.2. Let $L = \{0, 1, a^\perp, a, b^\perp, b\}$, where $a \not\leftrightarrow b$ and $a \wedge b = b \wedge a^\perp = b^\perp \wedge a = b^\perp \wedge a^\perp = 0$. Let us choose states $m_i$ $(i = 1, \ldots, 4)$ as follows:

$$\begin{array}{ll} m_1(a) = 0 & m_1(b) = 0.1 \\ m_2(a) = 1 & m_2(b) = 0.1 \\ m_3(a) = 0.9 & m_3(b) = 0 \\ m_4(a) = 0.9 & m_4(b) = 1. \end{array}$$

Then $(L, M)$ is *q.f.s.* for $M = \{m_1, \ldots, m_4\}$. Let $x(\{0\}) = a$, $x(\{2\}) = a^\perp$, $y(\{1\}) = b$, $y(\{3\}) = b^\perp$. Obviously $x \not\leqslant_1 y$. Now we have $m(x) = 2m(a^\perp)$, $m(y) = m(b) + 3m(b^\perp)$. Hence $m_1(x) = 2 \leqslant m_1(y) = 2.9$; $m_2(x) = 0 \leqslant m_2(y) = 2.9$; $m_3(x) = 0.2 \leqslant m_3(y) = 3$; $m_4(x) = 0.2 \leqslant m_4(y) = 1$. We can conclude that $m(x) \leqslant m(y)$ for all $m \in M$ but $x \not\leqslant_1 y$.

S. Gudder [6], [9] showed that if $(L, M)$ is *q.f.s.*, $x, y$ are bounded observables and the spectrum of $x$ has at most one limit point, then

$$m(x) = m(y) \quad \text{for any} \quad m \in M \quad \text{implies} \quad x = y.$$

Under the same assumption the implication

$$m(x) \leqslant m(y) \quad \text{for any} \quad m \in M \Rightarrow x \leqslant_1 y$$

does not hold, as Example 2.2, shows.

**Proposition 2.7.** *Let* $(L, M)$ *be q.f.s. and* $\sigma(x) = \{t_1, t_2\}$, $\sigma(y) = \{r_1, r_2\}$ $(t_1 < t_2, r_1 < r_2)$. *If* $t_2 \leqslant r_1$ *or* $t_1 = r_1$, *then* $x \leqslant_1 y$ *iff* $m(x) \leqslant m(y)$ *for any* $m \in M$.

Proof. Let us put $x(\{t_1\}) = a$, $y(\{r_1\}) = b$. If $t_2 \leqslant r_1$ we can use Proposition 2.6 for $r_0 = r_1$.

Now consider $t_1 = r_1$. Then

$$m(x) = t_1 + (t_2 - t_1)m(a^\perp), \quad m(y) = t_1 + (r_2 - t_1)m(b^\perp).$$

From the assumption we have

$$(t_2 - t_1)m(a^\perp) \leqslant (r_2 - t_1)m(b^\perp).$$

Let $n \in M$ be such that $n(b) = 1$. Obviously, $n(a^\perp) = 0$. Hence $n(b) = 1$ implies $n(a) = 1$. We conclude that $a \geqslant b$.

Let $m \in M$ be such that $m(a^\perp) = 1$. Since $b^\perp \leqslant a^\perp$, we have $m(b^\perp) = 1$. It follows that $t_2 \leqslant r_2$. Hence $x \leqslant_1 y$. The converse implication follows from Theorem 2.1. (Q.E.D.)

If $(L, M)$ is a *f.s.* and $x$, $y$ are proposition observables, then we have $x \leqslant_1 y$ iff $m(x) \leqslant m(y)$ for any $m \in M$ and moreover $x \leqslant_1 y$ implies $x \leftrightarrow y$ and $x_0 \leqslant x \leqslant x_1$ for all proposition observables $x$.

Let $(L, M)$ be *q.f.s.* Put $i(x) = inf\{r \in R | r \in \sigma(x)\}$. $s(x) = sup\{r \in R | r \in \sigma(x)\}$. It is clear that $m(x) \leqslant m(y)$ for all $m \in M$ implies $i(x) \leqslant i(y)$ and $s(x) \leqslant s(y)$. If $x \geqslant 0$ and $x \leqslant_1 y$, then $y \geqslant 0$. If $\sigma(y) = \{t\}$ and $m(x) \leqslant m(y)$ for all $m \in M$, then $x \leqslant_1 y$. Analogically, if $m(x) \geqslant m(y)$ for all $m \in M$, then $y \leqslant_1 x$.

Let $x$, $y$ be such observables that $x$ has a point spectrum and $y(\{i(y)\}) \wedge b \neq 0$ for $b \in R(x) \cap \{0\}^c$. Then $m(x) \leqslant m(y)$ for all $m \in M$ iff $x \leqslant_1 y$. Indeed, put $\sigma(x) = \{t_i\}_{i=1}^{\infty}$. Because $y(\{i(y)\}) \wedge x(\{t_j\}) \neq 0$ for each $j$, there is a state $m_j \in M$ such that $m_j(y(\{i(y)\}) \wedge x(\{t_j\})) = 1$. Now we have $m_j(y) = i(y)$, $m_j(x) = t_j$. From the assumption it follows that $i(y) \geqslant t_j$ for all $j$. Now we use Proposition 2.3 for $r_0 = i(y)$.

Now we consider a sum logic $(L, M)$ and $a \in L$, $a \neq 0$ such that, for any summable observables $x$, $y$ on $L$, the following conditions are satisfied:

$\alpha$) *if* $x \leftrightarrow a$, $y \leftrightarrow a$, *then* $x + y \leftrightarrow a$;

$\beta$) *if* $R(x) \cup R(y)$ *is p.c.* $[a]$, *then* $(x + y) \wedge a = x \wedge a + a \wedge y$.

For instance, Hilbert space logic $L(H)$ fulfils $\alpha$), $\beta$).

**Proposition 2.9.** *Let* $(L, M)$ *be a sum logic and* $a \in L$, $(a \neq 0)$ *such that* $\alpha$), $\beta$) *are fulfilled. Let* $m \in M$ *be such that* $m(a) = 1$. *Then for any pair observables* $x$, $y$ *on* $L$, *with* $R(x) \cup R(y)$ *p.c.* $[a]$ *there holds* $x \leqslant_1 y$ $[m]$ *iff* $x \leqslant_2 y$ $[m]$.

Proof. Since $x \wedge a \leftrightarrow y \wedge a$, we have $x \wedge a \leqslant_1 y \wedge a$ $[m]$ iff $x \wedge a \leqslant_2 y \wedge a$ $[m]$. But $m(x(E) \wedge y(F)) = m(x(E) \wedge a \wedge y(F) \wedge a)$ and $m(y(E)) = m(y(E) \wedge a)$, for any $E, F \in B(R)$: As $(-\infty, r) \in B(R)$ for each $r \in R$, we have $x \leqslant_1 y$ $[m]$ iff $x \leqslant_2 y$ $[m]$. (Q.E.D.)

**Corollary. 2.9.1.** *Let $x$, $y$ be p.c.* $[a]$, $m(a) = 1$ *and let* $x \geqslant 0$ $[m]$, $y \geqslant 0$ $[m]$. *Then*

(*i*) $x + y \geqslant 0$ $[m]$;

(*ii*) $x \cdot y \geqslant 0$ $[m]$;

(*iii*) *if* $x \wedge a = f \circ z$, $y \wedge a = g \circ z$, *where* $f$, $g$ *are Borel functions and* $z$ *is an observable on* $L_{[0, a]}$, *then* $x \leqslant y$ $[m]$ *iff* $f \leqslant g$ *a.e.* $[m_z]$.

## 3. Properties of functional representation for p.c. observables

In what follows we shall assume $(L, M)$ to be a sum logic with $\alpha$), $\beta$). Let $Q \subset L$ be a sublogic of $L$ which is p.c. $[a]$ for some $a \in L$ $(a \neq 0)$. From the properties of partial compatibility it follows that $Q \wedge a$ is a Boolean $\sigma$-algebra. Let $m \in M$ be such that $m(a) = 1$. Denote by $X(Q)$ the set of all bounded observables with $R(x) \subset Q$ and suppose that $x, y \in X(Q)$; then $x + y \in X(Q)$. Let us fix a measurable space $(\Omega, \mathscr{F})$, and a $\sigma$-homomorphism $h$ from $\mathscr{F}$ onto $Q \wedge a$, which exists by the Loomis-Sikorsky theorem (see [13], [22]). To any observable $x$ on $Q$ there is an $\mathscr{F}$-measurable function $f_x \colon \Omega \to R$ such that $x \wedge a = f_x \circ h$ [20]. We shall write $x \sim f_x$.

**Definition 3.1.** *Let* $x$, $y \in X(Q)$. *We shall say that* $x \simeq y$ $[m]$ ($x$ *is equal to* $y$ *modulo* $m$) *if for any* $E \in B(R)$

$$m(x(E) \mathbin{\triangle} y(E)) = 0,$$

*where* $a \mathbin{\triangle} b = (a \wedge b^{\perp}) \vee (a^{\perp} \wedge b)$ $(a, b \in L)$.

**Lemma 3.1.** *Let* $x$, $y \in X(Q)$. *Then* $x \simeq y$ $[m]$, $y \simeq z[m]$ *imply* $x \simeq z$ $[m]$.

Proof. We have $m(x(E) \mathbin{\triangle} y(E)) = m(x(E) \wedge a \mathbin{\triangle} y(E) \wedge a)$. But $R(x \wedge a) \cup R(y \wedge a) \cup R(z \wedge a) \subset Q \wedge a$. The statements follows from the properties of the symmetric difference on a Boolean-$\sigma$-algebra. (Q.E.D.)

**Lemma 3.2.** *For* $x$, $y \in X(Q)$, $x \simeq y$ $[m]$ *iff* $f_x = f_y$ *a.e.* $[m_h]$, (*where* $m_h(E) = m(h(E))$, *for all* $E \in \mathscr{F}$).

Proof. We have

$$m(x(E) \mathbin{\triangle} y(E)) = m(x(E) \wedge a \mathbin{\triangle} y(E) \wedge a) =$$
$$= m(h(f_x^{-1}(E)) \mathbin{\triangle} h(f_y^{-1}(E))) = m_h(f_x^{-1}(E) \mathbin{\triangle} f_y^{-1}(E)).$$

It was shown by S. Gudder and J. Zerbe [10] that $f_x = f_y$ a.e. $[m_h]$ iff $m_h(f_x^{-1}(E) \mathbin{\triangle} f_y^{-1}(E)) = 0$ for all $E \in B(R)$. (Q.E.D.)

**Lemma 3.3.** *Let* $x$, $y \in X(Q)$ *and* $g$ *be any Borel real function. Then*

(*i*) $h(\{\omega \in \Omega | f_{x+y} = f_x + f_y\}) = 1$;

(*ii*) $h(\{\omega \in \Omega | f_{g \circ x}(\omega) = g(f_x(\omega))\}) = 1$;

(*iii*) $h(\{\omega \in \Omega | f_{x \cdot y}(\omega) = f_x(\omega) \cdot f_y(\omega)\}) = 1.$

Proof. (*i*) We have $f_{x+y}h(E) = (x + y) \wedge a(E) = (x \wedge a + y \wedge a)(E) = f_x \circ h(E) + f_y \circ h(E) = (f_x + f_y) \circ h(E)$ for any $E \in B(R)$. It means that

$$h(\{\omega \in \Omega | f_{x+y}(\omega) = f_x(\omega) + f_y(\omega)\}) = 1.$$

(*ii*) $f_{g \circ x} \circ h(E) = g \circ x(E) \wedge a = x(g^{-1}(E)) \wedge a = f_x \circ h(g^{-1}(E))) = g \circ f_x \circ h(E)$ for any $E \in B(R)$.

(*iii*) follows from (*i*) and (*ii*). (Q.E.D.)

Let $x$ be an observable on $Q$ such that $|m(x)| < \infty$. For $b \in Q$ let us denote by the symbol $\int_b x \, dm$ the following integral

$$\int_b x \, dm = \int rm(x(dr) \wedge b).$$

The integral on the right side exists, because $m(x(E) \wedge b) = m(x(E) \wedge b \wedge a) = m(x \wedge a(E) \wedge b \wedge a) = m_h(f_x^{-1}(E) \cap B)$, where $h(B) = b \wedge a$, $B \in \mathscr{F}$. Therefore

$$\int rm(x(dr) \wedge b) = \int rm_h(f_x^{-1}(dr) \cap B) = \int_B f_x(r) \, m_h(dr).$$

Especially for $b = 1$, $\int x \, dm = \int rm(x(dr))$.

**Lemma 3.4.** *For any $x \in X(Q)$ and $b \in Q$*

$$\int_b x \, dm = \int x \cdot x_b \, dm = \int rm(x \cdot x_b(dr)).$$

Proof. We have

$$\int_b x \, dm = \int_B f_x \, dm_h = \int \chi_B \cdot f_x \, dm_h.$$

We have $\chi_B \circ h(\{1\}) = h(B) = b \wedge a = x_b \wedge a(\{1\})$ and $\chi_B \circ h(\{0\}) = h(B^c) = b^\perp \wedge a = x_b \wedge a(\{0\})$. Then $x_b \sim \chi_B$. From this we obtain

$$\int \chi_B \cdot f_x \, dm_h = \int tm_h((f_x \cdot \chi_B)^{-1}(dt)) = \int tm((x \cdot x_b) \wedge a(dt)) = \int x \cdot x_b \, dm. \qquad \text{(Q.E.D.)}$$

**Lemma 3.5.** *Let $n$ be a finite measure on $Q$. If $n \ll m$, there is an observables $y$ on $L$, $y \leftrightarrow b$, for any $b \in Q$, such that $R(x) \wedge a \subset Q \wedge a$ and for any $x \in X(Q)$, $\int x \, dn = \int x \cdot y \, dm$.*

Proof. Let $x \in X(Q)$. We have

$$\int x \, \mathrm{d}n = \int t n(x(\mathrm{d}t)) = \int t n_h(f_x^{-1}(\mathrm{d}t)),$$

where $n_h(B) = n(h(B))$, $B \in \mathscr{F}$ $(h(B) = b \wedge a,\ b \in Q)$. If $m_h(B) = 0$, then $m(h(B)) = 0$ implies $n(h(B)) = 0$. But $n(h(B)) = n_h(B)$ and from $n_h \ll m_h$. By the Radon-Nikodým theorem there is a function $g: \Omega \to R$, $\mathscr{F}$-measurable, such that $n_h(B) = \int g \, \mathrm{d}n_h$. Put $y = g \circ h \vee x_0 \wedge a^{\perp}$. Then $y$ is an observable on $L$ and $R(y) \wedge a = R(g \circ h) \subset Q \wedge a$. Moreover, $y \leftrightarrow x$ for any $x \in X(Q)$. Now we have

$$\int x \, \mathrm{d}n = \int f_x \, \mathrm{d}n_h = \int g \cdot f_x \, \mathrm{d}m_h = \int x \cdot y \, \mathrm{d}m. \qquad \text{(Q.E.D.)}$$

Let $\{x_n\}_{n=1}^{\infty} \subset X(Q)$. We say that $x_n \to x$ a.e. $[m]$ if

$$m\left(\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} (x_n - x)[-\varepsilon, \varepsilon]\right) = 1$$

for all $\varepsilon > 0$. We say that $x_n \to x$ in $L_p$-mean $(x_n \xrightarrow{p} x)$ if $m(|x_n - x|^p) \to 0$ [8].

**Lemma 3.6.** *Let $\{x_n\}_{n=1}^{\infty}$, $x \subset X(Q)$.*

*(i) $x_n \to x$ a.e. $[m]$ iff $f_{x_n} \to f_x$ a.e. $[m_h]$;*

*(ii) $x_n \xrightarrow{p} x$ iff $f_{x_n} \xrightarrow{p} f_x$ in $L_p(\Omega, \mathscr{F}, m_h)$.*

Proof. (*i*) We have

$$m\left(\bigvee_{i=1}^{\infty} \bigwedge_{k=i}^{\infty} (x_k - x)[-\varepsilon, \varepsilon]\right) = m\left(\bigvee_{i=1}^{\infty} \bigwedge_{k=i}^{\infty} (x_k \wedge a - x \wedge a)[-\varepsilon, \varepsilon]\right) =$$

$$= m_h\left(\bigcup_{i=1}^{\infty} \bigcap_{k=1}^{\infty} (f_{x_k} - f_x)^{-1}[-\varepsilon, \varepsilon]\right) = m_h\left(\bigcup_{i=1}^{\infty} \bigcap_{k=1}^{\infty} \{t | f_{x_k}(t) - f_x(f) < \varepsilon\}\right).$$

The last expression equals 1 iff $f_{x_n} \to f_x$ a.e. $[m_h]$.

(*ii*)

$$m(|x_n - x|^p) = \int t m(|x_n - x|^p(\mathrm{d}t)) = \int |t|^p m((x_n \wedge a - x \wedge a)(\mathrm{d}t)) =$$

$$= \int |t|^p m_h((f_{x_n} - f_x)^{-1}(\mathrm{d}t)) = \int t m_h(((|f_{x_n} - f_x|^p)^{-1}(\mathrm{d}t)) =$$

$$= \int |f_{x_n}(t) - f_x(t)|^p m_h(\mathrm{d}t).$$

The last expression equals 0 iff $f_{x_n} \to f_x$ in $L_p(\Omega, \mathscr{F}, m_h)$. (Q.E.D.)

## 4. Conditional expectation

Let $P \subset Q$ be a sublogic. We define a *conditional expectation with respect to P* as follows.

**Definition 4.1.** *Let $x \in X(Q)$. We say that an observable y on Q is a conditional expectation of x with respect to P if*

(*i*) $R(y) \wedge a \subset P \wedge a$;

(*ii*) $\int_b x \, \mathrm{d}m = \int_b y \, \mathrm{d}m$ *for all* $b \in P$.

Any observable $y$, which satisfies (*i*), (*ii*) is called a *version of conditional expectation*. The case of $a = 1$ (i.e. $Q$ and $P$ are Boolean-$\sigma$-algebras) was studied in [12]. For any $x \in X(Q)$ this definition is a restriction on $Q$ of the definition of a conditional expectation of $x$ with respect to $P$ relativized by $a$ in the state $m$ ($E_m(x/P, a)$), which has been studied in [14]. Because we have a fixed $m$ and $a$, we write $E(x/P)$ in the sequel.

**Theorem 4.1.** *To any $x \in X(Q)$ there is a version of conditional expectation, then $y \simeq z$ [m].*

Proof. Since $x \in X(Q)$ then $x \wedge a$ is bounded on $L_{[0, a]}$. Then $f_x$ is bounded. Put $\mathbf{F}_0 = \{B \in \mathscr{F} | h(B) \in P \wedge a\}$. Since $\mathbf{F}_0$ is a sub-$\sigma$-algebra of $\mathscr{F}$ then there is a $\mathbf{F}_0$-measurable function $g$: $E(f_x/\mathbf{F}_0)$ which is bounded [2]. Put $y = g \circ h \vee \vee a^{\perp} \wedge x_0$. $R(y) \wedge a \subset P \wedge a$ and $y$ is a bounded observable. Let $b \in P$ and $B \in \mathscr{F}$ be such that $h(B) = b \wedge a$, clearly $B \in \mathbf{F}_0$. Now we have

$$\int_b x \, \mathrm{d}m = \int_B f_x \, \mathrm{d}m = \int_B g \, \mathrm{d}m = \int_b y \, \mathrm{d}m.$$

If $y$, $z$ are versions of conditional expectation $E(x/P)$, then their functional representations $f_y$, $f_z$ are versions of $E(f_x/\mathbf{F}_0)$. This implies $f_y = f_z$ $[m_h]$ and therefore $y \simeq z$ [$m$]. (Q.E.D.)

**Corollary 4.1.1.** (*i*) *If $a \in Q$, $x \in X(Q)$, then there is a version of conditional expectation of x which belongs to $X(Q)$;*

(*ii*) *If $a \in P$, $x \in X(Q)$; then there is a version of conditional expectation of x which belongs to $X(P)$.*

In what follows we shall write $x = y$ if $x \simeq y$ $[m]$; i.e. $f_x = f_y$ a.e. $[m_h]$. Then for example, $x_a = x_1$ if $a \in Q$. From Proposition 2.9 it follows that if $x, y \in X(Q)$, then $x \leqslant_1 y$ $[m]$ iff $x \leqslant_2 y$ $[m]$. In the following we shall write $x \leqslant y$. From Corollary 2.9.1 it is obvious that $x \leqslant y$ iff $f_x \leqslant f_y$ a.e. $[m_h]$.

**Theorem 4.2.** *A conditional expectation has the following properties:*

1) *If* $\alpha \in R$, *then* $E(\alpha x/P) = \alpha E(x/P)$.
2) *If* $x, y \in X(Q)$, *then* $E(x + y/P) = E(x/P) + E(y/P)$.
3) *If* $x \leqslant y$, *then* $E(x/P) \leqslant E(y/P)$.
4) *If* $x, y \in X(Q)$ *and* $R(x) \wedge a \subset P \wedge a$, *then* $E(x \cdot y/P) = x \cdot E(y/P)$.
5) *If* $x_1 \leqslant x_2 \leqslant \ldots$, $x$ *belong to* $X(Q)$ *and* $x_n \to x$ *a.e.* $[m]$, *then* $E(x_n/P) \to$ $\to E(x/P)$ *a.e.* $[m]$.
6) *If* $p \geqslant 1$, *then* $|E(x/P)|^p \leqslant E(|x|^p/P)$ $(x \in X(Q))$.
7) *If* $x_1, x_2, \ldots, x \in X(Q)$ *and* $x_n \xrightarrow{p} x$, *then* $E(x_n/P) \xrightarrow{p} E(x/P)$.

Proof. Follows from the fact that $x \sim f_x$, $E(x/P) \sim E(f_x/\mathbf{F}_0)$ and from the properties of $E(\cdot/\mathbf{F}_0)$ ([12], [14], [19]) (Q.E.D.)

## 5. Characterization of conditional expectation

In what follows we shall suppose that $a \in Q$. Let $Y(Q)$ be subset of $X(Q)$ such that $x \in Y(Q)$ if $x \geqslant 0$. Due to Corollary 2.9.1, it is clear that $Y(Q)$ is closed under the formulation of the product and the sum of observables. It is easy to see that $x \geqslant 0$ iff $f_x \geqslant 0$ $[m_h]$.

Let $T$ be a transformation of $Y(Q)$ into $Y(Q)$ satisfying the following conditions:

*T*1) *For* $x, y \in Y(Q)$, $\alpha > 0$, $\beta > 0$ $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

*T*2) *For* $x, y \in Y(Q)$ $T(x \cdot T(y)) = T(x) \cdot T(y)$.

*T*3) *If* $x_1, x_2, \ldots, x \in Y(Q)$, $x_n \to x$ *a.e.* $[m]$, $x_n \leqslant x_{n+1}$ *for each* $n$, *then* $T(x_n) \to$ $\to T(x)$ *a.e.* $[m]$.

By Theorem 4.2 and Corollary 4.1.1 the transformation which transfers $x$ to $E(x/P)$ is a transformation of $Y(Q)$ into $Y(Q)$ satisfying $T1)$, $T2)$, $T3)$.

**Lemma 5.1.** *If* $x, y \in Y(Q)$ *and* $x \geqslant y$, *then* $T(x) \geqslant T(y)$.

Proof. We have $x \geqslant y$ iff $f_x \geqslant f_y$ a.e. $[m_h]$. The transformation $T$ induces a transformation $T_h$ of the set of all bounded $\mathscr{F}$-measurable functions $f \geqslant 0$ a.e. $[m_h]$ on $(\Omega, \mathscr{F})$ into itself. Indeed, for any element $x \in Y(Q)$ there is $f_x$ such that $x \sim f_x$, where $f_x \geqslant 0$ a.e. $[m_h]$. Let $T(x) = y$. We put $T_h(f_x) = f_y$. By this definition $T(x) \geqslant T(y)$ iff $T_h(f_x) \geqslant T_h(f_y)$ a.e. $[m_h]$. Now we have $x \geqslant y$ iff $f_x \geqslant f_y$ a.e. $[m_h]$. But $T_h(f_x) = T_h(f_y + (f_x - f_y)) = T_h(f_y) + T_h(f_x - f_y) = T_h(f_y)$, because $f_x - f_y \geqslant$ $\geqslant 0$ a.e. $[m_h]$ and then $T_h(f_x - f_y) \geqslant 0$ a.e. $[m_h]$. Then we have $T(x) \geqslant T(y)$. On the other hand, to any bounded $\mathscr{F}$-measurable function $f \geqslant 0$ a.e. $[m_h]$ there is an observable $x \in Y(Q)$ such that $x \sim f$. In fact, put $x = f \circ h \vee a^{\perp} \wedge x_0$. (Q.E.D.)

**Lemma 5.2.** *Denote by* $Z = \{x \in Y(Q) | T(y) \cdot x = T(x \cdot y)$ *for all* $y \in Y(Q)\}$. *Then the following statements are true.*

1) *If* $x, y \in Z$, *then* $x + y \in Z$, $x \cdot y \in Z$ *and if* $x \geqslant y$, *then* $x - y \in Z$.
2) *For* $a > 0$, $x \in Z$, *we have* $ax \in Z$.
3) *If* $\{x_1, x_2, \ldots\} \subset Z$ *such that* $x_n \leqslant x_{n+1}$ *for all n and* $x_n \to x$ *a.e.* $[m]$, *where* $x \in Y(Q)$, *then* $x \in Z$.
4) *If* $\{x_1, x_2, \ldots\} \subset Z$ *and there is* $y \in Y(Q)$ *such that* $x_n \leqslant y$ *for all n and, moreover, if* $x_n \to x$ *a.e.* $[m]$, *then* $x \in Z$.

Proof. Let $\mathscr{E}$ be the set of all bounded $\mathscr{F}$-measurable functions $f \geqslant 0$ a.e. $[m_h]$ for which $T_h(f \cdot g) = f \cdot T_h(g)$ for any $\mathscr{F}$-measurable bounded function $g \geqslant 0$ a.e. $[m_h]$. By repeating the arguments Shu-Ten Chen Moy — if we restrict our considerations to bounded functions only — we can prove that the following statements hold:

1′) If $g_1, g_2 \in \mathscr{E}$, then $g_1 + g_2 \in \mathscr{E}$ and $g_1 \cdot g_2 \in \mathscr{E}$, and if $g_1 \leqslant g_2$ a.e. $[m_h]$, then $g_2 - g_1 \in \mathscr{E}$.
2′) If $a > 0$, $g \in \mathscr{E}$, then $ag \in \mathscr{E}$.
3′) If $\{g_1, g_2, \ldots\} \subset \mathscr{E}$, $g_n \nearrow g$ a.e. $[m_h]$ (where $g$ is bounded), then $g \in \mathscr{E}$.
4′) If $\{g_1, g_2, \ldots\} \subset \mathscr{E}$, $g_n \to g$ a.e. $[m_h]$ and there is a bounded function $k$ for which $g_n \leqslant k$ a.e. $[m_h]$ for any $n$, then $g \in \mathscr{E}$.

If we pass from functional representation to observables, we obtain 1), 2), 3), 4). (Q.E.D.)

**Lemma 5.2.** *If* $x \in Y(Q)$, *then* $T(x) \in Z$.
Proof. Follows from T2). (Q.E.D.)

**Lemma 5.3.** *Define* $P = \{d \in L | x_d \in Z\}$. *Then P is a sublogic of Q.*

Proof. If $x_1 \in Z$, then $1 \in P$. Let $d \in P$. Then $x_{d^{\perp}} = x_1 - x_d \in Z$. This implies by Lemma 5.2.1 that $d^{\perp} \in P$.

Let $d, b \in P$; then $x_d, x_b \in Z$ and $x_d \sim \chi_D$, $x_b \sim \chi_B$, where $D, B \in \mathscr{F}$ and $h(B) = b \wedge a$, $h(D) = d \wedge a$. Then $x_d \cdot x_b \sim \chi_B \cdot \chi_D = \chi_{B \cap D}$ and $\chi_{B \cap D} \sim x_{d \wedge b}$. But $x_d \cdot x_b \in Z$ according to Lemma 5.2.1; then $d \wedge b \in P$. Then also $d \wedge b \in P$. By induction it can be proved that $\{d_1, \ldots, d_n\} \subset P$ implies $\bigvee_{i=1}^{n} d_i \in P$.

Without loss of generality we can assume that $\{d_1, d_2, \ldots,\} \subset P$ are mutually orthogonal elements. Denote by $y_n = x_{d_1 \vee \ldots \vee d_n}$ for all $n$. It is sufficient to prove that $y_n \to x_{\bigvee_n d_n}$ a.e. $[m]$, ($y_n \leqslant y_{n+1}$ for all $n$). Put $x = x_{\bigvee_n d_n}$. Because $d_n \leftrightarrow d_m$ for all $n, m$ there is an observable $z$ such that $\{d_1, d_2, \ldots\} \subset R(z)$. Suppose $\{B_1, B_2,$

$\ldots\} \subset B(R)$, $z(B_n) = d_n$ and $D_n = \bigcup_{i=1}^{n} B_i$, $D = \bigcup_n B_n = \bigcup_n D_n$. Put $f_n = \chi_{D_n}$, $f = \chi_D$. We obtain $x = f \circ z$, $y_n = f_n \circ z$. Hence

$$m(\lim (x - y_n)[-\varepsilon, \varepsilon]) = m_z(\lim (f - f_n)^{-1}[-\varepsilon, \varepsilon]).$$

Consider $\varepsilon \geqslant 1$. Then

$$m_z(\lim (f - f_n)^{-1}[-\varepsilon, \varepsilon]) = m_z(R) = 1.$$

Further, if $\varepsilon < 1$, then

$$m_z(\lim (f - f_n)^{-1}[-\varepsilon, \varepsilon]) = m_z(\lim \chi_{D-D_n}^{-1}(\{0\})) =$$
$$= m_z(\lim \{t | \chi_{D-D_n}(t) = 0\}) = m_z(\lim D^c \cup D_n) m_z(R) = 1.$$

Then $x \in Z$. Thus $\bigvee_n d_n \in P$. (Q.E.D.)

**Corollary 5.3.1.** (*i*) *For all* $x \in Y(Q)$ *we have* $R(T(x)) \wedge a \subset P \wedge a$.
(*ii*) *Since* $x_a = x_1$ *and* $1 \in P$, *we have* $a \in P$.

**Theorem 5.5.** *Let* $T$ *be a transformation of the set* $Y(Q)$ *into* $Y(Q)$ *satisfying* $T1)$, $T2)$, $T3)$; *then* $T$ *is of the form* $T(x) = E(x \cdot y / P)$, *where* $y \geqslant 0$ *such that* $y \leftrightarrow Q$.

Proof. Define $\beta$: $d \mapsto \int T(x_d) \, \mathrm{d}m$, $d \in Q$. Then $\beta$ is a measure on $Q$ by $T1)$, $T2)$, $T3)$, and if $m(b) = 0$ ($b \in Q$), then $x_b = x_0$. And

$$\beta(b) = \int T(x_b) \, \mathrm{d}m = \int T(x_0) \, \mathrm{d}m = \int T(x_0 \cdot x_1) \, \mathrm{d}m = \int x_0 \cdot T(x_1) \, \mathrm{d}m = 0.$$

so that $\beta \ll m$. Moreover $\beta(a) = \beta(1)$. By Lemma 3.5, there is an observable $y$, $y \geqslant 0$ and $y \leftrightarrow Q$ such that for $x \in Y(Q)$

$$\int x \, \mathrm{d}\beta = \int x \cdot y \, \mathrm{d}m.$$

Let $\{\alpha_1, \ldots, \alpha_n\} \subset R$ ($\alpha_i \geqslant 0$ for all $i$), $\{B_1, \ldots, B_n\} \subset \mathscr{F}$, $B_i \cap B_j = \emptyset$, for $i \neq j$ and $h(B_i) = b_i \wedge a$, ($\{b_1, \ldots, b_n\} \subset Q$). If we put $f = \sum_{i=1}^{n} \alpha_i \chi_{B_i}$, then $f \circ h \vee a^\perp \wedge \wedge x_0 \in Y(Q)$ and

$$\int T\left(\sum_{i=1}^{n} \alpha_i x_{b_i}\right) \mathrm{d}m = \sum_{i=1}^{n} \alpha_i \int T(x_{b_i}) \, \mathrm{d}m = \sum_{i=1}^{n} \alpha_i \int T_h(\chi_{B_i}) \, \mathrm{d}m_h =$$
$$= \sum_{i=1}^{n} \alpha_i \int T(x_{b_i} \wedge a) \, \mathrm{d}m = \sum_{i=1}^{n} \alpha_i \int T(x_{b_i \wedge a}) \, \mathrm{d}m = \sum_{i=1}^{n} \alpha_i \beta(b_i \wedge a).$$

As

$$\beta(b_i \wedge a) = \int x_{b_i \wedge a}\, \mathrm{d}\beta, \quad \text{and} \quad \beta(a^{\perp}) = 0,$$

$$\sum_{i=1}^{n} \alpha_i \int x_{b_i \wedge a}\, \mathrm{d}\beta = \sum_{i=1}^{n} \alpha_i \int x_{b_i}\, \mathrm{d}\beta,$$

i.e. $\int T\left(\sum_{i=1}^{n} \alpha_i x_{b_i}\right) \mathrm{d}m = \int \left(\sum_{i=1}^{n} \alpha_i x_{b_i}\right) \mathrm{d}\beta$. Since any $\mathscr{F}$-measurable function $f$ can be described as a limit of a nondecreasing sequence of a simple functions, we have for all $x \in X(Q)$

$$\int T(x)\, \mathrm{d}m = \int T(f_x \circ h \vee a^{\perp} \wedge x_0)\, \mathrm{d}m = \int f_x \circ h \vee a^{\perp} \wedge x_0\, \mathrm{d}\beta =$$

$$= \int x\, \mathrm{d}\beta = \int x \cdot y\, \mathrm{d}m.$$

Let $d \in P$; then

$$\int_d T(x)\, \mathrm{d}m = \int x_d \cdot T(x)\, \mathrm{d}m = \int T(x_d \cdot x)\, \mathrm{d}m = \int x_d \cdot x \cdot y \mathrm{d}m = \int_d x \cdot y\, \mathrm{d}m.$$

Because $R(T(x) \wedge a \subset P \wedge a$ we have $E(x \cdot y/P) = T(x)$. (Q.E.D.)

Let us consider a transformation $S$ of $X(Q)$ into $X(Q)$ with the following properties:

$S1)$ *$\alpha, \beta \in R$ $S(\alpha x + \beta y) = \alpha S(x) + \beta S(y)$ for $x, y \in X(Q)$.*

$S2)$ *$S(x \cdot S(y)) = S(x) \cdot S(y)$ for $x, y \in X(Q)$.*

$S3)$ *If $x_n \xrightarrow{1} x$, then $S(x_n) \xrightarrow{1} S(x)$, $(\{x_n\}_{n=1}^{\infty}, x \subset X(Q))$.*

$S4)$ *$m(|S(x)|) \leqslant m(|x|)$, $(x \in X(Q))$.*

As before, $x_n \xrightarrow{1} x$ means that $m(|x_n - x|) \to 0$. Moreover, $x_n \xrightarrow{1} x$ iff $f_{x_n} \to f_x$ in $L_1(\Omega, \mathscr{F}, m_h)$, i.e. $\int |f_{x_n} - f_x|\, \mathrm{d}m_h \to 0$.

**Lemma 6.1.** *Let $K = \{y \in X(Q) | S(x \cdot y) = y \cdot S(x)$, for all $x \in X(Q)\}$. Then the following holds:*

*(i) If $y_1, y_2 \in K$, then $\alpha y_1 + \beta y_2 \in K$ $(\alpha, \beta \in R)$.*

*(ii) $y_1, y_2 \in K$ implies $y_1 \cdot y_2 \in K$.*

*(iii) If $\{y_n\}_{n=1}^{\infty} \subset K$ $y_n \xrightarrow{1} y$ $(y \in X(Q))$, then $y \in K$.*

Proof. Statements (*i*), (*ii*) follow immediately from $S1)$, $S2)$. To prove (*iii*) let us observe that for any $\{y_n\}_{n=1}^{\infty}$, $y$, $x$ from $X(Q)$, $y_n \xrightarrow{1} y$; then $x \cdot y_n \xrightarrow{1} x \cdot y$. Indeed, $y_n \xrightarrow{1} y$ iff $f_{y_n} \to f_y$ in $L_1(\Omega, \mathscr{F}, m_h)$. This implies $f_{y_n} \cdot f_x \to f_y \cdot f_x$ in

$L_1(\Omega, \mathcal{F}, m_h)$, and then $x \cdot y_n \xrightarrow{1} x \cdot y$. Now, let $\{y_n\}_{n=1}^{\infty} \subset K$ and $y_n \xrightarrow{1} y$. Then $S(x \cdot y_n) \xrightarrow{1} S(x \cdot y)$ by $S3$). Hence $S(x \cdot y)$ for any $x \in X(Q)$, i.e. $y \in K$. (Q.E.D.)

**Lemma 6.2.** *Define $V = \{d \in L | x_d \in K\}$. Then $V$ is a sublogic of $L$.*

Proof. Similarly, as in Lemma 5.3 we prove that if $d \in V$, then $d^{\perp} \in V$ and if $\{d_1, \ldots, d_n\} \subset V$, then $\vee d_i \in V$. Suppose now that $\{d_1, d_2, \ldots\} \subset V$ are mutually orthogonal elements. We have

$$m(|x_{\vee d_n} - x_{b_r}|) = m_h(|\chi_{D - D_r}|),$$

where $\chi_D \circ h = x_{\vee d_n} \wedge a$, $\chi_{D_n} \circ h = x_{b_n} \wedge a$, $b_n = \bigvee_{i=1}^{n} d_i$ $\left(\text{i.e. } h(D) = (\vee d_n) \wedge a = = h(\cup D_n),\ h(D_n) = \bigvee_{i=1}^{n} d_i \wedge a\right)$. But $\lim_{n \to \infty} m_h(|\chi_{D - D_n}|) = m_h(|\chi_0|) = 0$. Then $\vee d_n \in V$. (Q.E.D.)

**Corollary 6.2.1.** *Fol all $x \in X(Q)$ we have $R(S(x)) \wedge a \subset V \wedge a$.*

**Lemma 6.3.** *If $y \in X(Q)$ and $R(y) \in V$, then $y \in K$.*

Proof. If $y = x_b$, then $x_b \in K$. Linear combinations of proposition observables are in $K$. Any bounded observable can be written as a limit in $L_1$ of a sequence of linear combinations of proposition observables, which implies that $y \in K$. (Q.E.D.)

**Theorem 6.4.** *If $S$ is a transformation on $X(Q)$ into $X(Q)$ satisfying $S1$)—$S4$) and $S(x_1) = x_1$, then $S(x) = E(x/V)$ for all $x \in X(Q)$.*

Proof. Put $\eta(d) = m(S(x_d))$ $(d \in Q)$. Then $\eta(1) = \int x_1 \, dm = 1$. Let $\{b_i\}_{i=1}^{\infty} \subset Q$, be mutually orthogonal. Put $b = \vee b_i$. Then

$$x_{c_n} = \sum_{i=1}^{n} x_{b_i} \xrightarrow{1} \sum_{i=1}^{\infty} x_{b_i} = x_{\vee b_i} \quad \left(c_n = \bigvee_{i=1}^{n} b_i\right),$$

which implies $S(x_{c_n}) \xrightarrow{1} S(x_b)$. Therefore $\eta(\vee b_n) = \vee \eta(b_n)$. Then $\eta$ is a $\sigma$-additive function on $Q$. Further $|\eta(b)| = |m(S(x_b))| \leqslant m(|S(x_b)|) \leqslant m(|x_b|) = m_h(\chi_B)$, where $h(b) = b \wedge a$. Now we have $m_h(\chi_B) = m(b)$. Then $m(b) \geqslant |\eta(b)| \geqslant \eta(b)$, for all $b \in Q$. If $m(b) > \eta(b)$ and $m(b^{\perp}) \geqslant \eta(b^{\perp})$, then $1 = m(b) + m(b^{\perp}) > > \eta(b) + \eta(b^{\perp}) = 1$. Then $m(b) = \eta(b)$ for all $b \in Q$. This fact implies $\eta(b) = = m(x_b)$ for all $b \in Q$.

If $\{\alpha_1, \ldots, \alpha_n\} \subset R$, $\{b_1, \ldots, b_n\} \subset Q$, then

$$m\left(S\left(\sum_{i=1}^{n} \alpha_i x_{b_i}\right)\right) = \sum_{i=1}^{n} \alpha_i m(S(x_{b_i})) = \left(\sum_{i=1}^{n} \alpha_i x_{b_i}\right).$$

Let $b \in V$; then

$$\int_b S(x)\,\mathrm{d}m = \int S(x)\cdot x_b\,\mathrm{d}m = \int S(x\cdot x_b)\,\mathrm{d}m = \int x\cdot x_b\,\mathrm{d}m = \int_b x\,\mathrm{d}m,$$

i.e.

$$S(x) = E(x/V). \qquad \text{(Q.E.D.)}$$

We note that the sublogic $V$ (resp. $P$) is not uniquely defined. In fact, if $P_1$, $P_2$ are sublogics of $Q$ such that $P_1 \wedge a = P_2 \wedge a$, then the conditional expectations with respect to $P_1$ and $P_2$ are equal to $m$ for any $x \in X(Q)$. Also, if $a \notin Q$, we put $Q_0 = \{b \wedge a \vee c \wedge a^{\perp} | b, c \in Q\}$. Then $Q_0$ is a sublogic of $L$, $a \in Q_0$, and for any $x \in X(Q)$, the functional representation $f_x$ depends only on $Q \wedge a = Q_0 \wedge a$.

Acknowledgement. The author is very indebted to Dr. S. Pulmannová, DrSc. for her interest in this work.

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Received November 7, 1984

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## УПОРЯДОЧЕНИЕ НАБЛЮДАЕМЫХ И ХАРАКТЕРИСТИКА УСЛОВНОГО МАТЕМАТИЧЕСКОГО ОЖИДАНИЯ

Oľga Nánásiová

### Резюме

В первой части этой статьи рассматриваются два способа упорядочения наблюдаемых и исследуются отношения «релативных условных ожиданий» для частично компатибильных наблюдаемых на квантовой логике. Эти «релативные ожидания» были определены в [14]. Главный результат — характеристика «релативных условных ожиданий» в смысле Шу-Тен Хен Мой [19] на квантовой логике.