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ON THE LEBESGUE DECOMPOSITION OF A FUNCTION RELATIVE TO A P-IDEAL OF AN ORTHOMODULAR LATTICE

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ABSTRACT. In this paper we established a decomposition theorem in which a finitely additive group-valued function defined in an orthomodular lattice is decomposed with respect to a p-ideal.

It is well known how interesting it is to obtain a non commutative version of the Lebesgue decomposition theorem ([6] III.4.14) also because in many questions it is important to have a function absolutely continuous with respect to another (e.g. [9]). Recently many results have been obtained in this direction ([16], [17], [12], [5], [18], [15]).

In this paper, following the method used by V. Ficker [7] and P. Capek ([3], [4]) to obtain a decomposition theorem for a real function defined on a Boolean algebra (cf. also V. Palko [13]) and by C. Tarantino [19] for the group-valued case, we establish a decomposition theorem in which a finitely additive group-valued function defined in an orthomodular lattice is decomposed with respect to a p-ideal.

Obviously this decomposition theorem generalizes the classical one and it is analogous to the theorem proved in [5], where the decomposition was established with respect to an orthoideal contained in the centre of an orthomodular poset.

In this context, having an orthomodular lattice L and an ideal I, it is useful to study the orthosublattices of L of which I is a p-ideal. This enables us to obtain a decomposition in a more restricted lattice. In Part 3 we prove that between such lattices there is always a maximal one.

1.

Let $(L, \leq)$ be a lattice with 0 and 1. In the following we employ the usual notations to indicate the supremum or the infimum of a subset of L, if they exist.

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Thus the rule R_2 *is valid in* LMC (2.3).

Proof. With respect to Lemma 1.2 (R_2), the class $\mathcal{U}_0$ in LMC is $\left\{\boldsymbol{L}_0'(\boldsymbol{Y}-\mathbf{X}\beta_0)\colon \boldsymbol{L}_0 \in \mathcal{M}\left(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\right)\right\}$. Let $(\boldsymbol{L}_{01}', \boldsymbol{L}_{02}')' \in \mathcal{M}\left(\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right) \iff \mathbf{X}'\boldsymbol{L}_{01} + \mathbf{B}'\boldsymbol{L}_{02} = \boldsymbol{O} \implies \mathbf{K}_\mathbf{B}'\mathbf{X}'\boldsymbol{L}_{01} = \boldsymbol{O} \iff \boldsymbol{L}_{01} \in \mathcal{M}\left(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\right)$; further $\boldsymbol{L}_{01}'\boldsymbol{Y} + \boldsymbol{L}_{02}'(-\boldsymbol{b}) = \boldsymbol{L}_{01}'\boldsymbol{Y} + \boldsymbol{L}_{02}'\mathbf{B}\beta_0 = \boldsymbol{L}_{01}'\boldsymbol{Y} + (-\boldsymbol{L}_{01}'\mathbf{X})\beta_0 = \boldsymbol{L}_{01}'(\boldsymbol{Y} - \mathbf{X}\beta_0)$. Let $\boldsymbol{L}_0 \in \mathcal{M}\left(\mathbf{M}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\right) \iff \mathbf{K}_\mathbf{B}'\mathbf{X}'\boldsymbol{L}_0 = \boldsymbol{O} \iff \mathbf{X}'\boldsymbol{L}_0 \in \mathcal{M}(\mathbf{B}') \iff \exists\{\boldsymbol{v} \in R^q\}\mathbf{X}'\boldsymbol{L}_0 + \mathbf{B}'\boldsymbol{v} = \boldsymbol{O} \iff \binom{\boldsymbol{L}_0}{\boldsymbol{v}} \in \mathcal{M}\left(\mathbf{M}_{\binom{\mathbf{X}}{\mathbf{B}}}\right) \implies \boldsymbol{L}_0'(\boldsymbol{Y} - \mathbf{X}\beta_0) = \boldsymbol{L}_0'\boldsymbol{Y} + \boldsymbol{v}'\mathbf{B}\beta_0 = \boldsymbol{L}_0'\boldsymbol{Y} + \boldsymbol{v}'(-\boldsymbol{b})$. □

The following lemma is useful before studing the rule R_3 in LMC (2.3).

Lemma 2.4. *Let* $\mathbf{W}$ *be an* $n\times n$ p.s.d. *matrix and let* $\mathcal{M}(\mathbf{X}) \subset (\mathbf{W})$. *Then*
(a)

$$\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \begin{cases} \mathbf{P}^{\mathbf{W}}_{\mathbf{X}} - \mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^-\mathbf{B}'} & \text{for } \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}'),\\ \mathbf{P}^{\mathbf{W}}_{\mathbf{X}} - \mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}'} & \text{otherwise,} \end{cases}$$

where $\mathbf{V}$ *is any* $q\times q$ *matrix with the property* $\mathcal{M}(\mathbf{B}'\mathbf{V}\mathbf{B}) = \mathcal{M}(\mathbf{B}')$.
(b)

$$\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^-\mathbf{B}'} = \mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X})^-\mathbf{B}'}\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \mathbf{O} \qquad \text{if } \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}')$$

and

$$\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}}\mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}'} = \mathbf{P}^{\mathbf{W}}_{\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}'}\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \mathbf{O} \qquad \text{otherwise.}$$

(c)

$$\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{X}'\mathbf{W} - \mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}' \cdot \\ \cdot[\mathbf{B}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{B}']^-\mathbf{B}(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{B}'\mathbf{V}\mathbf{B})^-\mathbf{X}'\mathbf{W}.$$

Proof. The first equality in (a) can be proved directly; as $\mathcal{M}(\mathbf{K}_\mathbf{B}) = \mathcal{M}(\mathbf{M}_{\mathbf{B}'})$, $\mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{K}_\mathbf{B}} = \mathbf{P}^{\mathbf{W}}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}} = \mathbf{X}\mathbf{M}_{\mathbf{B}'}(\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}\mathbf{X}\mathbf{M}_{\mathbf{B}'})^-\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}$.
Now the equality $\mathbf{M}_{\mathbf{B}'}(\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}\mathbf{X}\mathbf{M}_{\mathbf{B}'})^+\mathbf{M}_{\mathbf{B}'} = (\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}\mathbf{X}\mathbf{M}_{\mathbf{B}'})^+$
and the implication $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}'\mathbf{W}\mathbf{X}) \implies (\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{W}\mathbf{X}\mathbf{M}_{\mathbf{B}'})^+ = (\mathbf{X}'\mathbf{W}\mathbf{X})^+ - (\mathbf{X}'\mathbf{W}\mathbf{X})^+\mathbf{B}'[\mathbf{B}(\mathbf{X}'\mathbf{W}\mathbf{X})^+\ \mathbf{B}']^-\mathbf{B}'(\mathbf{X}'\mathbf{W}\mathbf{X})^+$ from Lemma 1.4 is to be used; thus

$P^{W}_{XK_B} = X(M_{B'}X'WXM_{B'})^{+}X'W = X(X'WX)^{+}X'W - X(X'WX)^{+}B'[B(X'WX)^{+}\cdot$
$\cdot X'WX(X'WX)^{+}B']^{-}B(X'WX)^{+}X'W = P^{W}_{X} - P^{W}_{X(X'WX)^{-}B'}\cdot$

In the case of the second equality in (a), it is sufficient to prove $R(X)$ – $R(XK_B) + R[X(X'WX+B'VB)^{-}B']$ and $\mathcal{M}(XK_B) \perp_W \mathcal{M}[X(X'WX+B'VB)^{-}B']$, where $\perp_W$ means the orthogonality with respect to W, i.e. $x, y \in \mathbb{R}^n$, $x \perp_W y \Leftrightarrow x'Wy = O$. Let $\mathcal{M}_1 = \mathcal{M}(X)$, $\mathcal{M}_2 = \mathcal{M}(XK_B) = \mathcal{M}(XM_B)$ and $\mathcal{M}_3$ $\mathcal{M}[X(X'WX+B'VB)^{-}B']$. As $M_{B'}X'WX(X'WX+B'VB)^{-}B' - M_{B'}(X'WX + B'VB)(X'WX+B'VB)^{-}B' = M_{B'}B' = O$, $\mathcal{M}_2 \perp_W \mathcal{M}_3$. To prove $R(X) = R(XK_B) + R[X(X'WX+B'VB)^{-}B']$ we proceed as follows:
$P^{W}_{XK_B} = P^{W}_{XM_{B'}} = XM_{B'}(M_{B'}X'WXM_{B'})^{+}M_{B'}X'W = X[M_{B'}(X'WX + B'VB)\cdot$
$\cdot M_B]^{+}X'W = X(X'WX + B'VB)^{+}X'W - X(X'WX + B'VB)^{+}B'[B(X'WX + B'VB)^{+}B']^{+}B(X'WX + B'VB)^{+}X'W$ (Lemma 1.4 is used)

$$WX(X'WX + B'VB)^{+}X'W = WP^{W}_{XK_B} + WM_3,$$

where
$M_3 = X(X'WX + B'VB)^{+}B'[B(X'WX + B\ VB)^{+}B']^{+}B(X'WX + B'VB)^{+}X'W$.
Both matrices $WP^{W}_{XK_B}$, WM_3 are p.s.d. and $\left(WP^{W}_{XK_B}\right)'W^{+}WM_3 = O$ (it is a consequence of $\mathcal{M}_2 \perp_W \mathcal{M}_3$); thus with respect to Lemma 1.1, we have $R[WX(X'WX+B'VB)^{+}X'W] = R\left(WP^{W}_{XK_B} + WM_3\right) = R\left(WP^{W}_{XK_B}, WM_3\right) = R\left(WP^{W}_{XK_B}\right) + R(WM_3)$. Further $R[WX(X'WX + B'VB)^{+}X'W] = R(X)$, $R\left(WP^{W}_{XK_B}\right) = R(XK_B)$ and $R(M_3) = R\ WM_3) = R[X(X'WX + B'VB)^{+}B']$.
The last three equalities are consequences of the following relations, cf. Lemma 1.5 : $X'WX + B'VB = JJ'$, $(X'WX + B'VB)^{+} = KK'$, $\mathcal{M}(X'W) \subset \mathcal{M}(J) \iff \exists\{F\colon X'W = JF\}$, thus $WXKK'X'W = F'J'KK'JF = F'F \implies R[WX(X'WX + B'VB)^{+}X'W] = R(F') \geq R(F'J) = R(X'W) \geq R(X'WW^{+}) = R(X')$; the inequality $R[WX(X'WX + B'VB)^{+}X'W] \leq R(X)$ is obvious.
Sim larly $R\left(WP^{W}_{XK_B}\right) = R(WXK_B) \geq R(W^{+}WXK_B) = R(XK_B) \geq R(WXK_B)$ and $R(M_3) \geq R(WM_3) \geq R(W^{+}WM_3) = R(M_3)$ (here the implication $\mathcal{M}(X) \subset \mathcal{M}(W) = \mathcal{M}(W^{+}) \implies W^{+}(W^{+})^{+}X = X$ was used).

The statement (b) is a consequence of the equalities $K'_B X'WX(X'WX)^{-}B' = K'_B B' = O$ and $K'_B X'WX(X'WX + B'VB)^{-}B' = K'_B(X'WX + B'VB)(X'WX + B'VB)^{-}B' = K'_B B' = O$, respectively.

(c) is implied by the equality $(M_{B'}X'WXM_{B'})^{+} = [M_{B'}(X'WX+B'VB)M_{B'}]^{+}$ and by the last statement of Lemma 1.4. □

Theorem 2.5. *In* LMC (2.3) *the rule* R_3 *is valid.*

(2.2). *Let L be an orthomodular lattice, k an infinite cardinal number, M a k-orthocomplete p-ideal, N a subset of M containing $\{0\}$. If $M \setminus N$ satisfies the α_k-condition then there is an element $c \in M$ such that $M = N_{c'} = M_{c'}$.*

P r o o f. Let H be an orthogonal maximal subset of $M \setminus N$, as $\mathrm{card}(H) \leq k$ then there exists $c = \bigvee H \in M$, and, for the lemma above, $M = N_{c'}$.

If an element a belongs to $M_{c'} \setminus N_{c'}$ then $\{a \wedge c'\} \cup H$ is an orthogonal subset of $M \setminus N$, a contradiction because of the maximality of H as an orthogonal subset of L. Then $M_{c'} = N_{c'}$.

(2.3). *Let k be a cardinal number and L an orthomodular lattice, and let L be k-orthocomplete if k is infinite. If $(x_i)_{i \in I}$ is an orthogonal family of elements of L with cardinality k and c is an element of L, we have*

$$\left(\bigvee\{x_i :\ i \in I\}\right) \wedge (\wedge\{x'_j \vee c :\ j \in I\}) = \bigvee\{x_i \wedge (x'_i \vee c) :\ i \in I\}.$$

P r o o f. It is sufficient to observe that the set

$$\{x_i \wedge c' :\ i \in I\} \cup \{x_i \wedge (x_i \wedge c')' :\ i \in I\}$$

forms an orthogonal family of cardinality k.

(2.4). *Let L be an orthomodular lattice, G a commutative topological group, M a p-ideal of L, μ an element of $\mathrm{a}(L, G)$. If c is an element of M such that $M \subseteq \mathcal{N}(\mu_{c'})$ then $\mu_{c'} \in \mathrm{a}(L, G)$.*

P r o o f. It suffices to observe that if $x, y \in L$ with $x \perp y$ the set

$$\{x \wedge c',\ y \wedge c',\ c' \wedge (x \vee y) \wedge (x' \vee c) \wedge (y' \vee c)\}$$

is an orthogonal subset of L and therefore we have

$$c' \wedge (x \vee y) = (x \wedge c') \vee (y \wedge c') \vee (c' \wedge (x \vee y) \wedge (x' \vee c) \wedge (y' \vee c)).$$

For (2.3) and [11] 2.6.4 we find that

$$(x \vee y) \wedge (x' \vee c) \wedge (y' \vee c) = (x \wedge (x' \vee c)) \vee (y \wedge (y' \vee c)) \in M.$$

Then

$$\mu_{c'}(x \vee y) = \mu_{c'}(x) + \mu_{c'}(y).$$

In the same way we prove that:

(2.5). *Let L be an orthomodular σ-orthocomplete lattice, G a topological commutative group, M a σ-orthocomplete p-ideal of L, μ an element of* $\mathrm{ca}(L,G)$. *If c is an element of M such that $M \subseteq \mathcal{N}(\mu_{c'})$ then $\mu_{c'} \in \mathrm{ca}(L,G)$.*

(2.6). *Let L be an orthomodular lattice, M a p-ideal of L, G a topological commutative group. Let μ, ξ, η be elements of* $\mathrm{a}(L,G)$ *such that*

i) $\mu = \xi + \eta$,

ii) $M \subseteq \mathcal{N}(\eta)$,

iii) $\exists\, c \in M$ *such that* $c' \in \mathcal{N}(\xi)$;

then we find, for every $x \in L$,

$$\xi(x) = \mu(x \wedge (x' \vee c)), \qquad \eta(x) = \mu(x \wedge c').$$

P r o o f. Since $x \wedge (x' \vee c) \in M$ for every $x \in L$, we find that

$$\eta(x \wedge (x' \vee c)) = 0 \qquad \text{for every } x \in L$$

and by hypothesis,

$$\xi(x \wedge c') = 0 \qquad \text{for every } x \in L,$$

therefore

$$\eta(x) = \eta(x \wedge c') + \eta(x \wedge (x' \vee c)) = \eta(x \wedge c') = \\ = \eta(x \wedge c') + \xi(x \wedge c') = \mu(x \wedge c'),$$

$$\xi(x) = \xi(x \wedge c') + \xi(x \wedge (x' \vee c)) = \xi(x \wedge (x' \vee c)) = \\ = \xi(x \wedge (x' \vee c)) + \eta(x \wedge (x' \vee c)) = \mu(x \wedge (x' \vee c)).$$

(2.7). *Let L be an orthomodular lattice, M a p-ideal of L, G a commutative topological group, μ an element of* $\mathrm{a}(L,G)$. *Moreover let c and d be two elements of M and*

$$\mu_1 : x \in L \to \mu(x \wedge c'), \qquad \mu_2 : x \in L \to \mu(x \wedge (x' \vee c)),$$
$$\nu_1 : x \in L \to \mu(x \wedge d'), \qquad \nu_2 : x \in L \to \mu(x \wedge (x' \vee d)).$$

If $M \subseteq \mathcal{N}(\mu_1) \cap \mathcal{N}(\nu_1)$, then $\mu_1 = \nu_1$ and $\mu_2 = \nu_2$.

P r o o f. M is a p-ideal, $c \vee d$ belongs to M, hence, for every $x \in L$,

$$\mu_1(x \wedge (x' \vee c \vee d)) = 0.$$

Then

$$\mu_1(x) = \mu(x \wedge c') = \mu(x \wedge c' \wedge d') + \mu(x \wedge c' \wedge (x' \vee c \vee d)) = \\ = \mu(x \wedge c' \wedge d') + \mu_1(x \wedge (x' \vee c \vee d)) = \mu(x \wedge c' \wedge d').$$

In the same way, we have

$$\nu_1(x) = \mu(x \wedge c' \wedge d') \qquad \text{for every } x \in L,$$

therefore $\mu_1 = \nu_1$.

Theorem I. *Let L be an orthomodular lattice, G a commutative topological group, μ an element of* $\mathrm{a}(L,G)$ *(resp.* $\mathrm{sa}(L,G)$*). Moreover let k be an infinite cardinal, M a k-orthocomplete p-ideal of L such that $M \setminus \mathcal{N}(\mu)$ satisfies the α_k-condition. Then μ can be uniquely represented as the sum of two elements ξ, η of* $\mathrm{a}(L,G)$ *(resp.* $\mathrm{sa}(L,G)$*) such that η is M-continuous and ξ is M-singular.*

P r o o f. Since $M \setminus \mathcal{N}(\mu) = M \setminus (M \cap \mathcal{N}(\mu))$, because of (2.2), $c \in M$ exists such that

$$M = (M \cap \mathcal{N}(\mu))_{c'} = M_{c'} \cap (\mathcal{N}(\mu))_{c'}$$

therefore

$$M \subseteq (\mathcal{N}(\mu))_{c'} = \mathcal{N}(\mu_{c'}). \tag{1}$$

Then the function $\eta = \mu_{c'}$, because of (2.4), is an element of $\mathrm{a}(L,G)$ (resp. $\mathrm{sa}(L,G)$) and because of (1), is also M-continuous.

Let ξ be the function

$$\xi : x \in L \to \mu(x \wedge (x' \vee c)),$$

obviously $\mu = \xi + \eta$, then ξ belongs to $\mathrm{a}(L,G)$ (resp. $\mathrm{sa}(L,G)$), moreover c' belongs to $\mathcal{N}(\xi)$, then ξ is M-singular.

The uniqueness of the decomposition follows from (2.6) and (2.7).

In the same way as in Theorem I, but using (2.5) instead of the (2.4), the following is proved

Theorem II. *Let L be a σ-orthocomplete orthomodular lattice, G a commutative topological group, μ an element of* $\mathrm{ca}(L,G)$. *Let k be an infinite cardinal, M a k-orthocomplete p-ideal of L such that $M \setminus \mathcal{N}(\mu)$ satisfies the α_k-condition. Then μ can be uniquely represented as the sum of two elements ξ, η of* $\mathrm{ca}(L,G)$ *such that η is M-continuous and ξ is M-singular.*

We observe that from Theorem II it is easy to obtain Theorem 2.11 of [5] and subsequently to arrive at the classical Lebesgue decomposition theorem.

We note also that proposition (2.2) is true if we suppose that M is a p-ideal and that every orthogonal subset of $M \setminus N$ is finite; then also Theorem I and Theorem II are true with the hypothesis

i) M is a p-ideal,
ii) every orthogonal subset of $M \setminus \mathcal{N}(\mu)$ is finite.

3.

(3.1). *Let L be an orthomodular lattice, H an ideal of L. Then we have an orthosublattice L_1 of L such that:*

i) *H is a p-ideal of L_1,*

ii) *there is no orthosublattice of L that strictly contains L_1 and for which H is a p-ideal within it.*

Proof. Let

$$\widehat{H} = H \cup H', \qquad \text{where} \quad H' = \{a : \ a' \in H\}.$$

Obviously $\widehat{H}$ contains H and is contained in each orthosublattice of L which contains H. Moreover

$$x \in \widehat{H} \qquad \text{implies} \quad x' \in \widehat{H}. \tag{1}$$

Let x, y be two elements of $\widehat{H}$. If they both belong to H, it is obvious that $x \vee y \in H$, if $x \notin H$, for (1), then $x' \wedge y' \in H$ and also $x \vee y = (x' \wedge y')' \in \widehat{H}$.

For every $x \in \widehat{H}$ and for every $a \in H$

$$\{x, x' \vee a\} \cap H \neq \emptyset,$$

therefore $x \wedge (x' \vee a) \in H$. Then (cf. 2.6.4 of [11]) H is a p-ideal of $\widehat{H}$.

The proof is completed by Zorn's Lemma.

If L is an orthomodular lattice obtained by Greechie's method (cf. [8] theor. 3) the results of (3.1) can be improved proving that L_1 is an orthosublattice such that

i) H is a p-ideal of L_1

ii) If Λ is an orthosublattice of L such that H is a p-ideal of Λ, then Λ is contained in L_1.

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