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PLANAR ORDERED SETS OF WIDTH TWO

JUREK CZYZOWICZ*—ANDRZEJ PELC**[+]—IVAN RIVAL**[++]

ABSTRACT. We give finite list of orders such that an order of width two is planar if and only if it does not contain a subdiagram a homeomorph of a member of this list. The proof of our result yields an $O(n^{3/2})$ algorithm to test planarity of orders of width two with n vertices.

As ordered sets occur widely in computation, bearing for instance, on problems of scheduling, sorting and searching, there has emerged a need for efficient data structures to code and store ordered sets. Among these data structures, graphical ones, in particular, may play a decisive role especially in human decision-making problems. Chief among graphical data structures for ordered sets is the 'diagram'. For elements a, b of an ordered set P, say that *a covers b* (or a is an *upper cover* of b), and write $a \succ b$, if, for each x in P, $a > x \geq b$ implies $x = b$. A *diagram* of an ordered set P is a pictorial representation of P on the plane in which small circles, corresponding to elements of P, are arranged in such a way that, any circle corresponding to an upper cover a of b is situated higher in the plane than the circle corresponding to b and is joined to it by an edge which is a monotonic arc (that is, an arc with no repeated y-coordinates). As a diagram of P is a drawing of it there is, of course, considerable variation possible in the actual rendering; nonetheless, any diagram of P determines it. It is therefore common practice to identify P with the diagram itself.

Diagrams are drawn to be read. Of course, the foremost practical quality of a diagram of an ordered set P is that, for elements a and b in P, we may readily decide whether or not $a < b$. Many graphical schemes (e.g. comparability graph) and incidence structures (e.g. incidence matrix) solve such questions efficiently. There are many other qualities though, especially of a structural character, such as whether P has a decomposition as a direct product, linear sum, lexicographic product, etc., none of which are necessarily apparent from

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a full listing of the comparabilities in either a comparability graph or incidence matrix representation. Each such feature may, however, be conveniently read from a diagram appropriate to it.

Fig. 1

A (nonplanar) diagram of ordered set.

A planar representation of the same ordered set.

What then are criteria for a 'good' diagram? We have given extensive attention to several recently. How can a diagram be produced (if at all) with all lower covers of every element horizontally aligned [A. Pelc and I. Rival (1987)]? How can a diagram be produced using few slopes for the edges [J. Czyzowicz, A. Pelc and I. Rival (1987); J. Czyzowicz, A. Pelc, I. Rival, and J. Urrutia (1987)]? Our aim in this article is to consider perhaps the most obvious criterion of all — planarity. We say that P is *planar* if it has a diagram in which none of the lines corresponding to the covering pairs intersect, except possibly at an endpoint, where they may meet a small circle corresponding to an element of P. Such a rendering of P we call a *planar representation* of it. Our problem is this.

Find an efficient procedure to decide whether an ordered set is planar.

Indeed, it is not at all clear that there ever will be one. Our main result is this.

We say that an ordered set is *homeomorphic* to any ordered set obtained by adding vertices to its edges and adding only the comparabilities induced along these edges. Thus, each new vertex has precisely one upper cover and precisely one lower cover. If P is a homeomorph of Q then Q is a *homeomorph* of P. A *subdiagram* of an ordered set P is any subset of the elements of P together with all of the edges joining them in P.

Theorem. *A finite ordered set P of width two is planar if and only if no member, or its dual, of the list $\mathfrak{P}$, is a homeomorph of a subdiagram of P* (cf. Figure 2).

Moreover, the construction used to prove this theorem provides an effective procedure to test planarity of a width two ordered set.

Corollary. *For any ordered set of width two with n vertices there is a decision procedure to test whether it is planar in $O(n^{3/2})$ time.*

What are the prospects for an efficient planarity-testing decision procedure for an arbitrary ordered set? Recently, D. Kelly (1987) has shown that *any* *planar ordered set always has a representation using only straight line edges,* a result analogous to the well-known result of *K.* Wagner (1936) and I. Fary (1948) for (undirected) graphs. In the process Kelly established this characterization. *An n-element ordered set* ($n \geq 3$) *is planar if and only if it is a*

The list $\mathfrak{P}$

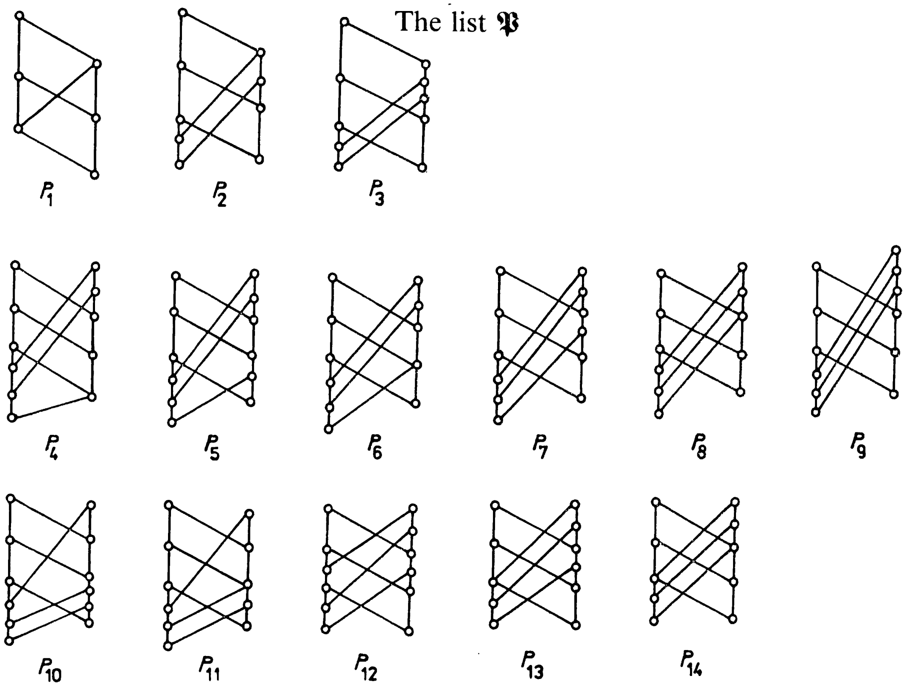


Fig. 2

homeomorph of a subdiagram of a planar lattice with at most $3n$-5 *elements.* Apart from its intrinsic interest as a characterization of planar ordered sets this does provide a decision procedure — albeit far from polynomial — for planarity-testing. (There are finitely many nonisomorphic planar lattices with $3n$-5 elements and hence finitely many nonisomorphic subdiagrams). A simpler though no more efficient procedure to test for planarity is this. It is fairly easy to see that, for any planar representation of an ordered set P we may associate another planar representation in which no two elements have the same y-coordinate. ('Shake' the original planar representation!) Indeed, the precise y-coordinates play no decisive role, only their relative ordering is important. Now fix a vertical line somewhere to the right of this planar representation of P. Starting with any element on the right boundary of this representation 'drag' it horizontally until the vertex coincides with the vertical line. The edges incident with this element may be stretched, retaining their monotonicity, so that we still have a diagram and a planar representation too. We continue in this way, successively dragging elements in the representation of P on the right boundary (not yet on the vertical line) to the vertical line and stretching the corresponding edges. Finally, we obtain a planar representation of P in which all elements are vertically aligned. The corresponding y-coordinates induce a linear extension. Given an arbitrary n-element ordered set we may consider each of its linear extensions (each one corresponds to a feasible arrangement of y-coordinates of its points). Now adjoin arcs one at a time in all possible ways. Thus, if a covers b in P and, if there are k intervening vertices in the designated linear extension of P, consider all 2^k types of monotonic arcs joining a and b which surround each of the k intervening vertices from the left or from the right. Do this in succession for each covering pair. If one such arrangement can be effected with no intersections at all then P has a planar representation. If not then P doesn't.

Most of what is known about planar ordered sets is actually related to planar lattices, that is, planar ordered sets in which, for every pair a, b of elements there is the supremum $a + b$ and the infimum $a \cdot b$, both belonging to the ordered set. For instance, it is a well-known fact (cf. [D. Kelly and I. Rival (1975)] that any planar ordered set with a top and a bottom element must be a (planar) lattice. It is precisely this fact that Kelly exploited in his characterization of planarity, mentioned above. There is, in turn, even a linear time planarity-testing procedure for lattices. This follows from the result of C. Platt (1976) that *a finite lattice is planar if and only if the undirected graph corresponding to its diagram, with an additional edge joining the top and bottom elements, is actually a planar graph.* Thus, planarity-testing for lattices reduces to planarity-testing for graphs, which is well known to be linear (cf. [J. Hopcroft and R. Tarjan (1974)]).

K P

Fig. 3

P contains no homeomorph of K yet there is an edge covering chain embedding of K to P.

D. Kelly and I. Rival (1975) found a minimal (although infinite) list $\mathfrak{L}$ of nonplanar lattices such that a lattice L is planar if and only if no member (or its dual) of $\mathfrak{L}$ can be order embedded in L. (While there seems to be a superficial resemblance between the relations of 'order embedding' and 'homeomorphism' neither accounts for the other, even for lattices.)
Using the theory of planar lattices there is also a simple characterization of planarity among ordered sets with a top and a bottom. Indeed, a finite ordered set with a top and a bottom is planar if and only if no member (or its dual) of $\mathfrak{L}$ $\{P_1\}$ can be order embedded in it. The reason is this. If P is a nonplanar ordered set with a top and a bottom but which is not a lattice then it contains a four-tuple of elements a, b, c, d such that both c and d are minimal upper bounds of $\{a, b\}$. Then this four-tuple of elements together with the top and the bottom of P forms a subset order-isomorphic to P_1.

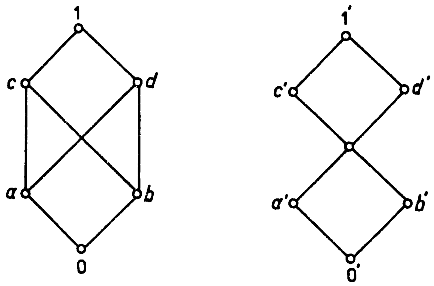
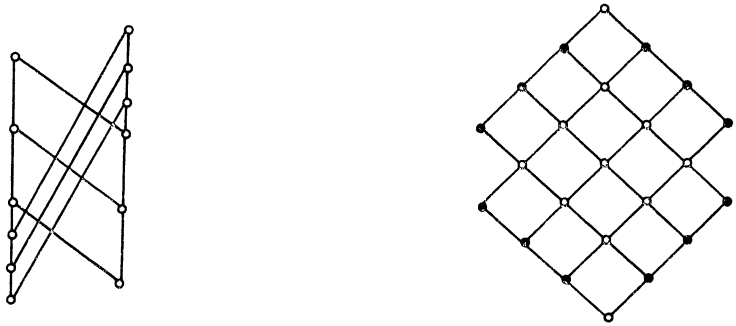


Fig. 4

An order-embedding of the nonplanar ordered set P_1 into a planar ordered set.

It is a curious twist in the study of planarity for ordered sets that, although a subdiagram of a planar order is obviously planar, an ordered set may be nonplanar, yet 'order-embeddable' in a planar ordered set. Thus, the ordered set P_1 (belonging to our list $\mathfrak{P}$) is nonplanar, yet it can be order embedded in a planar ordered set (see Figure 4). Actually, every one of the (nonplanar) ordered sets in the list $\mathfrak{P}$ can be order embedded in a planar ordered set. (The 'completion by cuts' of any member of $\mathfrak{P}$ is a planar lattice since, any width two ordered set has order-dimension two and the 'completion by cuts' preserves the order-dimension. Then couple these observations with the fact that a lattice has order-dimension at most two if and only if it is planar (cf. [D. Kelly and I. Rival (1975)]. See Figure 5)

It is natural, therefore, to consider the following associated concept. Call an ordered set P *essentially planar* if there is a planar ordered set Q and an order embedding of P into Q. Otherwise, call P *essentially nonplanar*. According to D. Kelly (cf. [D. Kelly and I. Rival (1975); R. Nowakowski, I. Rival and J. Urrutia (1987)]) every nonplanar lattice is essentially nonplanar. This fails for some ordered sets, as for example every nonplanar ordered set of order-dimension two. The following proposition shows that while planarity is a homeomorphism invariant, essential planarity is not.



P_q

Fig. 5

The (planar) completion by cuts of P_q.

Proposition. (i) *Every homeomorph of a nonplanar ordered set is nonplanar.* (ii) *Every nonplanar ordered set has an essentially nonplanar homeomorph.*

Proof. (i) is straightforward.

To prove (ii) let P be any nonplanar ordered set and let Q be obtained from P by adding a vertex to each edge of P with, in each case, only the comparabilities induced along these edges (cf. Figure 6). Suppose, on the contrary, that there is a planar ordered set R and an order embedding of Q to R.

Fig. 6

A nonplanar, essentially planar ordered set.

An essentially nonplanar homeomorph.

Consider a planar representation of R and identify the elements a', b', c', etc. in R corresponding to the elements a, b, c, etc. in Q. We may construct a diagram for Q by assigning to each covering pair $a \succ b$ of Q a distinct monotonic arc which follows a covering chain in R joining the corresponding endpoints a', b'. As Q is nonplanar, there must be distinct edges $a_1 \succ b_1$, $a_2 \succ b_2$ in Q whose corresponding covering chains cross in R, at an element belonging to R. In particular, $a_1 > b_2$ and $a_2 > b_1$. Then none of these four vertices can be new ones added along edges of P which is impossible since $a_1 \succ b_1$ and $a_2 \succ b_2$ in Q.

Since order embedding does not preserve planarity but subdiagrams and homeomorphs do, it seems natural to characterize planarity in terms of forbidden homeomorphs of subdiagrams. At this time we do not know whether there is, for the case of arbitrary orders, a similar list (perhaps infinite) of minimal forbidden subobjects.

Proof of the Theorem. We first consider the 'only if' part. According to the Proposition (i) it suffices to verify that each member of $\mathfrak{P}$ is nonplanar. We sketch an argument for P_1 only. (The others are similar — if tedious.)

Suppose P_1 is labelled as in Figure 7. If it has a planar representation then there must be two monotonic curves representing the covering chains $4 \prec 3 \prec 2 \prec 1$ and $4 \prec 6 \prec 5 \prec 1$ which intersect only at 4 and at 1. Their union is, therefore, a simple closed curve. As 3 and 5 both have larger y-coordinates than 4 and smaller than 1, a monotonic arc joining 3 and 5 cannot be entirely outside of the closed curve, whence it must be entirely inside. As the same must be true for any monotonic arc joining 2 and 6, these two monotonic curves must intersect, contradicting the planarity of the representation.

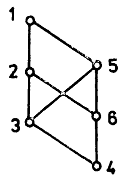


Fig. 7

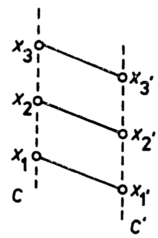


Fig. 8

We turn now to the 'if' part. As P has width two, we may represent P as the union of two maximal chains C, C'. There is no loss in generality to suppose that these maximal chains are disjoint (otherwise, we may consider the intervals between the intersection points). Let $x_1 \succ x_1'$, $x_2 \succ x_2'$, ..., $x_n \succ x_n'$ stand for all covering pairs in P such that each x_i belongs to C, x_i' to C', and $x_1 < x_2 < \ldots < x_n$. To fix the discussion more, we draw the chains C and C' along two verticals, C' to the right of C, locating the vertices x_i and x_i' so that x_i has bigger y-coordinate than x_i' and, all distances between x_i and x_{i+1}, and, x_i' and x_{i+1}' are equal. The corresponding covering pairs $x_i \succ x_i'$ may all be drawn as straight and parallel edges (cf. Figure 8).

We now focus our attention on the edges representing the covering pairs $y' \succ y$, where y' belongs to C' and y to C. We shall prove that, under the assumption that P contains no subdiagram which is a homeomorph of any member (or its dual) of $\mathfrak{P}$, then these additional edges can be drawn (usually, not as straight line segments, but rather general monotonic arcs) to complete the drawing (see Figure 8) without any crossing at all.

If there are at most two covering pairs $x_1 \succ x_1'$, $x_2 \succ x_2'$ and at most one covering pair $y' \succ y$ then Figure 8 can be easily completed by adding an edge for $y' \succ y$ without any crossing. Therefore, we may suppose that there are at least three covering pairs $x_i \succ x_i'$ or, two such pairs and two pairs $y_1' > y_1$, $y_2' \succ y_2$, where the y_i belong to C and y_i' to C'.

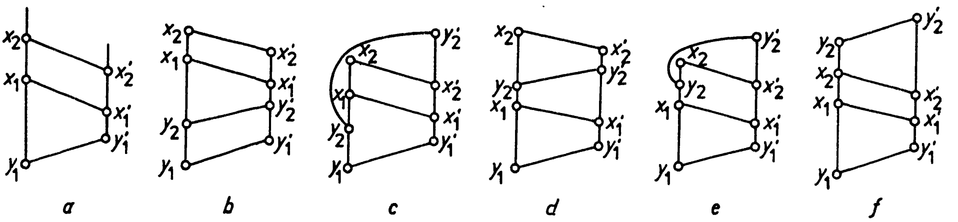


Fig. 9

We consider the second easier case first. Let $y_1 < y_2$ (and hence $y_1' < y_2'$). Suppose that $y_1' \leq x_1'$ (cf. Figure 9a). If $y_2' \leq x_1'$ then P is planar (cf. Figure 9b). If $y_2 < x_1$ and $x_1' < y_2' \leq x_2'$ then P contains as subdiagram a homeomorph of P_1. If $y_2 < x_1$ and $y_2' > x_2'$ then P is planar (cf. Figure 9c). If $y_2 \geq x_1$ then P is planar again (cf. Figure 9d, e, f,).

Next suppose that $y_1' > x_1'$. Then, apart from the case $x_1 \leq y_1 < x_2$, $y_1' > x_2'$ and $y_2 \geq x_2$ in which it contains as subdiagram a homeomorph of P_1, P is planar. Planar representations of the various subcases are illustrated in Figure 10.

We turn now to the case that there are at least three covering pairs $x_i \succ x_i'$. The 'ladder' between the chains C, C', as illustrated in Figure 8, is divided into $n + 1$ cells by the covering edges $x_i \succ x_i'$: the bottom open cell E_0, the subsequent parallelograms $E_1, E_2, \ldots, E_{n-1}$, and the top open cell E_n. According to our assumption, there are at least four cells.

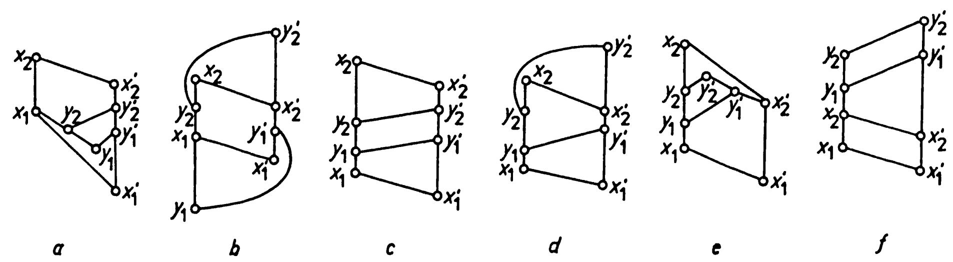


Fig. 10

For purposes of the proof, call a covering pair $y_i \prec y'_i$ *free* if both ends of it are in the same cell. Draw all edges corresponding to free covering pairs, as parallel segments, in the obvious way; no crossings are produced. We shall prove that this can be completed to a planar representation as long as no member of the list $\mathfrak{P}$ is homeomorphic to a subdiagram of P.

If all covering pairs $y_i \prec y'_i$ are free, then, clearly, P is planar: its planar representation has just been drawn. So suppose there exist a non-free covering pair. Consider the free covering pairs in cells E_0 and E_n. If both of these cells contain free covering pairs then, in view of the existence of a non-free covering pair P must contain a subdiagram which is a homeomorph of P_1. Suppose then, that E_0 contains such pairs and E_n does not. If there are non-free covering pairs $y_i \prec y'_i$ such that $y'_i \leq x'_n$ then P contains as subdiagram a homeomorph of P_1. Hence we may assume that no such pairs exist and so all non-free covering pairs $y_i \prec y'_i$ satisfy $y'_i > x'_n$. If there is only one such pair, the order P is planar (cf. Figure 11).

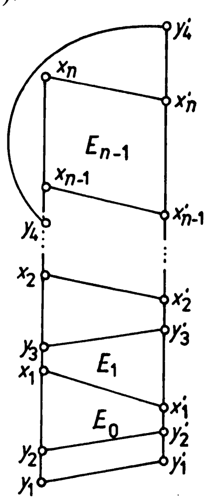


Fig. 11
The only non-free covering pair is $y_4 \prec y'_4$.
There are free edges in E_0 (and E_1) but not in E_n.

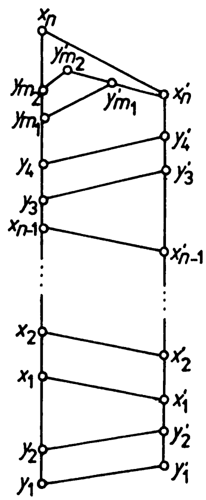


Fig. 12

Suppose there are at least two of them:

$$y_{m_1} \prec y'_{m_1}, \; \ldots, \; y_{m_k} \prec y'_{m_k}, \quad \text{where} \quad y_{m_1} < \ldots < y_{m_k}, \quad k \geq 2.$$

If for some j and i, $y_{m_j} < x_i \leq y_{m_{j+1}}$ then P contains as subdiagram a homeomorph of P_1. If $y_{m_k} < x_1$ then P contains as subdiagram a homeomorph of P_4 or to the dual P_3^d of P_3. If none of these is the case then $x_i \leq y_{m_1} < \ldots < y_{m_k} < x_{i+1}$ for some i. If $i = n - 1$ then the order P is planar (cf. Figure 12). If $i < n - 1$

then P contains as subdiagram a homeomorph of P_2^d or P_3^d. This completes the discussion of the case that the cell E_0 contains free covering pairs and the cell E_n does not. The case that E_n contains free covering pairs and E_0 does not is dual. Hence, we may assume that neither E_0 nor E_n contains free covering pairs.

First suppose that there is exactly one non-free covering pair $y \prec y'$. If $y < x_1$ or $y' > x_n'$ then P is planar (cf. Figure 13). On the other hand, if $y \geq x_1$ and $y' \leq x_n'$ then P contains as subdiagram a homeomorph of P_1.

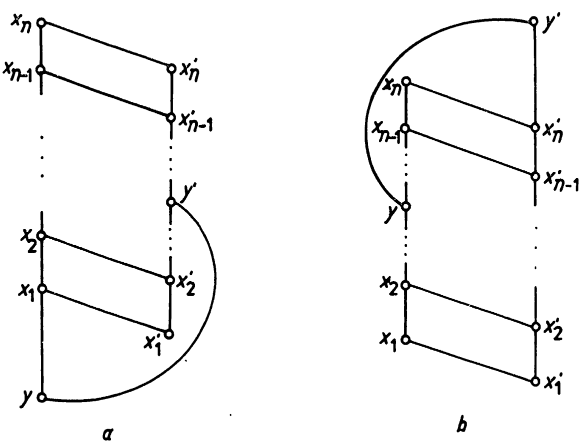


Fig. 13

Next suppose that there are exactly two non-free covering pairs $y_1 \prec y_1'$, $y_2 \prec y_2'$, (where $y_1 < y_2$). As before, if for either of them $y_i \geq x_1$ and $y_i' \leq x_n'$ then P contains as subdiagram a homeomorph of P_1. Suppose that $y_i < x_1$ or $y_i' > x_n$, for $i = 1, 2$.

Case 1. $y_1, y_2 < x_1$

If $x_1' < y_1' < y_2' \leq x_2'$ then P is planar (cf. Figure 14a). If $x_i' < y_1' < y_2' \leq x_{i+1}$ for any $i \geq 2$ then P contains as subdiagram a homeomorph of P_2 or P_3. If, for some i, $x_n' \geq y_2' > x_i' \geq y_1'$ then P contains as subdiagram a homeomorph of P_1. If $y_2' > x_n'$ then P is planar (cf. Figure 14b).

Case 2. $y_1 < x_1$, $y_2 \geq x_1$

Then $y_2' > x_n'$ by our assumption (otherwise P contains as subdiagram a homeomorph of P_1) and hence P is planar (cf. Figure 15).

Case 3. $y_1, y_2 \geq x_1$

Then $y_1', y_2' > x_n'$ by our assumption and the case is dual to case 1.

Finally suppose that there are at least three non-free covering pairs $y_{m_i} \prec y'_{m_i}$. If for any i, $x_1 \leq y_{m_i} \leq x_n$ and $x'_1 \leq y'_{m_i} \leq x'_n$ then P contains as subdiagram a homeomorph of P_1. Hence we may assume that for every i, $y_{m_i} < x_1$ or $y'_{m_i} > x'_n$. We distinguish three cases.

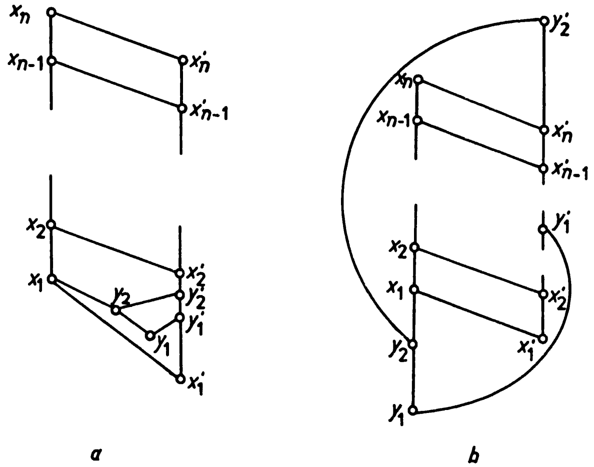


Fig. 14

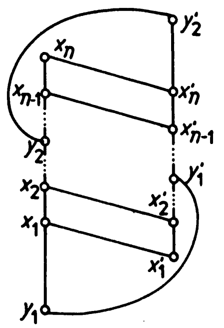


Fig. 15

Case 1. There are at least two covering pairs $y_i \prec y'_i$ and $y_j \prec y'_j$ for which $y_i < y_j < x_1$ and $x'_n < y'_i < y'_j$.
In this case P is always nonplanar: all possible positions of the third non-free covering pair $y \prec y'$ satisfying $y < x_1$ or $y' > x'_n$ yield as subdiagram a homeomorph of one of the orders P_5, P_6, P_7, P_8, P_9 or of a dual P_5^d, P_6^d, P_7^d, d_8^d, P_9^d.

Case 2. There is exactly one covering pair $y \prec y'$ for which $y < x_1$ and

$y' > x'_n$. First suppose that two other non-free covering pairs $y_i \prec y'_i$ and $y_j \prec y'_j$ satisfy $y_i < y_j < y$. If $x'_1 < y'_i < y'_j \leq x'_2$ then P contains as subdiagram homeomorpic to P_{10} or P_{11}. If $x'_t < y'_i < y'_j \leq x'_{t+1}$ for some $t > 1$ then P contains as subdiagram a homeomorph of P_2 or P_3. If $y'_i \leq x'_t < y'_j \leq x'_n$ for some $t > 1$ then P contains as subdiagram a homeomorph of P_1. Next suppose that the two other non-free covering pairs $y_i \prec y'_i$ and $y_j \prec y'_j$ satisfy $y' < y'_i < y'_j$. This is dual to the above.

Finally suppose that $y_i < y$ and $y'_j > y'$. If $y'_i \leq x'_2$ and $y_j \geq x_2$ then P contains as subdiagram a homeomorph of P_1. If $y_j > x_2$ and $y'_i > x'_2$ then P contains as subdiagram a homeomorph of P_3. If $y_j = x_2$ and $y'_i > x'_2$ then P contains as subdiagram a homeomorph of P_2. If $y_j < x_2$ and $y'_i = x'_2$ then P contains as subdiagram a homeomorph of P_2^d. If $y_j < x_2$ and $y'_i > x'_2$ there are four possibilities: if $y_j \neq x_1$ and $y'_i \neq x'_n$ then P contains as subdiagram homeormorphic to P_{12}; if $y_j = x_1$ and $y'_i \neq x_n$ then P contains as subdiagram a homeomorph of P_{13}; if $y_j \neq x_1$ and $y'_i = x'_n$ then P contains as subdiagram a homeomorph of P_{13}^d; if $y_j = x_1$ and $y'_i = x'_n$ then P contains as subdiagram a homeomorph of P_{14}.

Case 3. There is no covering pair $y \prec y'$ for which $y < x_1$ and $y' > x'_n$.

We can divide all non free covering pairs into two classes: the pairs $u_1 \prec u'_1, \ldots,$

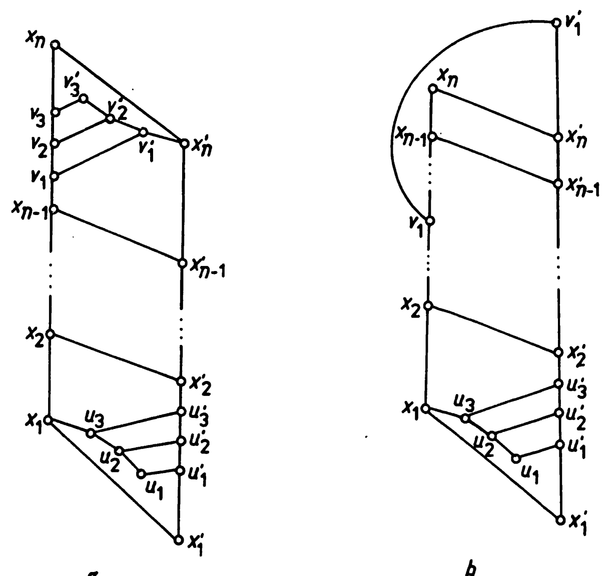


Fig. 16

$u_p \prec u'_p$, for which $u_1 < \ldots < u_p < x_1$ and $x'_1 < u'_1 < \ldots < u'_p \leq x'_n$ and the pairs $v_1 \prec v'_1, \ldots, v_q \prec v'_q$, for which $x'_n < v'_1 < \ldots < v'_q$ and $x_1 \leq v_1 < \ldots < v_q < x_n$. Since there are at least three non-free covering pairs $y \prec y'$ we may suppose that

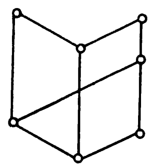
$p \geq 2$ (the case $q \geq 2$ is dual). If $u'_p \leq x'_2$ and $v_1 \geq x_{n-1}$ then P is planar (cf. Figure 16a). If $u'_j \leq x'_i < u'_{j+1}$ or $v_j < x_i \leq v_{j+1}$ for some i and j then P subdiagram a homeomorph of P_1. If $x'_i < u'_1 < \ldots < u'_p \leq x'_{i+1}$ for $i > 1$ then P contains as subdiagram homeomorphic to P_2 or P_3. If $u'_p \leq x'_2$ and $q = 1$ with v_1 situated arbitrarily between x_1 and x_n then P is planar (cf. Figure 16b). All remaining situations are dual to the above. This completes the discussion of case 3 and concludes the proof of the theorem.

Proof of the Corollary. We assume that for any vertex a list of its upper covers and lower covers is given. Thus finding an upper cover or a lower cover as well as plotting a vertex or a covering edge takes constant time. According to [S. Micali and V. V. Vazirani (1980)] there is an $O(\sqrt{ne})$ — time procedure to exhibit a minimum chain decomposition in an order with n vertices and e edges. Since every ordered set of width two has at most $2n$ edges, decomposing it into two chains takes $O(n^{3/2})$ time. It can be easily seen from the proof of our theorem that the rest of the planarity-testing as well as drawing a planar representation (if there is one) takes linear time.

Several remarks related to the theorem and its proof are in order.

As P has width two, in every occurrence of a forbidden order P_i, the 'diagonal' edges as illustrated in Figure 2, always occur as covering edges of P.

Notice that in checking whether a width two ordered set is nonplanar we set out with any two-chain decomposition and locate a forbidden P_i with respect to this initial choice.



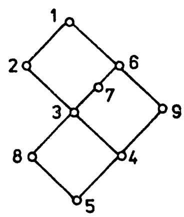


Fig. 17

A nonplanar ordered set of width three which contains no subdiagram homeomorphic to any width two nonplanar ordered set.

The three chains $A = \{1, 2, 3, 4, 5\}$, $B = \{6, 7, 8\}$, $C = \{9\}$ cannot be placed along three verticals to produce a planar representatio.

At this time we have little insight about a possible characterization of planarity for ordered sets of width at least three. Indeed, we have been unable even to settle whether there is a finite list of nonplanar ordered sets of width three such that an ordered set of width three is nonplanar if and only if it contains as subdiagram a homeomorph of an ordered set in the list. Moreover, in the width three case there may even be a decomposition of a planar ordered set into three chains which, if placed along vertical lines, cannot produce a planar representation.

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