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ON TWO SUMMABILITY METHODS

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ABSTRACT. The purpose of this paper is to establish some relations between the $|\mathbf{R},p_n|_k$ and $|\mathbf{C},\alpha|_k$ summability, where $\alpha > 0$ and $k \geq 1$.

1. Definitions and notations

Let $\sum a_n$ be an infinite series with sequence of its partial sums (s_n) and let $\mathbf{T} = (a_{nv})$ be an infinite matrix. Suppose that

$$T_n = \sum_{v=0}^{\infty} a_{nv} s_v \qquad (n = 0, 1, \dots) \tag{1}$$

exists (i.e., the series on the right-hand side converges for each n). If $(T_n) \in b_v$, i.e.,

$$\sum_{n=0}^{\infty} |T_n - T_{n-1}| < \infty \qquad (T_{-1} = 0) \tag{2}$$

then the series $\sum a_n$ is said to be absolutely summable by the matrix $\mathbf{T}$, or simple, summable $|\mathbf{T}|$. As known, the series $\sum a_n$ is said to be $|\mathbf{R},p_n|$ summable if (2) holds when $\mathbf{T}$ is a Riesz matrix. By a Riesz matrix we denote one that

$$a_{nv} = p_v/P_n \quad \text{for} \quad 0 \leq v \leq n\,, \qquad a_{nv} = 0 \quad \text{for} \quad v > n\,,$$

where (p_n) is a sequence of positive real numbers and $P_n = p_0 + p_1 + \dots + p_n$, $P_{-1} = 0$.

Let (T_n) be given by (1). If

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty\,, \tag{3}$$

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then $\sum a_n$ is said to be $|\mathbf{T}|_k$ summable, $k \geq 1$. As known, $|\mathbf{T}|_k$ summability reduces to $|\mathbf{C}, \alpha|_k$ summability whenever we put the Cesàro matrix of order α ($\alpha > -1$) in place of the matrix $\mathbf{T}$ (see [2]). And in this special case, condition (3) is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty, \tag{4}$$

where t_n^α denotes Cesàro means of order α of the sequence (na_n) (see [1]).

Flett [2], using (4), established comparison theorems between $|\mathbf{C}, \alpha|_k$, $|\mathbf{C}, \beta|_k$, and $|\mathbf{A}|_k$, where $\mathbf{A}$ denotes Abel summability.

Throughout the paper, the matrix $\mathbf{T} = (a_{nv})$ will be a Riesz matrix with $P_n \to \infty$ as $n \to \infty$. Hence, if there is no confusion, we say that $\sum a_n$ is summable $|\mathbf{R}, p_n|_k$, $k \geq 1$, if (3) holds.

Let α be any real number, and let

$$E_n^\alpha = \binom{\alpha + n}{n} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \quad \text{for} \quad n \geq 1, \qquad E_0^\alpha = 1. \tag{5}$$

We have immediately the following well-known identities:

$$\frac{1}{(1-x)^\alpha} = \sum_{v=0}^{\infty} E_v^{\alpha-1} x^v \qquad (|x| < 1), \tag{6}$$

$$\alpha > -1 \implies E_n^\alpha > 0, \tag{7}$$

$$|E_n^\alpha| \leq A(\alpha) n^\alpha \quad \text{for all} \quad \alpha, \qquad E_n^\alpha \geq A(\alpha) n^\alpha \quad \text{for} \quad \alpha > -1, \tag{8}$$

where $A(\alpha)$ is a positive constant depending on α.

$$E_n^{\alpha+\beta} = \sum_{v=0}^{n} E_{n-v}^{\alpha-1} E_v^\beta, \tag{9}$$

$$\frac{1}{nE_n^\alpha} = \int_0^1 (1-x)^\alpha x^{n-1}\, \mathrm{d}x, \qquad (\alpha > -1, \quad n \geq 1). \tag{10}$$

2. Comparison theorems

The purpose of this paper is to establish some comparison theorems for $|\mathbf{R}, p_n|_k$ and $|\mathbf{C}, \alpha|_k$ summabilities for $\alpha > 0$.

Theorem 2.1. *Let $0 < \alpha < 1$. Then $|\mathbf{R}, p_n|_k$ summability ($k \geq 1$) implies $|\mathbf{C}, \alpha|_k$ summability provided that*

$$P_n = O(n^\alpha p_n) \qquad \text{as} \quad n \to \infty. \tag{11}$$

Proof. Suppose that $\sum a_n$ is summable $|\mathbf{R}, p_n|_k$, $k \geq 1$. Then

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty, \tag{12}$$

where T_n denotes weighted means of $\sum a_n$, i.e.,

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v. \tag{13}$$

Hence, we have by (13), $s_n = T_{n-1} - \frac{P_n}{p_n} \Delta T_{n-1}$ and

$$a_n = -\Delta s_{n-1} = s_n - s_{n-1} = \begin{cases} T_0 & \text{for } n = 0, \\ \dfrac{1}{p_1}(P_1 T_1 - a_0 p_0 - p_1 a_0) & \text{for } n = 1, \\ -\Delta T_{n-2} + \Delta\left(\dfrac{P_{n-1}}{p_{n-1}} \Delta T_{n-2}\right) & \text{for } n \geq 2. \end{cases} \tag{14}$$

Now by (t_n^α) we denote the $(\mathbf{C}, \alpha)$ mean of the sequence (na_n), then it follows from (14) that

$$\begin{aligned}
t_n^\alpha &= \frac{1}{E_n^\alpha}\sum_{v=1}^{n} E_{n-v}^{\alpha-1} v a_v = \frac{E_{n-1}^{\alpha-1}}{E_n^\alpha} a_1 + \frac{1}{E_n^\alpha}\sum_{v=2}^{n} E_{n-v}^{\alpha-1} v a_v \\
&= \frac{E_{n-1}^{\alpha-1}}{E_n^\alpha} a_1 + \frac{1}{E_n^\alpha}\sum_{v=2}^{n} E_{n-v}^{\alpha-1} v \left\{ -\Delta T_{v-2} + \Delta\left(\frac{P_{v-1}}{p_{v-1}}\Delta T_{v-2}\right)\right\} \\
&= \frac{E_{n-1}^{\alpha-1}}{E_n^\alpha} a_1 + \frac{1}{E_n^\alpha}\left\{ \sum_{v=2}^{n} E_{n-v}^{\alpha-1} v(-\Delta T_{v-2}) + \sum_{v=2}^{n} E_{n-v}^{\alpha-1} v \frac{P_{v-1}}{p_{v-1}}\Delta T_{v-2} \right. \\
&\qquad \left. + \sum_{v=2}^{n} E_{n-v}^{\alpha-1} v\left(-\frac{P_v}{p_v}\Delta T_{v-1}\right)\right\} \\
&= \frac{E_{n-1}^{\alpha-1}}{E_n^\alpha} a_1 + \frac{1}{E_n^\alpha}\left\{ \sum_{v=2}^{n} E_{n-v}^{\alpha-1} v(-\Delta T_{v-2}) + \frac{2P_1}{p_1}\Delta T_0 E_{n-2}^{\alpha-1} \right. \\
&\quad \left. + \sum_{v=3}^{n} E_{n-v}^{\alpha-1} v \frac{P_{v-1}}{p_{v-1}}\Delta T_{v-2} + \sum_{v=2}^{n-1} E_{n-v}^{\alpha-1} v\left(-\frac{P_v}{p_v}\Delta T_{v-1}\right) - E_0^{\alpha-1}\frac{n P_n}{p_n}\Delta T_{n-1}\right\} \\
&= \left(E_{n-1}^{\alpha-1} a_1 + 2\frac{P_1}{p_1}\Delta T_0 E_{n-2}^{\alpha-1}\right)\frac{1}{E_n^\alpha} - \frac{n P_n \Delta T_{n-1}}{p_n E_n^\alpha} - \frac{1}{E_n^\alpha}\sum_{v=2}^{n} E_{n-v}^{\alpha-1} v \Delta T_{v-2} \\
&\qquad + \frac{1}{E_n^\alpha}\sum_{v=2}^{n-1}\left(E_{n-v-1}^{\alpha-1}(v+1) - E_{n-v}^{\alpha-1} v\right)\frac{P_v}{p_v}\Delta T_{v-1}\,.
\end{aligned}$$

On the other hand, since

$$E_{n-v-1}^{\alpha-1}(v+1) - E_{n-v}^{\alpha-1} v = v\left(E_{n-v-1}^{\alpha-1} - E_{n-v}^{\alpha-1}\right) + E_{n-v-1}^{\alpha-1} \qquad (1 \le v \le n-1),$$

and

$$E_{n-v-1}^{\alpha-1} - E_{n-v}^{\alpha-1} = -E_{n-v}^{\alpha-2} \qquad (\alpha \ne 1, \quad 1 \le v \le n-1),$$

we have

$$\begin{aligned}
t_n^\alpha &= \left(E_{n-1}^{\alpha-1} a_1 + 2\frac{P_1}{p_1}\Delta T_0 E_{n-2}^{\alpha-1}\right)\frac{1}{E_n^\alpha} - \frac{n P_n}{E_n^\alpha p_n}\Delta T_{n-1} - \frac{1}{E_n^\alpha}\sum_{v=2}^{n} E_{n-v}^{\alpha-1} v \Delta T_{v-1} \\
&\qquad + \frac{1}{E_n^\alpha}\sum_{v=2}^{n-1} E_{n-v-1}^{\alpha-1}\frac{P_v}{p_v}\Delta T_{v-1} + \frac{1}{E_n^\alpha}\sum_{v=2}^{n-1}\left(-E_{n-v}^{\alpha-2}\right) v \frac{P_v}{p_v}\Delta T_{v-1} \\
&=: w_{n,1}^\alpha + w_{n,2}^\alpha + w_{n,3}^\alpha + w_{n,4}^\alpha + w_{n,5}^\alpha\,.
\end{aligned}$$

To prove the theorem, by Minkowski's inequality and (4), it suffices to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |w_{n,i}^{\alpha}|^k < \infty \qquad \text{for} \quad i = 1,2,3,4,5\,. \tag{15}$$

Taking account of (8) and $\alpha > 0$, we have

$$\sum_{n=3}^{\infty} \frac{1}{n} |w_{n,1}^{\alpha}|^k = O\left\{\sum_{n=3}^{\infty} n^{-k-1}\right\} < \infty\,,$$

and by the fact that $P_n = O(n^{\alpha} p_n)$, and by (12)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} |w_{n,2}^{\alpha}|^k &= \sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{n\, P_n \Delta T_{n-1}}{E_n^{\alpha}\, p_n} \right|^k = \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n\, n^{\alpha}} \right)^k n^{k-1} |\Delta T_{n-1}|^k \\ &= O\left\{\sum_{n=1}^{\infty} n^{k-1} |\Delta T_{n-1}|^k\right\} < \infty\,. \end{aligned} \tag{16}$$

Now by Hölder's inequality and (9)

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{n} |w_{n,3}^{\alpha}|^k &\le \sum_{n=2}^{\infty} \frac{1}{n (E_n^{\alpha})^k} \left\{ \sum_{v=2}^{n} v\, E_{n-v}^{\alpha-1} |\Delta T_{v-2}| \right\}^k \qquad (\alpha > 0) \\ &\le \sum_{n=2}^{\infty} \frac{1}{n\, E_n^{\alpha}} \sum_{v=2}^{n} v^k E_{n-v}^{\alpha-1} |\Delta T_{v-2}|^k x \left\{ \frac{1}{E_n^{\alpha}} \sum_{v=2}^{n} E_{n-v}^{\alpha-1} \right\}^{k-1} \\ &\le \sum_{n=2}^{\infty} \frac{1}{n\, E_n^{\alpha}} \sum_{v=2}^{n} v^k E_{n-v}^{\alpha-1} |\Delta T_{v-2}|^k \\ &= \sum_{v=2}^{\infty} v^k |\Delta T_{v-2}|^k \sum_{n=v}^{\infty} \frac{E_{n-v}^{\alpha-1}}{n\, E_n^{\alpha}}\,. \end{aligned}$$

However, by (10)

$$\begin{aligned}\sum_{n=v}^{\infty} \frac{E_{n-v}^{\alpha-1}}{n\, E_n^{\alpha}} &= \sum_{i=0}^{\infty} \frac{E_i^{\alpha-1}}{(v+i) E_{v+i}^{\alpha}} = \sum_{i=0}^{\infty} E_i^{\alpha-1} \int_0^1 (1-x)^{\alpha} x^{v+i-1}\, \mathrm{d}x \\ &= \int_0^1 (1-x)^{\alpha} x^{v-1} \left(\sum_{i=0}^{\infty} E_i^{\alpha-1} x^i \right) \mathrm{d}x = \int_0^1 (1-x)^{\alpha} \frac{1}{(1-x)^{\alpha}} x^{v-1}\, \mathrm{d}x \\ &= \int_0^1 x^{v-1}\, \mathrm{d}x = \frac{1}{v}\,. \end{aligned}$$

(Term-by-term integration is legitimate since everything is positive.)

$$\sum_{n=2}^{\infty} \frac{1}{n} |w_{n,3}^{\alpha}|^k \le \sum_{v=2}^{\infty} v^{k-1} |\Delta T_{v-2}|^k = O\left\{ \sum_{v=1}^{\infty} v^{k-1} |\Delta T_{v-1}|^k \right\} < \infty ,$$

by (12), and we write

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{1}{n} |w_{n,4}^{\alpha}|^k &\le \sum_{n=3}^{\infty} \frac{1}{n(E_n^{\alpha})^k} \left\{ \sum_{v=2}^{n-1} E_{n-v-1}^{\alpha-1} \frac{P_v}{p_v} |\Delta T_{v-1}| \right\}^k \\
&\le \sum_{n=3}^{\infty} \frac{1}{nE_n^{\alpha}} \sum_{v=2}^{n-1} E_{n-v-1}^{\alpha-1} \left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k x \left\{ \frac{1}{E_n^{\alpha}} \sum_{v=2}^{n-1} E_{n-v-1}^{\alpha-1} \right\}^{k-1} \\
&\le \sum_{n=3}^{\infty} \frac{1}{n\, E_n^{\alpha}} \sum_{v=2}^{n-1} E_{n-v-1}^{\alpha-1} \left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k \\
&= \sum_{v=2}^{\infty} \left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{\infty} \frac{E_{n-v-1}^{\alpha-1}}{nE_n^{\alpha}} \\
&= \sum_{v=2}^{\infty} \frac{1}{(v+1)} \left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k \le \sum_{v=2}^{\infty} \frac{1}{v} \left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k . \qquad (17)
\end{aligned}$$

Therefore by the fact that condition $P_v = O(v^{\alpha} p_v)$ for $0 < \alpha < 1$ implies condition $P_v = O(vp_v)$ it follows by (12) that

$$\sum_{n=3}^{\infty} \frac{1}{n} |w_{n,4}^{\alpha}|^k = O\left\{ \sum_{v=2}^{\infty} v^{k-1} |\Delta T_{v-1}|^k \right\} < \infty .$$

Finally, applying Hölder's inequality for $k > 1$ (trivially for $k = 1$), we have

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{1}{n} |w_{n,5}^{\alpha}|^k &\le \sum_{n=3}^{\infty} \frac{1}{n(E_n^{\alpha})^k} \left\{ \sum_{v=2}^{n-1} \frac{v\, P_v}{p_v} |E_{n-v}^{\alpha-2}| |\Delta T_{v-1}| \right\}^k \\
&\le \sum_{n=3}^{\infty} \frac{1}{n(E_n^{\alpha})^k} \sum_{v=2}^{n-1} \left(\frac{v\, P_v}{p_v}\right)^k |\Delta T_{v-1}|^k |E_{n-v}^{\alpha-2}| x \left\{ \sum_{v=2}^{n-1} |E_{n-v}^{\alpha-2}| \right\}^{k-1} .
\end{aligned}$$

Now, if $0 < \alpha < 1$, i.e., $1 < 2 - \alpha < 2$, then (using (8)) we have

$$\sum_{v=2}^{n-1} |E_{n-v}^{\alpha-2}| \le A(\alpha) \sum_{v=2}^{n-1} (n-v)^{\alpha-2} = A(\alpha) \sum_{v=1}^{n-2} v^{\alpha-2} = O(1) \qquad \text{as} \quad n \to \infty .$$

So that it follows that

$$\sum_{n=3}^{\infty}\frac{1}{n}|w_{n,5}^{\alpha}|^k = O\Bigg\{\sum_{n=3}^{\infty}\frac{1}{n(E_n^{\alpha})^k}\sum_{v=2}^{n-1}\Big(\frac{vP_v}{p_v}\Big)^k|\Delta T_{v-1}|^k|E_{n-v}^{\alpha-2}|\Bigg\}$$

$$= O\Bigg\{\sum_{v=2}^{\infty}\Big(\frac{vP_v}{p_v}\Big)^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{\infty}\frac{|E_{n-v}^{\alpha-2}|}{n(E_n^{\alpha})^k}\Bigg\}$$

$$= O\Bigg\{\sum_{v=2}^{\infty}\Big(\frac{vP_v}{p_v}\Big)^k|\Delta T_{v-1}|^k\frac{1}{(E_v^{\alpha})^{k-1}}\sum_{n=v+1}^{\infty}\frac{|E_{n-v}^{\alpha-2}|}{nE_n^{\alpha}}\Bigg\}$$

$$= O\Bigg\{\sum_{v=2}^{\infty}\Big(\frac{vP_v}{p_v}\Big)^k|\Delta T_{v-1}|^k\frac{1}{(E_v^{\alpha})^{k-1}}\Bigg|\sum_{n=v+1}^{\infty}\frac{E_{n-v}^{\alpha-2}}{nE_n^{\alpha}}\Bigg|\Bigg\}$$

$$= O\Bigg\{\sum_{v=2}^{\infty}\Big(\frac{vP_v}{p_v}\Big)^k|\Delta T_{v-1}|^k\frac{1}{(E_v^{\alpha})^{k-1}}\Bigg|\sum_{n=v}^{\infty}\frac{E_{n-v}^{\alpha-2}}{nE_n^{\alpha}}-\frac{E_0^{\alpha-2}}{vE_v^{\alpha}}\Bigg|\Bigg\}.$$

On the other hand, by (8) and (10)

$$\Bigg|\sum_{n=v}^{\infty}\frac{E_{n-v}^{\alpha-2}}{n\,E_n^{\alpha}}\Bigg| = \Bigg|\sum_{i=0}^{\infty}\frac{E_i^{\alpha-2}}{(v+i)E_{v+i}^{\alpha}}\Bigg| = \Bigg|\sum_{i=0}^{\infty}E_i^{\alpha-2}\int_0^1(1-x)x^{v+i-1}\,\mathrm{d}x\Bigg|$$

$$= \Bigg|\int_0^1(1-x)^{\alpha}x^{v-1}\Big\{\sum_{i=0}^{\infty}E_i^{\alpha-2}x^i\Big\}\,\mathrm{d}x\Bigg| = \frac{1}{v\,E_v^1} = O\Big(\frac{1}{v^2}\Big),$$

(here term-by-term integration is legitimate since the terms are ultimately of constant sign), and by (11) we have

$$\sum_{n=3}^{\infty}\frac{1}{n}|w_{n,5}^{\alpha}|^k = O\Bigg\{\sum_{v=2}^{\infty}\Big(\frac{v\,P_v}{p_v}\Big)^k|\Delta T_{v-1}|^k\frac{1}{(E_v^{\alpha})^{k-1}}\Big(\frac{1}{v\,E_v^1}+\frac{1}{v\,E_v^{\alpha}}\Big)\Bigg\}$$

$$= O\Bigg\{\sum_{v=2}^{\infty}v^k\Big(\frac{P_v}{p_v}\Big)^k|\Delta T_{v-1}|^k\frac{1}{v^{\alpha(k-1)}}\frac{1}{v^{\alpha+1}}\Bigg\}$$

$$= O\Bigg\{\sum_{v=2}^{\infty}\Big(\frac{P_v}{v^{\alpha}p_v}\Big)^k v^{k-1}|\Delta T_{v-1}|^k\Bigg\} = O\Bigg\{\sum_{v=2}^{\infty}v^{k-1}|\Delta T_{v-1}|^k\Bigg\} < \infty.$$

This, together with (12), leads to the proof of the theorem.

The following result can be at once derived from the above theorem by taking $k = 1$.

COROLLARY 2.2. *If $P_n = O(n^\alpha p_n)$ for $0 < \alpha < 1$, then $|\mathbf{R}, p_n| \implies |\mathbf{C}, \alpha|$.*

Also, taking account of Corollary 2.2 and the following theorem of F l e t t [2], we can establish a relation between $|\mathbf{R}, p_n|$ and $|\mathbf{C}, \alpha|_k$, $k \geq 1$.

THEOREM 2.3. *If $\sum a_n$ is summable $|\mathbf{C}, \alpha|_k$, where $k > 1$, $\alpha > -1$, then it is summable $|\mathbf{C}, \beta|_r$, whenever $r \geq k$ and $\beta \geq \alpha + 1/k - 1/r$. If we take $k = 1$, the result holds when $\alpha > -1$, $\beta > \alpha + 1/k - 1/r$.*

COROLLARY 2.4. *If $P_n = O(n^\alpha p_n)$ for $0 < \alpha < 1$, then*

$$|\mathbf{R}, p_n| \implies |\mathbf{C}, \alpha + 1|_k, \qquad k \geq 1.$$

One may now ask such a question as under what condition does:
$|\mathbf{R}, p_n|_k \implies |\mathbf{C}, \beta|_k$, where $\beta \geq 1$. In fact, condition (11) is answer to this since, by Theorem 2.3 and Theorem 2.1

$$|\mathbf{R}, p_n|_k \implies |\mathbf{C}, \alpha|_k \implies |\mathbf{C}, 1|_k \implies |\mathbf{C}, \beta|_k.$$

However, we show that $|\mathbf{R}, p_n|_k \implies |\mathbf{C}, \alpha|_k$, $k \geq 1$, replacing by a weaker condition.

THEOREM 2.5. *Let $\alpha \geq 1$. Then $|\mathbf{R}, p_n|_k$ summability ($k \geq 1$) implies $|\mathbf{C}, \alpha|_k$ summability provided that*

$$P_n = O(np_n) \qquad as \quad n \to \infty. \tag{18}$$

P r o o f. The case $\alpha = 1$ is easy, so consider $\alpha > 1$, we only show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|w_{n,i}^\alpha\right|^k < \infty \qquad \text{for} \quad i = 2, 3, 5,$$

since the other is the same as in Theorem 2.1. By (16),

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|w_{n,2}^\alpha\right|^k \leq \sum_{n=1}^{\infty} \left(\frac{P_n}{n^\alpha p_n}\right)^k |\Delta T_{n-1}|^k n^{k-1}.$$

Thus by the fact that $P_n = O(np_n)$ implies $P_n = O(n^\alpha p_n)$ for $\alpha \geq 1$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|w_{n,2}^\alpha\right|^k = O\left\{\sum_{n=1}^{\infty} \left(\frac{P_n}{np_n}\right)^k n^{k-1} |\Delta T_{n-1}|^k\right\} = O\left\{\sum_{n=1}^{\infty} n^{k-1} |\Delta T_{n-1}|^k\right\} < \infty,$$

and by (17) and (18) it is clear that

$$\sum_{n=3}^{\infty} \frac{1}{n}|w_{n,3}^{\alpha}|^k \leq \sum_{v=2}^{\infty} \frac{1}{v}\left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k = O\left\{\sum_{v=2}^{\infty} v^{k-1}|\Delta T_{v-1}|^k\right\} < \infty .$$

Finally, by Hölder's inequality, for $\alpha > 1$,

$$\sum_{n=3}^{\infty} \frac{1}{n}|w_{n,5}^{\alpha}|^k \leq \sum_{n=3}^{\infty} \frac{1}{n(E_n^{\alpha})^k}\left\{\sum_{v=2}^{n-1} v\frac{P_v}{p_v}E_{n-v}^{\alpha-2}|\Delta T_{v-1}|\right\}^k$$

$$\leq \sum_{n=3}^{\infty} \frac{1}{n\,E_n^{\alpha}} \sum_{v=2}^{n-1} v\left(\frac{P_v}{p_v}\right)^k E_{n-v}^{\alpha-2}|\Delta T_{v-1}|^k x\left\{\frac{1}{E_n^{\alpha}}\sum_{v=2}^{n-1} vE_{n-v}^{\alpha-2}\right\}^{k-1}$$

Observe that for $\alpha > 1$

$$\frac{1}{E_n^{\alpha}}\sum_{v=1}^{n-1} v\,E_{n-v}^{\alpha-2} = O\left\{\frac{n}{E_n^{\alpha}}\sum_{v=1}^{n} E_{n-v}^{\alpha-2}\right\}^{k-1} = O\left(\frac{n}{E^{\alpha}}E_n^{\alpha-1}\right)^{k-1} = O(1).$$

So,

$$\sum_{n=3}^{\infty} \frac{1}{n}|w_{n,5}^{\alpha}|^k = O\left\{\sum_{v=2}^{\infty} v\left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{\infty} \frac{E_{n-v}^{\alpha-2}}{n\,E_n^{\alpha}}\right\}$$

$$= O\left\{\sum_{v=2}^{\infty} \frac{1}{v}\left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k\right\} = O\left\{\sum_{v=2}^{\infty} v^{k-1}|\Delta T_{v-1}|^k\right\} < \infty .$$

This completes the proof.

We note that, if we choose $p_n = 1$ for all n, then $P_n = n + 1$. In the case, $|\mathsf{R}, p_n|_k$ summability is the same as $|\mathsf{C}, 1|_k$ summability. Therefore, the following known result of [2] can be derived from the above theorem.

COROLLARY 2.6. *$|\mathsf{C}, 1|_k$ summability implies $|\mathsf{C}, \alpha|_k$ summability for $\alpha \geq 1$ and $k \geq 1$.*

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