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UNIFORM MODULAR INTEGRABILITY AND CONVERGENCE PROPERTIES FOR A CLASS OF URYSOHN INTEGRAL OPERATORS IN FUNCTION SPACES

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ABSTRACT. We obtain some modular equi-integrability properties for a class of integral operators of the form

$$(T_w f)(s) = \int_G K_w\big(s,t,f(t)\big)\,\mathrm{d}\mu(t)\,, \qquad s \in G\,,$$

in modular spaces, where G is a locally compact topological space provided with a regular measure μ defined on the Borel sets $\mathcal{B}$ of G. Then we obtain applications to modular convergence theorems.

1. Introduction

In [1] we have begun the study of approximation properties in modular spaces for classes of Urysohn integral operators of the form

$$(T_w f)(s) = \int_G K_w\big(s,t,f(t)\big)\,\mathrm{d}\mu(t)\,, \qquad s \in G\,,\ \ w > 0\,,$$

where G is a locally compact Hausdorff topological group with the Haar measure μ, using a notion of singularity for the kernel $(K_w)_{w>0}$ which involves strongly the algebraic structure of G.

More recently, in [3] we have studied the same family of operators where G is simply a locally compact topological space without any algebraic structure, by using a more general and, in some respects, more natural notion of singularity.

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However, the extension given in [3] is not complete because our approximation result is given in a local form or in non-local form when the kernel $(K_w)_{w>0}$ satisfies some compact support assumptions.

In the present paper, for the same class of operators, we obtain a full extension of the approximation result in [1], without any compact support assumptions on the kernel, by introducing a notion of local equi-integrability for families of functions related with the kernel $(K_w)_{w>0}$. This notion is very natural, as we show by examples, and enables us to state a uniform integrability result for the family $(T_w f)_{w>0}$ when f is bounded with compact support. This in turn allows us to obtain a modular convergence theorem of $(T_w f)_{w>0}$ towards f for any continuous function f with compact support by using a Vitali type convergence theorem. Then, using a modular density result (see [1] for Orlicz spaces and [9] for modular spaces), we obtain the general modular convergence theorem for every function f belonging to a suitable modular space.

In the first part of the paper, we work in an Orlicz space which represents the typical example of the general abstract modular function spaces (see [11], [8], [6]). In the second part, we show how to extend our theory in abstract modular function spaces. In this frame we have to modify the notion of local uniform integrability taking into consideration the modular ϱ and using the notion of compatibility of ϱ with respect to regular families of measures (see Section 4). This concept was introduced firstly in [9] and then used in [3], [10].

2. Notations and definitions

Let G be a locally compact Hausdorff topological space with the family of Borel sets $\mathcal{B}$. Let μ be a regular and σ-finite measure defined on $\mathcal{B}$. We will assume that the topology of G is uniformizable, i.e. there is a uniform structure $\mathcal{U} \subset 2^{G\times G}$ which generates the topology of G (see [13]). For every $U \in \mathcal{U}$, we put $U_s = \{t \in G : (s,t) \in U\}$. By local compactness, we will assume that for every $s \in G$, the base $\{U_s : U \in \mathcal{U}\}$ contains compact sets.

We will denote by $X(G)$ the space of all real-valued measurable functions $f\colon G \to \mathbb{R}$ provided with equality a.e., by $C(G)$ the space of all bounded and continuous functions $f\colon G \to \mathbb{R}$ and by $C_c(G)$ the space of all continuous functions with compact support.

Let us recall that a function $f\colon G \to \mathbb{R}$ is *uniformly continuous on* G if for every $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that $|f(t) - f(s)| < \varepsilon$ for every $s \in G$, $t \in U_s$.

Let $\mathcal{L}$ be the class of all globally measurable functions $L\colon G \times G \to \mathbb{R}_0^+$ such that the sections $L(\cdot, t)$ and $L(s, \cdot)$ belong to $L^1(G)$ for every $t, s \in G$ respectively.

Let Ψ be the class of all functions $\psi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that ψ is continuous (nondecreasing) function and $\psi(0)=0$; $\psi(u)>0$ for $u>0$. For a given $\psi \in \Psi$, let $\mathcal{K}_\psi$ be the class of all functions $K\colon G \times G \times \mathbb{R} \to \mathbb{R}$ such that the following conditions hold:

K.1) $K(\cdot,\cdot,u)$ is measurable on $G\times G$ for every $u \in \mathbb{R}$ and $K(s,t,0)=0$ for every $(s,t)\in G\times G$.

K.2) K is (L,ψ)-Lipschitz i.e. there are a function $L \in \mathcal{L}$ and a constant $D>0$ such that

$$0<\beta(s):=\int_G L(s,t)\,\mathrm{d}\mu(t)\le D$$

for all $s\in G$ and

$$|K(s,t,u)-K(s,t,v)|\le L(s,t)\psi\big(|u-v|\big)$$

for every $s,t\in G$, $u,v\in\mathbb{R}$.

Let $\Xi=(\psi_w)_{w>0}\subset\Psi$ be a family of functions such that the following two assumptions hold:

1. $(\psi_w)_{w>0}$ is equicontinuous at $u=0$,
2. for every $u\ge 0$ the net $\big(\psi_w(u)\big)_{w>0}$ is bounded.

We denote by $\mathcal{K}_\Xi$ the class of all families of functions $\mathbb{K}=(K_w)_{w>0}$ such that for every $w>0$ we have $K_w\in\mathcal{K}_{\psi_w}$. Let us denote by $\mathbb{L}=(L_w)_{w>0}\subset\mathcal{L}$ the corresponding class of functions such that the Lipschitz condition holds for any $w>0$, i.e.

$$|K_w(s,t,u)-K_w(s,t,v)|\le L_w(s,t)\psi_w\big(|u-v|\big)$$

for every $s,t\in G$, $u,v\in\mathbb{R}$ and $w>0$.

For a given $\mathbb{K}=(K_w)_{w>0}\in\mathcal{K}_\Xi$ we will take into consideration the following family of nonlinear integral operators $\mathbf{T}=(T_w)_{w>0}$ given by

$$(T_wf)(s)=\int_G K_w\big(s,t,f(t)\big)\,\mathrm{d}\mu(t)\,,\qquad s\in G\,,$$

where $f\in\operatorname{Dom}\mathbf{T}=\bigcap_{w>0}\operatorname{Dom}T_w$; here $\operatorname{Dom}T_w$ is the subset of $X(G)$ on which T_wf is well defined as a μ-measurable function of $s\in G$.

We will say that $\mathbb{K}$ is *singular* if the following assumptions hold:

1) There is $D>0$ such that for every $s,t\in G$ and $w>0$ we have

$$0<\beta_w(s)=\int_G L_w(s,t)\,\mathrm{d}\mu(t)\le D\,,\qquad \int_G L_w(s,t)\,\mathrm{d}\mu(s)\le D\,.$$

2) For every $s \in G$ and for every $U \in \mathcal{U}$ we have

$$\lim_{w\to+\infty} \int_{G\setminus U_s} L_w(s,t)\ \mathrm{d}\mu(t) = 0\,.$$

3) For every $s \in G$ and for every $u \in \mathbb{R}$ we have

$$\lim_{w\to+\infty} \int_G K_w(s,t,u)\ \mathrm{d}\mu(t) = u\,.$$

We will say that $\mathbb{K}$ is *uniformly singular* if conditions 2) and 3) are replaced by the following ones:

2') For every $U \in \mathcal{U}$ we have

$$\lim_{w\to+\infty} \int_{G\setminus U_s} L_w(s,t)\ \mathrm{d}\mu(t) = 0$$

uniformly with respect to $s \in G$,

3') we have

$$\lim_{w\to+\infty} \int_G K_w(s,t,u)\ \mathrm{d}\mu(t) = u$$

uniformly with respect to $s \in G$ and $u \in C$, where C is any compact subset of $\mathbb{R}\setminus\{0\}$.

For the above concepts see [3].

We have the following theorem:

Theorem 1. *Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$. Then $L^\infty(G) \subset \mathrm{Dom}\,\mathbf{T}$ and for every $w > 0$, $T_w f \in L^\infty(G)$ whenever $f \in L^\infty(G)$.*

Proof. We obtain easily the measurability of $K_w\big(s,\cdot,f(\cdot)\big)$ and $T_w f$, with $f \in L^\infty(G)$. Moreover we obtain

$$|(T_w f)(s)| \le \int_G L_w(s,t)\psi_w\big(|f(t)|\big)\ \mathrm{d}\mu(t) \le \psi_w\big(\|f\|_\infty\big)D \le M'D\,,$$

being $M' = \sup_{w>0} \psi_w\big(\|f\|_\infty\big)$. □

As to the pointwise convergence, in [3] we proved the following result:

Theorem 2. *Let $f \in L^\infty(G)$ and $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. Then $T_w f \to f$ pointwise at every continuity point of f. Moreover, if $\mathbb{K}$ is uniformly singular, then $\|T_w f - f\|_\infty \to 0$ as $w \to +\infty$, whenever $f \in C(G)$ is uniformly continuous.*

3. Convergence in Orlicz spaces

Let Φ be the class of all functions $\varphi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

i) φ is continuous, nondecreasing,

ii) $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim\limits_{u\to+\infty} \varphi(u) = +\infty$.

Moreover, we denote by $\widetilde{\Phi}$ the subspace of Φ whose elements are convex functions.

For $\varphi \in \Phi$, we define the functional

$$\varrho^\varphi(f) = \int_G \varphi\bigl(|f(s)|\bigr)\,\mathrm{d}\mu(s)$$

for every $f \in X(G)$.

As it is well known, ϱ^φ is a modular on $X(G)$ and the subspace

$$L^\varphi(G) = \bigl\{f \in X(G):\ \varrho^\varphi(\lambda f) < +\infty \ \text{for some}\ \lambda > 0\bigr\}$$

is the Orlicz space generated by φ (see [11]). If $\varphi \in \widetilde{\Phi}$, then ϱ^φ is a convex modular. The subspace of $L^\varphi(G)$, defined by

$$E^\varphi(G) = \bigl\{f \in X(G):\ \varrho^\varphi(\lambda f) < +\infty \ \text{for every}\ \lambda > 0\bigr\}\,,$$

is called the *space of finite elements of* $L^\varphi(G)$. For example every bounded function with compact support belongs to $E^\varphi(G)$.

In order to establish a convergence result in Orlicz spaces, we consider the following notion of convergence (see [11]). We say that a sequence $(f_w)_{w>0} \subset L^\varphi(G)$ is *modularly convergent to* $f \in L^\varphi(G)$ if there is $\lambda > 0$ such that

$$\lim_{w\to+\infty} \varrho^\varphi\bigl[\lambda(f_w - f)\bigr] = 0\,.$$

This notion of convergence induces a topology on $L^\varphi(G)$, called *modular topology*.

In the following we will need a link between the modular ϱ^φ and the family of functions $\Xi = (\psi_w)_{w>0}$ used in the Lipschitz condition of $(K_w)_{w>0}$. Given a function $\eta \in \Phi$, we introduce the modular $\varrho^\eta(f) = \int_G \eta\bigl(|f(s)|\bigr)\,\mathrm{d}\mu(s)$ and we will say that the triple $(\varrho^\varphi, \psi_w, \varrho^\eta)$ is *properly directed* if the following condition holds: there exists a net $(c_w)_{w>0}$, with $c_w \to 0$ as $w \to +\infty$, for which for every $\lambda \in\,]0,1[$ there exists $C_\lambda \in\,]0,1[$ with

$$\int_G \varphi\Bigl(C_\lambda \psi_w\bigl(|f(s)|\bigr)\Bigr)\,\mathrm{d}\mu(s) \le \int_G \eta\bigl(\lambda|f(s)|\bigr)\,\mathrm{d}\mu(s) + c_u \tag{1}$$

for every function $f \in X(G)$.

PROPOSITION 1. ([1]) *Let $\varphi \in \Phi$. Then*

$$\overline{C_c(G)} = L^{\varphi}(G)\,,$$

where $\overline{C_c(G)}$ denotes the closure of $C_c(G)$ with respect to the modular topology in $L^{\varphi}(G)$.

We will need the following definition.

We say that the family

$$\big(L_w(\cdot,t)\big)_{t\in G,\, w > 0}$$

satisfies *property* $(*)$ if for every compact $C \subset G$ and $\varepsilon > 0$ there exists a compact subset $B \subset G$ such that

$$\int_{G\setminus B} L_w(s,t)\ \mathrm{d}\mu(s) < \varepsilon \tag{2}$$

for every $t \in C$, and sufficiently large $w > 0$.

EXAMPLES 1.

1. Classical kernels satisfy property $(*)$. Indeed, let $(\widetilde{K}_w)_{w>0}$ be an approximate identity on $G = \mathbb{R}$. For $\delta > 0$, we have that (see [7])

$$\lim_{w\to+\infty} \int_{|t|>\delta} |\widetilde{K}_w(t)|\ \mathrm{d}t = 0\,.$$

Let $C = [-M, M]$. Assume that the kernel $K_w(s,t,u)$ has the form

$$K_w(s,t,u) = \widetilde{K}_w(s-t)u$$

and so (2), with $B = [-\delta,\delta]$ and $L_w(s,t) = |\widetilde{K}_w(s-t)|$, becomes

$$\int_{|s|>\delta} |\widetilde{K}_w(s-t)|\ \mathrm{d}s = \int_{-\infty}^{-\delta} |\widetilde{K}_w(s-t)|\ \mathrm{d}s + \int_{\delta}^{+\infty} |\widetilde{K}_w(s-t)|\ \mathrm{d}s$$

$$= \int_{-\infty}^{-\delta-t} |\widetilde{K}_w(u)|\ \mathrm{d}u + \int_{\delta-t}^{+\infty} |\widetilde{K}_w(u)|\ \mathrm{d}u\,.$$

Then if δ is such that $-\delta - t < 0$ and $\delta - t > 0$ for all $t \in C$, both integrals tend to zero by the classical previous property of $\widetilde{K}_w$. In an analogous way, we can prove the same property for approximate identities defined in $G = \mathbb{R}^N$.

2. More generally, let G be a locally compact and σ-compact (Hausdorff) topological group with operation $+$ and neutral element θ. Let μ be the Haar

measure on the Borel σ-algebra of G. Let us consider a kernel $(K_w)_{w>0}$ defined for every $w > 0$ by

$$K_w(s,t,u) = \widetilde{K}_w(s-t,u)\,, \qquad s,t \in G\,, \ \ u \in \mathbb{R}\,,$$

where for every $w > 0$, $\widetilde{K}_w \colon G \to \mathbb{R}$ belongs to $L^1(G)$ and satisfies a Lipschitz condition of the form:

$$|\widetilde{K}_w(t,u) - \widetilde{K}_w(t,v)| \le \widetilde{L}_w(t)\psi\big(|u-v|\big)\,, \qquad t \in G\,, \ \ u,v \in \mathbb{R}\,,$$

and where for every $w > 0$ the function $\widetilde{L}_w \in L^1(G)$, and $\|\widetilde{L}_w\|_1 \le D$ for some absolute constant $D > 0$. Assume that for every compact, symmetric neighbourhood of θ there results

$$\lim_{w\to+\infty} \int\limits_{G\setminus U} \widetilde{L}_w(t)\ \mathrm{d}\mu(t) = 0\,.$$

Then, the family of functions $(L_w)_{w>0}$, $L_w \colon G \times G \to \mathbb{R}$, defined by

$$L_w(s,t) = \widetilde{L}_w(s-t)\,, \qquad s,t \in G\,,$$

satisfies property $(*)$. Thus, the present theory is applicable to families of non-linear integral operators of convolution type, with singular kernels (see [6]).

3. Let $G = \mathbb{R}$ provided with Lebesgue measure. Let us consider the family of functions

$$L_w(s,t) = \frac{2w}{\pi}\frac{\mathrm{e}^{w(s+t)}}{\mathrm{e}^{2ws} + \mathrm{e}^{2wt}}$$

for $s,t \in \mathbb{R}$ and $w \ge 1$. Then for every $t \in \mathbb{R}$ and $r > 0$ we have

$$\int\limits_{\mathbb{R}\setminus[t-r,t+r]} L_w(s,t)\ \mathrm{d}s = 1 + \frac{2}{\pi}(\arctan \mathrm{e}^{-wr} - \arctan \mathrm{e}^{wr})\,,$$

and so (2) follows. This is an example in non-convolution case (see [4]).

We have the following theorem.

THEOREM 3. *Let $\varphi \in \widetilde{\Phi}$, $\Xi = (\psi_w)_{w>0} \subset \Psi$ and let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. Let f be a bounded function with compact support and let C be the support of f. If the family $\big(L_w(\cdot,t)\big)_{t\in G,\, w>0}$ satisfies property $(*)$, then there exists a constant $\alpha > 0$, independent of f, such that following properties hold:*

i) *for every $\varepsilon > 0$ there is a compact set B such that*

$$\varrho^\varphi\big[\alpha(T_w f)\chi_{G\setminus B}\big] < \varepsilon$$

for sufficiently large $w > 0$,

ii) *for every sequence $(B_k)_{k\in\mathbb{N}} \subset \mathcal{B}$ with $B_{k+1} \subset B_k$ and $\mu(B_k) \to 0$ we have*

$$\lim_{k\to+\infty} \varrho^\varphi\big[\alpha(T_w f)\chi_{B_k}\big] = 0\,,$$

uniformly with respect to $w > 0$.

P r o o f . Let $\alpha > 0$ be fixed such that $\alpha D \leq 1$. Let $\varepsilon > 0$ be fixed. For every measurable subset $B \subset G$, we have

$$\begin{aligned} \varrho^{\varphi}\big[\alpha(T_w f)\chi_{G\setminus B}\big] &= \int_G \varphi\big(\alpha|(T_w f)(s)|\chi_{G\setminus B}(s)\big)\, \mathrm{d}\mu(s) \\ &\leq \int_{G\setminus B} \varphi\bigg(\alpha \int_G \big|K_w(s,t,f(t))\big|\, \mathrm{d}\mu(t)\bigg)\, \mathrm{d}\mu(s) \\ &\leq \int_{G\setminus B} \varphi\bigg(\alpha \int_G L_w(s,t)\psi_w\big(|f(t)|\big)\, \mathrm{d}\mu(t)\bigg)\, \mathrm{d}\mu(s)\,. \end{aligned}$$

If we consider

$$\nu_w^s(A) = \int_A L_w(s,t)\, \mathrm{d}\mu(t)\,, \qquad A \in \mathcal{B}\,,$$

we have $\nu_w^s(A) \leq \nu_w^s(G) = \beta_w(s) \leq D$. By using the Jensen inequality, the Fubini-Tonelli theorem and taking into account that every function $f \in L^\infty(G$ with compact support C is contained in $E^\varphi(G)$, we obtain

$$\begin{aligned} \varrho^{\varphi}\big[\alpha(T_w f)\chi_{G\setminus B}\big] &\leq \int_{G\setminus B} \varphi\bigg(\alpha D \int_C \frac{L_w(s,t)}{D}\psi_w\big(|f(t)|\big)\, \mathrm{d}\mu(t)\bigg)\, \mathrm{d}\mu(s) \\ &\leq \int_{G\setminus B} \bigg[\int_C \varphi\big(\alpha D\psi_w(|f(t)|)\big)\frac{L_w(s,t)}{D}\, \mathrm{d}\mu(t)\bigg]\, \mathrm{d}\mu(s) \\ &= \frac{1}{D}\int_{G\setminus B} \bigg[\int_C \varphi\big(\psi_w(|f(t)|)\big)L_w(s,t)\, \mathrm{d}\mu(t)\bigg]\, \mathrm{d}\mu(s) \\ &\leq \frac{1}{D}\int_G \varphi\big(\psi_w(|f(t)|\chi_C(t))\big)\bigg[\int_{G\setminus B} L_w(s,t)\, \mathrm{d}\mu(s)\bigg]\, \mathrm{d}\mu(t)\,. \end{aligned}$$

By property $(*)$ we can consider a compact subset $B \subset G$ such that

$$\int_{G\setminus B} L_w(s,t)\, \mathrm{d}\mu(s) < \frac{\varepsilon D}{\varrho^{\varphi}[M'\chi_C]}\,,$$

where $M' = \sup_{w>0} \psi_w\big(\|f\|_\infty\big)$. Then

$$\varrho^{\varphi}\big[\alpha(T_w f)\chi_{G\setminus B}\big] < \varepsilon\,.$$

Now we can prove ii). Let $(B_k)_{k\in\mathbb{N}}$ be a sequence of measurable sets with $B_{k+1} \subset B_k$ and $\lim_{k\to+\infty} \mu(B_k) = 0$. Then, considering a constant α such that $\alpha D \leq 1$, we obtain

$$\varrho^{\varphi}\big[\alpha(T_w f)\chi_{B_k}\big] = \int_G \varphi\big(\alpha|(T_w f)(s)|\chi_{B_k}(s)\big)\,\mathrm{d}\mu(s)$$
$$\leq \int_{B_k} \varphi\big(\alpha|(T_w f)(s)|\big)\,\mathrm{d}\mu(s) \leq \varphi\Big(\psi_w\big(\|f\|_\infty\big)\Big)\mu(B_k)\,.$$

Since $\big(\psi_w\big(\|f\|_\infty\big)\big)$ is bounded, we have easily the assertion. □

Using Theorems 2 and 3, we obtain the following theorem.

THEOREM 4. *Let* $\varphi \in \widetilde{\Phi}$, $\Xi = (\psi_w)_{w>0} \subset \Psi$ *and* $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ *be singular. If the family* $\big(L_w(\cdot,t)\big)_{t\in G,\, w>0}$ *satisfies property* $(*)$, *then there exists a constant* $\alpha > 0$ *such that*

$$\lim_{w\to+\infty} \varrho^{\varphi}\big[\alpha(T_w f - f)\big] = 0$$

for every $f \in C_c(G)$.

P r o o f. From Theorem 2, $T_w f$ converges pointwise to f, so by continuity of φ we have that $\varphi\big(\alpha|T_w f - f|\big)$ tends to zero pointwise, too. Moreover, since for every subset $A \subset G$ we have

$$\int_A \varphi\big(\alpha|(T_w f)(s) - f(s)|\big)\,\mathrm{d}\mu(s)$$
$$\leq \int_A \varphi\big(2\alpha|(T_w f)(s)|\big)\,\mathrm{d}\mu(s) + \int_A \varphi\big(2\alpha|f(s)|\big)\,\mathrm{d}\mu(s)$$

and $f \in E^{\varphi}(G)$, we deduce by Theorem 3 that the integrals

$$\int_{(\cdot)} \varphi\big(\alpha|(T_w f)(s) - f(s)|\big)\,\mathrm{d}\mu(s)$$

are equiabsolutely continuous, by taking α such that $2\alpha D \leq 1$. So the assertion follows from the Vitali convergence Theorem. □

In order to give a modular convergence theorem for every function $f \in L^{\varphi+\eta}(G) \cap \operatorname{Dom}\mathbf{T}$, we prove the following continuity result for the family $(T_w)_{w>0}$.

THEOREM 5. *Let $\varphi \in \widetilde{\Phi}$, $\eta \in \Phi$ and $\Xi = (\psi_w)_{w>0} \subset \Psi$ be such that the triple $(\varrho^\varphi, \psi_w, \varrho^\eta)$ is properly directed. Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. If $f, g \in X(G) \cap \operatorname{Dom} \mathbf{T}$, then, given $\lambda > 0$, there exists $\alpha > 0$ such that*

$$\varrho^\varphi\big[\alpha(T_w f - T_w g)\big] \leq \varrho^\eta\big[\lambda(f-g)\big] + c_w\,.$$

P r o o f. Let $\lambda > 0$ be fixed and let α be a positive constant with $\alpha D \leq C_\lambda$, being C_λ the constant in (1). By using the Jensen inequality and the Fubini-Tonelli theorem, we have

$$\begin{aligned}
\varrho^\varphi\big[\alpha(T_w f - T_w g)\big] &= \int_G \varphi\big(\alpha|(T_w f)(s) - (T_w g)(s)|\big)\, \mathrm{d}\mu(s) \\
&\leq \int_G \varphi\bigg(\alpha \int_G L_w(s,t)\,\psi_w\big(|f(t)-g(t)|\big)\, \mathrm{d}\mu(t)\bigg)\, \mathrm{d}\mu(s) \\
&\leq \int_G \bigg(\int_G \varphi\big(\alpha D \psi_w(|f(t)-g(t)|)\big)\frac{L_w(s,t)}{D}\, \mathrm{d}\mu(t)\bigg)\, \mathrm{d}\mu(s) \\
&\leq \frac{1}{D}\int_G \bigg(\int_G \varphi\big(C_\lambda \psi_w(|f(t)-g(t)|)\big) L_w(s,t)\, \mathrm{d}\mu(t)\bigg)\, \mathrm{d}\mu(s) \\
&= \frac{1}{D}\int_G \varphi\big(C_\lambda \psi_w(|f(t)-g(t)|)\big)\bigg(\int_G L_w(s,t)\, \mathrm{d}\mu(s)\bigg)\, \mathrm{d}\mu(t) \\
&\leq \int_G \varphi\big(C_\lambda \psi_w(|f(t)-g(t)|)\big)\, \mathrm{d}\mu(t) \\
&\leq \int_G \eta\big(\lambda|f(t)-g(t)|\big)\, \mathrm{d}\mu(t) + c_w \\
&= \varrho^\eta\big[\lambda(f-g)\big] + c_w\,.
\end{aligned}$$

□

Now we are ready to prove a general modular convergence result.

THEOREM 6. *Let $\varphi \in \widetilde{\Phi}$, $\eta \in \Phi$ and $\Xi = (\psi_w)_{w>0} \subset \Psi$ be such that the triple $(\varrho^\varphi, \psi_w, \varrho^\eta)$ is properly directed. Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. Let us assume that the family $\big(L_w(\cdot,t)\big)_{t\in G,\, w>0}$ satisfies property $(*)$. Then, for every $f \in L^{\varphi+\eta}(G) \cap \operatorname{Dom} \mathbf{T}$, there exists a constant $\alpha > 0$ such that*

$$\lim_{w\to+\infty} \varrho^\varphi\big[\alpha(T_w f - f)\big] = 0\,.$$

P r o o f . Let $f \in L^{\varphi+\eta}(G) \cap \operatorname{Dom} \mathbf{T}$. By Proposition 1 there is a constant $\lambda' \in]0,1[$ and a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(G)$ such that for every $\varepsilon > 0$ there exists $\overline{n} \in \mathbb{N}$ with

$$\varrho^{\varphi+\eta}\big[\lambda'(f_n - f)\big] < \varepsilon$$

for every $n \geq \overline{n}$. Now we can consider the corresponding constant $C_{\lambda'}$ in (1). Moreover, given $n \geq \overline{n}$, by Theorem 4 we have that there is $\lambda'' > 0$, independent of n, such that for every $\varepsilon > 0$ there exists $\overline{w} > 0$ with

$$\varrho^{\varphi}\big[\lambda''(T_w f_n - f_n)\big] < \varepsilon$$

for every $w > \overline{w}$. So, if we take α such that $3\alpha \leq \big\{\lambda', \frac{C_{\lambda'}}{D}, \lambda''\big\}$, we obtain

$$\begin{aligned}\varrho^{\varphi}\big[\alpha(T_w f - f)\big] &\leq \frac{1}{3}\varrho^{\varphi}\big[3\alpha(T_w f - T_w f_n)\big] + \frac{1}{3}\varrho^{\varphi}\big[3\alpha(T_w f_n - f_n)\big] \\ &\quad + \frac{1}{3}\varrho^{\varphi}\big[3\alpha(f_n - f)\big] = I_1 + I_2 + I_3 .\end{aligned}$$

By Theorem 5 we have that

$$I_1 \leq \frac{1}{3}\varrho^{\eta}\big[\lambda'(f - f_n)\big] + \frac{c_w}{3} ,$$

and so we obtain for $n \geq \overline{n}$ and $w \geq \overline{w}$

$$\begin{aligned}\varrho^{\varphi}\big[\alpha(T_w f - f)\big] &\leq \frac{1}{3}(\varrho^{\varphi} + \varrho^{\eta})\big[\lambda'(f - f_n)\big] + \frac{c_w}{3} + \frac{\varepsilon}{3} \\ &\leq \frac{2\varepsilon}{3} + \frac{c_w}{3} .\end{aligned}$$

The assertion follows since ε is arbitrary. □

4. Extension to modular spaces

Now we will introduce a general class of functional spaces.

A functional $\varrho\colon X(G) \to \overline{\mathbb{R}_0^+}$ is said to be a *modular* on $X(G)$ if

i) $\varrho(f) = 0 \iff f = 0$ a.e. in G,
ii) $\varrho(-f) = \varrho(f)$ for every $f \in X(G)$,
iii) $\varrho(\alpha f + \beta g) \leq \varrho(f) + \varrho(g)$ for every $f, g \in X(G)$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$,
iv) $\varrho\big(F(t,\cdot)\big)$ is a μ-measurable function of $t \in G$ for any globally measurable function $F\colon G \times G \to \mathbb{R}_0^+$.

By means of the functional ϱ, we introduce the vector subspace of $X(G)$, denoted by $L^{\varrho}(G)$, defined by

$$L^{\varrho}(G) = \Big\{ f \in X(G) : \lim_{\lambda \to 0+} \varrho(\lambda f) = 0 \Big\} .$$

The subspace $L^\varrho(G)$ is called the *modular space generated by* ϱ. The subspace of $L^\varrho(G)$ defined by

$$E^\varrho(G) = \{f \in L^\varrho(G) : \ \varrho(\lambda f) < +\infty \ \text{for all} \ \lambda > 0\}$$

is called the *space of the finite elements* of $L^\varrho(G)$, see [11].

The following assumptions on measurable modulars will be used

a) ϱ is *monotone*, i.e. if $f, g \in X(G)$ and $|f| \leq |g|$, then $\varrho(f) \leq \varrho(g)$.

b) ϱ is *finite*, i.e. if A is a measurable subset of G such that $\mu(A) < +\infty$, then $\chi_A \in L^\varrho(G)$.

c) ϱ is *absolutely finite*, i.e. ϱ is finite and for every $\varepsilon > 0$, $\lambda > 0$ there is $\delta > 0$ such that $\varrho(\lambda\chi_B) < \varepsilon$ for any measurable subset $B \subset G$ with $\mu(B) < \delta$.

d) ϱ is *strongly finite*, i.e. if A is a measurable subset of G such that $\mu(A) < +\infty$, then $\chi_A \in E^\varrho(G)$.

e) ϱ is *absolutely continuous*, i.e. there exists $\alpha > 0$ such that, for every $f \in X(G)$ with $\varrho(f) < +\infty$, the following two conditions are satisfied

(e.1) for every $\varepsilon > 0$ there is a compact subset $V \subset G$ such that $\varrho(\alpha f \chi_{G\setminus V}) < \varepsilon$;

(e.2) for every $\varepsilon > 0$ there is $\delta > 0$ such that $\varrho(\alpha f \chi_B) < \varepsilon$ for every measurable subset $B \subset G$ with $\mu(B) < \delta$.

For the above notions see [12], [6].

A classical example of modular space where the generating modular functional satisfies all the above assumptions, as well as Orlicz spaces introduced before, is given, more generally, by any Musielak-Orlicz space generated by a φ-function φ depending on a parameter, satisfying some growth conditions with respect to the parameter (see [11], [8], [6]).

Analogously to Section 3, we say that a net of functions $(f_w)_{w>0} \subset L^\varrho(G)$ is *modularly convergent* to a function $f \in L^\varrho(G)$ if there exists $\lambda > 0$ such that

$$\lim_{w\to+\infty} \varrho[\lambda(f_w - f)] = 0.$$

Moreover we will say that the net $(f_w)_{w>0} \subset L^\varrho(G)$ is *strongly convergent* to $f \in L^\varrho(G)$ if the above limit relation is satisfied for every $\lambda > 0$. This kind of convergence is equivalent to the norm convergence induced by the Luxemburg F-norm generated by the modular ϱ, see [11].

The modular convergence induces a topology on $L^\varrho(G)$, called *modular topology*. Given a subset $\mathcal{A} \subset L^\varrho(G)$, we will denote by $\overline{\mathcal{A}}$ the closure of $\mathcal{A}$ with respect to the modular topology. Then $f \in \overline{\mathcal{A}}$ if there is a net $(f_w)_{w>0} \subset \mathcal{A}$ such that f_w is modularly convergent to f.

Let us remark that $C_c(G) \subset L^\varrho(G)$ whenever ϱ is monotone and finite Indeed, if $C = \operatorname{supp} f$, then

$$\varrho(f) \leq \varrho(\|f\|_\infty \chi_C),$$

and so, since $\chi_C \in L^\varrho(G)$, we have

$$\lim_{\lambda\to 0^+} \varrho(\lambda f) = 0,$$

that is $f \in L^\varrho(G)$.

Analogously, if ϱ is monotone and strongly finite, then $C_c(G) \subset E^\varrho(G)$.

We have the following proposition:

PROPOSITION 2. *Let ϱ be a monotone, strongly finite, absolutely finite and absolutely continuous modular on $X(G)$. Then $\overline{C_c(G)} = L^\varrho(G)$.*

Proof. The proof is essentially the same as in [9; Theorem 1]. □

As in Section 3, we will need a link between the modular ϱ and the family $\Xi = (\psi_w)_{w>0} \subset \Psi$ given in the Lipschitz condition of $(K_w)_{w>0}$. This link is based on the introduction of another modular η on $X(G)$ such that the following condition holds: there exists a net $(c_w)_{w>0}$ with $c_w \to 0$ as $w \to +\infty$ for which for every $\lambda \in]0,1[$ there exists $C_\lambda \in]0,1[$ satisfying the inequality

$$\varrho\big(C_\lambda(\psi_w \circ g)\big) \le \eta(\lambda g) + c_w \tag{3}$$

for every nonnegative $g \in X(G)$. As in the case of the Orlicz spaces we will say that the triple (ϱ, ψ_w, η) is *properly directed* (see [2]).

We refer now to the notations and definitions of Section 2. We will say that the family of functions $\big(L_w(\cdot,t)\big)_{t\in G,\, w>0}$ satisfies *property ϱ-$(*)$* if for every compact $C \subset G$ and $\varepsilon > 0$ there exists a compact subset $B \subset G$ such that

$$\varrho\left[\lambda \int_C L_w(\cdot,t)\ \mathrm{d}\mu(t)\, \chi_{G\setminus B}(\cdot)\right] < \varepsilon$$

for every $\lambda > 0$ and sufficiently large $w > 0$.

EXAMPLES 2. As a first example, note that the above property is satisfied for any Orlicz spaces where the generating function φ is convex (see the proof of (i) in Theorem 3).

Another interesting example is given by some kind of a modular generated by a family of measures $\{m_j\}_{j>0}$ on an interval $I = [a,b[$, $b \in \overline{\mathbb{R}^+}$.

For example let us suppose that the family of measures $\{m_j\}_{j>0}$ is equibounded, i.e. there is a constant $H > 0$ such that $m_j\big([a,b[\big) \le H$ for every $j > 0$. Next, let $\Phi\colon I \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function such that Φ is a (continuous) convex function of $u \ge 0$ for every $x \in I$, $\Phi(x,0) = 0$, $\Phi(x,u) > 0$ for $u > 0$ and moreover $\Phi(x,u)$ is a Lebesgue measurable function of x in $[a,b[$ for every $u \ge 0$. Moreover let us assume that

$$\sup_{j>0} \int_a^b \Phi(x,r)\ \mathrm{d}m_j(x)$$

is finite for every $r > 0$. Finally, putting

$$\mathcal{J}_\Phi(x,f) = \int_G \Phi\big(x, |f(t)|\big)\ \mathrm{d}\mu(t)\,, \qquad f \in X(G)\,,\ \ x \in I\,,$$

we define the modular (see [5], [6])

$$\varrho(f) = \sup_{j>0} \int_a^b \mathcal{J}_\Phi(x,f)\ \mathrm{d}m_j(x)\,, \qquad f \in X(G)\,.$$

Then, if the family $\big(L_w(\cdot,t)\big)_{t\in G,\, w>0}$ satisfies property $(*)$, then it satisfies property ϱ-$(*)$, too.

We prove now a modular extension of Theorem 3 of Section 3.

THEOREM 7. *Let ϱ be a monotone, absolutely finite and strongly finite modular. Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. Let f be a bounded function with compact support and let C be the support of f. If the family $\big(L_w(\cdot,t)\big)_{t\in G,\, w>0}$ satisfies property ϱ-$(*)$, then for every constant $\alpha > 0$ the following properties hold:*

i) *for every $\varepsilon > 0$ there is a compact set B such that*

$$\varrho\big[\alpha(T_w f)\chi_{G\setminus B}\big] < \varepsilon$$

for sufficiently large $w > 0$,

ii) *for every sequence $(B_k)_{k\in\mathbb{N}}$ with $B_k \in \mathcal{B}$, $B_{k+1} \subset B_k$ and $\mu(B_k \to 0$ we have*

$$\lim_{k\to+\infty} \varrho\big[\alpha(T_w f)\chi_{B_k}\big] = 0$$

uniformly with respect to sufficiently large $w > 0$.

P r o o f. Let $\alpha > 0$ and $\varepsilon > 0$ be fixed and let B be the set in the property ϱ-$(*)$. Then we have

$$\varrho\big[\alpha(T_w f)\chi_{G\setminus B}\big] \le \varrho\Big[\alpha M' \int_C L_w(\cdot,t)\ \mathrm{d}\mu(t)\,\chi_{G\setminus B}(\cdot)\Big] < \varepsilon\,,$$

being $M' = \sup_{w>0} \psi_w\big(\|f\|_\infty\big)$. Moreover, let $(B_k)_{k\in\mathbb{N}}$ be a sequence of measurable subsets of G such that $B_{k+1} \subset B_k$ and $\lim_{k\to+\infty} \mu(B_k) = 0$. Then given $\varepsilon > 0$, let $\delta > 0$ be the constant in the definition of absolute finiteness of the modular ϱ. Let k be so large that $\mu(B_k) < \delta$, so

$$\varrho\big[\alpha(T_w f)\chi_{B_k}\big] \le \varrho\big[\alpha D M' \chi_{B_k}\big] < \varepsilon\,.$$

□

As in the case of Orlicz spaces, using Proposition 2 and Theorem 7, we now prove the following theorem.

THEOREM 8. *Let ϱ be a monotone, strongly finite, absolutely finite and absolutely continuous modular. Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. If the family $\big(L_w(\cdot,t)\big)_{t\in G,\, w>0}$ satisfies property ϱ-$(*)$, then there exists a constant $\alpha > 0$ such that*

$$\lim_{w\to+\infty} \varrho\big[\alpha(T_w f - f)\big] = 0$$

for every $f \in C_c(G)$.

Proof. Let $\alpha > 0$ be a constant such that the definition of absolute continuity of the modular is satisfied with 8α and let $\varepsilon > 0$ be fixed. Let B be the compact set in property ϱ-$(*)$ of the family $\big(L_w(\cdot,t)\big)_{t\in G,\, w>0}$. We write

$$\begin{aligned}\varrho\big[\alpha(T_w f - f)\big] &\le \varrho\big[2\alpha(T_w f - f)\chi_B\big] + \varrho\big[2\alpha(T_w f - f)\chi_{G\setminus B}\big]\\ &= I_1 + I_2\,.\end{aligned}$$

Now we estimate I_1. Let $(B_k)_{k\in\mathbb{N}}$ be as in previous theorem. There results

$$\begin{aligned}I_1 &\le \varrho\big[4\alpha(T_w f - f)\chi_{B\cap B_k}\big] + \varrho\big[4\alpha(T_w f - f)\chi_{B_k^c\cap B}\big]\\ &\le \varrho\big[8\alpha(T_w f)\chi_{B_k\cap B}\big] + \varrho\big[8\alpha f\chi_{B_k\cap B}\big] + \varrho\big[4\alpha(T_w f - f)\chi_{B_k^c\cap B}\big]\\ &\le \varrho\big[8\alpha DM'\chi_{B_k\cap B}\big] + \varrho\big[8\alpha f\chi_{B_k\cap B}\big] + \varrho\big[4\alpha(T_w f - f)\chi_{B_k^c\cap B}\big]\\ &= I_1^1 + I_1^2 + I_1^3\,.\end{aligned}$$

By [3; Corollary 1], I_1^3 tends to zero, since $\mu(B) < \infty$. As to I_1^1, this is less than ε by the absolute finiteness of the modular, for sufficiently large k. Finally I_1^2 is less or equal to ε by absolute continuity of the modular, for sufficiently large k. Now we estimate I_2. We write

$$I_2 = \varrho\big[2\alpha(T_w f - f)\chi_{G\setminus B}\big] \le \varrho\big[4\alpha(T_w f)\chi_{G\setminus B}\big] + \varrho\big[4\alpha f\chi_{G\setminus B}\big]\,.$$

The assertion follows from Theorem 7 and the absolute continuity of the modular, taking eventually $B \supset V$, where V is the compact subset in the definition of absolute continuity of the modular. □

Now, in order to give a modular convergence theorem for functions in the modular space, we need the following notions of regular families of measures and the compatibility with respect to the modular functional. The first notion in this respect was given in [9] and then used in [10]. Here we will use it in the form given in [3].

Let us consider a family of functions $(\mu_w)_{w>0}$ with $\mu_w\colon G\times\mathcal{B}\to\mathbb{R}_0^+$ such that $\mu_w(\cdot,A)$ is μ-measurable for every $A\in\mathcal{B}$, and $\mu_w(s,\cdot)$ is a measure on $\mathcal{B}$ for every $s\in G$. We will say that $(\mu_w)_{w>0}$ is a *regular family* if the following assumptions hold:

(i) $\mu_w^s(\cdot) \equiv \mu_w(s,\cdot) \ll \mu$ for every $w > 0$, $s \in G$.

(ii) There is a constant $D > 0$ such that, putting $\beta_w(\cdot) = \mu_w(\cdot, G)$, we have $0 < \beta_w(\cdot) \leq D$ for every $s \in G$ and for every sufficiently large $w > 0$.

(iii) Putting $\xi_w(s,t) = \mathrm{d}\mu_w^s / \mathrm{d}\mu$, we have that ξ_w is a globally measurable function on $G \times G$ and $\|\xi_w(\cdot,t)\|_1 \leq D$ for every $t \in G$ and sufficiently large $w > 0$.

For example, given a singular kernel $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$, the net

$$\mu_{\mathbb{L},w}(s,A) = \int_A L_w(s,t)\,\mathrm{d}\mu(t)\,, \qquad s \in G\,,\ \ w > 0\,, \tag{4}$$

where $(L_w)_{w>0} \subset \mathcal{L}$ is the family of functions corresponding to the (L,ψ)-Lipschitz condition, is a regular family.

We will say that a regular family $(\mu_w)_{w>0}$ is *compatible with the modular* ϱ if there are two constants $D', N > 0$ and a net $(b_w)_{w>0}$ of nonnegative real numbers with $\lim_{w\to+\infty} b_w = 0$, such that

$$\varrho\left[\int_G g(t)\,\mathrm{d}\mu_w^{(\cdot)}(t)\right] \leq N\varrho(D'g) + b_w \tag{5}$$

for every $g \in X(G)$, $g \geq 0$.

Examples of regular families and compatibility can be found in [9], [3], [10].

Now we are ready to give an extension to modular spaces of Theorem 5.

THEOREM 9. *Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. Let ϱ, η be two monotone modulars such that the triple (ϱ, ψ_w, η) is properly directed. Let us assume that the family $(\mu_{\mathbb{L},w})_{w>0}$ is compatible with the modular ϱ. Let $f, g \in X(G) \cap \mathrm{Dom}\,\mathbf{T}$; then given $\lambda > 0$, there exists $\alpha > 0$ such that*

$$\varrho[\alpha(T_w f - T_w g)] \leq N\eta[\lambda(f-g)] + Nc_w + b_w\,, \tag{6}$$

where N, D', $(b_w)_{w>0}$ and $(c_w)_{w>0}$ are given in (3) *and* (5).

P r o o f . Let N, D' be the constants in (5) and for a fixed $\lambda > 0$ let $C_\lambda \in]0,1[$ be a constant such that (3) holds. Let $0 < \alpha \le C_\lambda/D'$. Then we have

$$\begin{aligned}\varrho\big[\alpha(T_w f - T_w g)\big] &\le \varrho\Bigg[\alpha \int_G \big|K_w(\cdot,t,f(t)) - K_w(\cdot,t,g(t))\big| \,\mathrm{d}\mu(t)\Bigg] \\ &\le \varrho\Bigg[\alpha \int_G L_w(\cdot,t)\psi_w\big(|f(t)-g(t)|\big) \,\mathrm{d}\mu(t)\Bigg] \\ &= \varrho\Bigg[\alpha \int_G \psi_w\big(|f(t)-g(t)|\big) \,\mathrm{d}\mu^{(\cdot)}_{\mathbb{L},w}(t)\Bigg] \\ &\le N\varrho\big[\alpha D'\psi_w\big(|f(\cdot)-g(\cdot)|\big)\big] + b_w \\ &\le N\eta\big[\lambda(f-g)\big] + Nc_w + b_w\,,\end{aligned}$$

i.e. the assertion. □

Finally, we can formulate the main theorem of this section.

THEOREM 10. *Let ϱ, η be monotone, strongly finite, absolutely finite and absolutely continuous modulars and $\Xi = (\psi_w)_{w>0} \subset \Psi$ be such that the triple (ϱ, ψ_w, η) is properly directed. Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. Let us assume that the family $\big(\mu_{\mathbb{L},w}\big)_{w>0}$ is compatible with the modular ϱ and we assume that the family $\big(L_w(\cdot,t)\big)_{t\in G,\, w>0}$ satisfies property ϱ-$(*)$. Then for every $f \in L^{\varrho+\eta}(G) \cap \operatorname{Dom}\mathbf{T}$ there exists $\lambda > 0$ such that*

$$\lim_{w\to+\infty} \varrho\big[\lambda(T_w f - f)\big] = 0\,.$$

P r o o f . The proof is now essentially the same as in [3; Theorem 2] following the same method of Theorem 6. □

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