Jiří Rachůnek

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POLARS AND ANNIHILATORS IN REPRESENTABLE DRl-MONOIDS AND MV-ALGEBRAS

JIŘÍ RACHŮNEK

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ABSTRACT. Representable DRl-monoids form a class of algebras containing MV-algebras which are known to be an algebraic counterpart of the infinite valued propositional logic. In this paper, connections between polars and prime ideals and properties of sets of annihilators in representable DRl-monoids, and consequently also in MV-algebras, are shown.

Polars, called also annihilators, (i.e. sets of elements orthogonal to all elements of given subsets) in MV-algebras were introduced by Belluce in [2] and further studied by Hoo in [11], by Di Nola, Liguori and Sessa in [8] and by Belluce and Sessa in [3]. In [2] it is proved that every polar is an ideal and that the polar of a non-trivial linearly ordered ideal is a prime ideal, and by [11], the converse assertion is also true. In [8] the properties of polars of prime ideals are examined.

By [23] and [24], MV-algebras are in a one-to-one correspondence with special kinds of (bounded) dually residuated lattice ordered commutative monoids (DRl-monoids) introduced by Swamy in [26]. The connections between MV-algebras and DRl-monoids are summarized in Theorems 1, 2 and 3 below.

Ideals, prime ideals and polars in DRl-monoids were studied in [29], [19], [20], [21], [22], [10].

In this paper it is shown that prime ideals in any MV-algebra A and in the bounded DRl-monoid induced by A coincide and moreover that this DRl-monoid is representable. Therefore we will study connections between some types of prime ideals and polars in representable DRl-monoids, and the corresponding results for MV-algebras including also the results of [2], [11] and [8]

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will be then obtained as direct consequences. Similarly we will get properties of annihilators (which generalize polars and also further types of ideals) and relative annihilators in MV-algebras by specializing of those in DRl-monoids.

The notion of an MV-algebra was introduced by C. C. Chang in [5] and [6] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. There are various (but mutually equivalent) definitions of these algebras. For instance (see [7]):

DEFINITION. An algebra $A = (A, \oplus, \neg, 0)$ of signature $\langle 2, 1, 0\rangle$ is called an *MV-algebra* if it satisfies the following identities:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(MV2) $x \oplus y = y \oplus x$;
(MV3) $x \oplus 0 = x$;
(MV4) $\neg\neg x = x$;
(MV5) $x \oplus \neg 0 = \neg 0$;
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

If A is an MV-algebra, set $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$ for any $x, y \in A$. Then $(A, \vee, \wedge, 0, \neg 0)$ is a bounded distributive lattice (0 is the smallest and $\neg 0$ is the greatest element in A) and $(A, \oplus, \vee, \wedge)$ is a lattice ordered commutative monoid (l-monoid).

By the work of D. Mundici [8], MV-algebras can be viewed as intervals of abelian lattice ordered groups (l-groups). Namely, let $G = \big(G, +, 0, -(\cdot), \vee, \wedge\big)$ be an abelian l-group and $0 \le u \in G$. For any $x, y \in [0, u] = \{x \in G : 0 \le x \le u\}$ set $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$. Put $\Gamma(G, u) = \big([0, u], \oplus, \neg, 0\big)$. Then $\Gamma(G, u)$ is an MV-algebra. In [8] it is proved that MV-algebras of this form are completely representative because for every MV-algebra A there exist an abelian l-group G and an element u, $0 \le u \in G$, such that A is isomorphic to $\Gamma(G, u)$. Using this correspondence between MV-algebras and abelian l-groups, many results concerning classes of MV-algebras have been obtained by Jakubík in [12], [13], [14] and [15].

Another type of lattice ordered commutative monoids called *dually residuated lattice ordered monoids* (*DRl-monoids*) was introduced and studied by K. L. N. Swamy in [26], [27] and [28] as a mutual generalization of abelian l-groups and Brouwerian algebras.

DEFINITION. A *DRl-monoid* is an algebra $A = (A, +, 0, \vee, \wedge, -)$ of signature $\langle 2, 0, 2, 2, 2\rangle$ such that:

(1) $(A, +, 0)$ is a commutative monoid.
(2) $(A, \vee, \wedge)$ is a lattice.

(3) $(A, +, \vee, \wedge)$ is a lattice ordered semigroup (l-semigroup), i.e. A satisfies the identities

$$x + (y \vee z) = (x + y) \vee (x + z)\,,$$
$$x + (y \wedge z) = (x + y) \wedge (x + z)\,.$$

(4) If $\leq$ denotes the order on A induced by the lattice $(A, \vee, \wedge)$, then for each $x, y \in A$, the element $x - y$ is the smallest $z \in A$ such that $y + z \geq x$.

(5) A satisfies the identity

$$\big((x - y) \vee 0\big) + y \leq x \vee y\,.$$

As is shown in [26], condition (4) is equivalent to the following system of identities:

$$x + (y - x) \geq y\,,$$
$$x - y \leq (x \vee z) - y\,,$$
$$(x + y) - y \leq x\,,$$

and hence DRl-monoids form a variety of algebras of type $\langle 2, 0, 2, 2, 2\rangle$.

Remark.

a) In Swamy's original definition of a DRl-monoid (called there a *DRl-semigroup*), the identity $x - x \geq 0$ is also required. But by [16], in any algebra satisfying (1) – (4) the identity $x - x = 0$ is always satisfied.

b) If $G = \big(G, +, 0, -(\cdot), \vee, \wedge\big)$ is an abelian l-group and $-$ denotes the group subtraction, then $(G, +, 0, \vee, \wedge, -)$ is a DRl-monoid. Brouwerian algebras are other examples of DRl-monoids. Recall that a Brouwerian algebra $B = (B, \vee, \wedge)$ is a dually relative pseudocomplemented lattice (this means that for arbitrary $a, b \in B$ there exists a least $x \in B$ with $b \vee x \geq a$) with greatest element. It is obvious that B has smallest element 0. Denote by $a - b$ the relative pseudocomplement of b with respect to a. If we denote by $+$ the lattice join $\vee$, then $(B, +, 0, \vee, \wedge, -)$ is really a DRl-monoid.

In [23] and [24] the connections between DRl-monoids and MV-algebras are described. We have the following theorem:

THEOREM 1. ([23; Corollary 2], [24; Note]) *Let $A = (A, \oplus, \neg, 0)$ be an MV-algebra. For any $x, y \in A$ set $x \leq y \iff \neg(\neg x \oplus y) \oplus y = y$. Then $\leq$ is a lattice order on A (with the lattice operations $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$), for any $r, s \in A$ there exists a least element $r \ominus s$ with the property $s \oplus (r \ominus s) \geq r$, and $(A, \oplus, 0, \vee, \wedge, \ominus)$ is a DRl-semigroup with smallest element 0 and greatest element $1 = \neg 0$ satisfying the identity*

(i) $1 \ominus (1 \ominus x) = x$.

THEOREM 2. ([23; Theorem 3], [24; Note]) *Let $A = (A, +, 0, \vee, \wedge, -)$ be a bounded DRl-monoid with smallest element 0 and greatest element 1 satisfying the identity*

(i) $1 - (1 - x) = x$.

Set $\neg x = 1 - x$ for any $x \in A$. Then $(A, +, \neg, 0)$ is an MV-algebra.

Let us extend the language of bounded DRl-monoids to $\langle +, 0, \vee, \wedge, -, 1\rangle$ and then denote by $\mathcal{DR}l_{1(\mathrm{i})}$ the equational category of bounded DRl-monoids satisfying (i) and by $\mathcal{MV}$ the equational category of MV-algebras. Then there holds the following theorem:

THEOREM 3. ([24; Theorem 3]) *The categories $\mathcal{DR}l_{1(\mathrm{i})}$ and $\mathcal{MV}$ are isomorphic.*

Using the above results, many properties of MV-algebras will be derived directly from those of DRl-monoids.

Recall that if $A = (A, \oplus, \neg, 0)$ is an MV-algebra and $\emptyset \neq I \subseteq A$, then I is called an *ideal* of A if

(a) $(\forall a, b \in I)(a \oplus b \in I)$,

(b) $(\forall a \in I)(\forall x \in A)\big(a \wedge x = \neg(\neg(a \oplus \neg x) \oplus \neg x) \in I\big)$.

Let $B = (B, +, 0, \vee, \wedge, -)$ be a DRl-monoid and let $c * d = (c - d) \vee (d - c)$ for any $c, d \in B$. Then $\emptyset \neq J \subseteq B$ is called an *ideal* of B if

(c) $(\forall a, b \in J)(a + b \in J)$,

(d) $(\forall a \in J)(\forall x \in B)(x * 0 \leq a * 0 \implies x \in J)$.

If a DRl-monoid B is induced by an MV-algebra, then $x * 0 = x$ for any $x \in B$, and hence condition (d) can be replaced by

(d′) $(\forall a \in J)(\forall x \in B)(x \leq a \implies x \in J)$.

Therefore it is obvious that in MV-algebras ideals in the sense of MV-algebras (MV-ideals) and those in the sense of DRl-monoids (DRl-ideals) coincide.

Let us denote by $\mathcal{I}(A)$ the set of all ideals of a DRl-monoid or of an MV-algebra A. The prime ideals play a fundamental role in both theories.

An ideal I of an MV-algebra A is called *prime* if for any $x, y \in A$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$. (See e.g. [8].) On the other hand. if I is an ideal of a DRl-monoid A, then it is called *prime* in A if for arbitrary $J, K \in \mathcal{I}(A)$, $J \cap K = I$ implies $J = I$ or $K = I$. (See [19].) For comparing both notions of prime ideals, and also both types of algebras, the following notion will be useful.

A DRl-monoid A is called *representable* (see [30]) if $(x - y) \wedge (y - x) \leq 0$ for each $x, y \in A$.

It is known ([30]) that a DRl-monoid is representable if and only if it is isomorphic to a subdirect product of linearly ordered DRl-monoids. Every abelian l-group and every Boolean algebra (as a special case of a Brouwerian algebra) are representable DRl-monoids.

PROPOSITION 4. *Let A be a representable DRl-monoid and $I \in \mathcal{I}(A)$. Then the following conditions are equivalent:*

1. *I is prime (as a DRl-ideal).*
2. $(\forall J, K \in \mathcal{I}(A))\big(J \cap K \subseteq I \implies (J \subseteq I$ *or* $K \subseteq I)\big)$.
3. $(\forall x, y \in A)\big(0 \leq x \wedge y \in I \implies (x \in I$ *or* $y \in I)\big)$.
4. $(\forall x, y \in A)\big(x \wedge y = 0 \implies (x \in I$ *or* $y \in I)\big)$.
5. $(\forall x, y \in A)\big(x \wedge y \in I \implies (x \in I$ *or* $y \in I)\big)$.
6. $\{J \in \mathcal{I}(A) : I \subseteq J\}$ *is linearly ordered.*

Proof. The equivalence of conditions 1, 2 and 3 is proved (even more generally for the class of so called autometrized algebras) in [19; Theorem 1], the equivalence 1 and 4 is proved in [10; Proposition 3.1], and the equivalence of 1 and 6 in [21; Theorem 6].

$4 \implies 5$: Let $x, y \in A$ and $x \wedge y \in I$. Then by [19; Lemma 6], $\big(x - (x \wedge y)\big) \wedge \big(y - (x \wedge y)\big) = 0$, hence $x - (x \wedge y) \in I$ or $y - (x \wedge y) \in I$. Let $x - (x \wedge y) \in I$. By [19; Lemma 6], we have $x = (x \wedge y) + \big(x - (x \wedge y)\big)$, therefore by the assumption, $x \in I$.

$5 \implies 4$: Let $x, y \in A$, $x \wedge y = 0$. Since $0 \in I$, we get $x \in I$ or $y \in I$. □

PROPOSITION 5. *If A is an MV-algebra, then the DRl-monoid induced by A is representable.*

Proof. Let $x, y \in A$. Then by [23; Theorem 1, Corollary 2], in the induced DRl-monoid it holds that

$$(x - y) \wedge (y - x) = \neg(y \oplus \neg x) \wedge \neg(x \oplus \neg y)$$

and by [5; Theorem 3.3 (dual)], the identity

$$\neg(y \oplus \neg x) \wedge \neg(x \oplus \neg y) = 0$$

is satisfied in each MV-algebra. □

Remark.

a) By Propositions 4 and 5, the prime ideals in any MV-algebra and in its induced DRl-monoid coincide.

b) Moreover we get as a consequence the well-known fact that every MV-algebra is a subdirect product of linearly ordered MV-algebras. ([6; Lemma 3])

Let us now consider the set $\mathcal{I}(A)$ of all ideals of an MV-algebra A. It is obvious that $\mathcal{I}(A)$, ordered by set inclusion, is a complete lattice (in which infima coincide with set intersections). Applying [29; Theorem 6] and using [23; Theorem 7], one can show that the lattice $\big(\mathcal{I}(A), \subseteq\big)$ is algebraic and Brouwerian. By Propositions 4 and 5, the prime ideals in A are exactly the finitely meet-irreducible elements in the lattice $\mathcal{I}(A)$.

Let B be a subset of an MV-algebra A. Denote by $B^{\perp} = \{x \in A : x \wedge b = 0$ for all $b \in B\}$. Then $B^{\perp}$ will be called the *polar* of B. If $a \in A$, then we denote by $a^{\perp}$ the polar $\{a\}^{\perp}$ of the singleton $\{a\}$. (Note that $B^{\perp}$ is in [2], [8] and [11] called the *annihilator* of B. In accordance with the theories of lattices and autometrized algebras, we reserve this notion for other types of subsets.) A subset $C \subseteq A$ is called a *polar in* A if there exists $B \subseteq A$ such that $C = B^{\perp}$. Set $B^{\perp\perp} = (B^{\perp})^{\perp}$. Obviously, $B \subseteq A$ is a polar in A if and only if $B = B^{\perp\perp}$. If we denote by $\mathcal{P}(A)$ the set of all polars in A, then by [2; Theorem 25] (and also as a consequence of [20; Theorem 1]), $\mathcal{P}(A) \subseteq \mathcal{I}(A)$. By [20; Corollary of Theorem 2], any polar in A is the polar of an ideal of A. Hence by [29; Lemma 7], the polars in A are exactly the pseudocomplements of elements of the Brouwerian lattice $\mathcal{I}(A)$. Therefore by Glivenko's theorem (see e.g. [1]) the set $\mathcal{P}(A)$ of all polars of an arbitrary MV-algebra ordered by set inclusion is a complete Boolean algebra. (See also [23; Theorem 10].)

Remark. If A is a general DRl-monoid, then for any $B \subseteq A$, the polar $B^{\perp}$ of B is defined by $B^{\perp} = \{x \in A : (x * 0) \wedge (a * 0) = 0\}$. For DRl-monoids corresponding to MV-algebras this definition is equivalent to the one above.

Both polars and prime ideals in MV-algebras are not only special cases of ideals, but also there are closer connections between them. Some of these connections are shown in [2], [8] and [11]. Using the theory of DRl-monoids, we shall now describe this situation in further results.

Let A be a DRl-monoid. If I is a prime ideal in A, then I is called a *minimal prime ideal* in A if it is a minimal element in the set of all prime ideals in A ordered by set inclusion. By [20], every prime ideal contains a minimal prime ideal. One can prove ([21; Theorem 3]) that every ideal in A is the intersection of prime ideals, and in particular, every polar P in A is, by [10; Corollary 2.5], the intersection of minimal prime ideals not containing $P^{\perp}$.

Let $0 \neq a \in A$. Denote by $\mathrm{val}(a)$ the set of all ideals in A maximal with respect to the property of not containing the element a. It is obvious that $\mathrm{val}(a) \neq \emptyset$. If $I \in \mathrm{val}(a)$, then I is called a *value* of a. Every value of any element $0 \neq a \in A$ is by [21] a prime ideal in A.

Now we can prove the following theorem.

THEOREM 6. *Let A be a representable DRl-monoid and $\{0\} \neq I \in \mathcal{I}(A)$. Then the following conditions are equivalent.*

1. *I is linearly ordered.*
2. *If $0 < a \in I$, then $a^{\perp} = I^{\perp}$.*
3. *$I^{\perp}$ is a prime ideal.*
4. *$I^{\perp}$ is a minimal prime ideal.*
5. *$I^{\perp\perp}$ is a maximal linearly ordered ideal in A.*
6. *$I^{\perp\perp}$ is a minimal polar.*

7. *$I^{\perp}$ is a maximal polar.*
8. *Every $0 < a \in I$ has exactly one value.*

Proof. $\mathcal{I}(A)$ is a Brouwerian lattice, and by [17; Lemma 2.1], the equivalence of conditions 3, 4, 6, and 7 is valid in any of such lattices.

$1 \implies 2$: It is easy to verify that if A is a representable DRl-monoid, $I \in \mathcal{I}(A)$ and $x \in A$, then $x \in I$ if and only if $x * 0 \in I$. Now let $I \in \mathcal{I}(A)$ be linearly ordered and $0 < a \in I$. Let $x \in a^{\perp} \setminus I^{\perp}$. Then $a \wedge (x * 0) = 0$ and, at the same time, there exists $0 < b \in I$ such that $b \wedge (x * 0) > 0$. Let us denote $y = b \wedge (x * 0)$. Obviously $y \in I$. Moreover $0 \leq a \wedge y \leq a \wedge (x * 0) = 0$, hence $a \wedge y = 0$. Therefore, if $a \leq y$, then $a \wedge y = a$, and thus $a = 0$, and if $a > y$, then $y = 0$, a contradiction. Hence $a^{\perp} = I^{\perp}$.

$2 \implies 3$: For every $0 < a \in I$, let $a^{\perp} = I^{\perp}$. Let $x, y \in A$, $x \wedge y = 0$. If $x \notin I^{\perp}$, then there exists $0 < b \in I$ such that $x \wedge b = 0$, and thus for any $0 < a \in I$ we have $x_a = x \wedge a > 0$ and $x_a \in I$. Let $y \notin I^{\perp}$. Then $y \notin x_a^{\perp}$, and hence $z = x_a \wedge y > 0$. Evidently $z \leq x$, $z \leq y$, therefore $z = 0$, a contradiction. This means that $x \in I^{\perp}$ or $y \in I^{\perp}$.

$3 \implies 1$: Let $I^{\perp}$ be a prime ideal in A. Let us suppose that $a, b \in I$ and $a \wedge b = 0$. Then $a \in I^{\perp}$ or $b \in I^{\perp}$, and hence $a = 0$ or $b = 0$. Therefore by [10; Proposition 3.7], we get that I is linearly ordered.

$1 \implies 5$: Let I be linearly ordered. Then by 3, $I^{\perp\perp\perp} = I^{\perp}$ is a prime ideal in A, and thus the polar $I^{\perp\perp}$ is linearly ordered. Let $J \in \mathcal{I}(A)$, $J \supseteq I^{\perp\perp}$ and let J be linearly ordered. Then by 3, $J^{\perp}$ is prime. Moreover $J \supseteq I^{\perp\perp}$ implies $J^{\perp} \subseteq I^{\perp}$, hence by 4 we get $J^{\perp} = I^{\perp}$, and so $J = I^{\perp\perp}$.

$5 \implies 1$: Let $I^{\perp\perp}$ be a (maximal) linearly ordered ideal in A. Then $I \subseteq I^{\perp\perp}$ implies that I is also linearly ordered.

$3 \implies 8$: Let us suppose that $I^{\perp}$ is prime. Let $0 < a \in I$ and $U, V \in \operatorname{val}(a)$. Hence (since, by [21; Theorem 1], U and V are prime ideals) $a^{\perp} \subseteq U$ and $a^{\perp} \subseteq V$, therefore also $I^{\perp} \subseteq U$ and $I^{\perp} \subseteq V$. Thus by Proposition 4, U and V are comparable, and so $U = V$.

$8 \implies 1$: Let us suppose that I is not linearly ordered. Then by [10; Proposition 3.7], there exist $0 < a, b \in I$ such that $a \wedge b = 0$. Then $a \in b^{\perp}$, $b \in a^{\perp}$, but $b \notin b^{\perp}$, $a \notin a^{\perp}$. I is an ideal, hence $a \vee b \in I$. Since $b^{\perp} \in \mathcal{I}(A)$, (by [21; Theorem 2]) there exists $P \in \operatorname{val}(b)$ such that $b^{\perp} \subseteq P$. Obviously $a \vee b \notin P$, hence there exists $P_b \in \operatorname{val}(a \vee b)$ with $P \subseteq P_b$. Then P_b is a proper prime ideal in A, and because, by [22; Lemma 1], for arbitrary $0 \leq x, y \in A$ and proper prime ideal Q in A, $x \vee y \notin Q$ if and only if $x \notin Q$ or $y \notin Q$, we obtain $b \notin P_b$.

Similarly there exists $P_a \in \operatorname{val}(a \vee b)$ such that $a \notin P_a$. Since $P_a \neq P_b$, $a \vee b$ has two different values. □

Remark. In the paper [25], the class $\mathcal{IRN}$ was studied of all algebraic distributive lattices L such that the join-subsemilattice $\mathrm{Con}(L)$ of compact elements of L is a sublattice of L and the lattice $\mathrm{Con}(L)$ is relatively normal. Structure properties of the lattices of $\mathcal{IRN}$ have been further developed in [9]. (Recall that a distributive lattice is called *relatively normal* if the set of all its prime ideals is a root-system under set-inclusion.) By [25; Corollary 3.2], an algebraic distributive lattice L such that $\mathrm{Con}(L)$ is a sublattice of L belongs to the class $\mathcal{IRN}$ if and only if the meet-prime elements of L form a root-system.

By [29; Theorems 1, 6, Lemma 4], for an arbitrary DRl-monoid A, the lattice $\mathcal{I}(A)$ of ideals of A is a complete algebraic Brouwerian lattice in which the compact elements are exactly the principal ideals. Moreover, by [19; Propositions 2, 3], the principal ideals of an arbitrary DRl-monoid A form a sublattice of $\mathcal{I}(A)$ and, by our Proposition 4, the prime ideals of representable DRl-monoids form a root-system. Hence for any representable DRl-monoid A we have that $\mathcal{I}(A)$ belongs to the class $\mathcal{IRN}$. We say that $\{0\} \neq I \in \mathcal{I}(A)$ is a *linear element* in $\mathcal{I}(A)$ if the set $\{K \in \mathcal{I}(A) : K \subseteq I\}$ is a chain. Therefore some of results of [25] can also be applied in our situation. For example, by [25; Proposition 5.2], we obtain that conditions $1-8$ of Theorem 6 are also equivalent to each of the following conditions ($\{0\} \neq I \in \mathcal{I}(A)$):

9. *I is a linear element in $\mathcal{I}(A)$.*
10. *$I^{\perp\perp}$ is a maximal linear element in $\mathcal{I}(A)$.*

Since ideals (prime ideals, polars, respectively) in any MV-algebra and in the induced DRl-monoid coincide, we get as an immediate consequence:

COROLLARY 7. *If A is an MV-algebra and $\{0\} \neq I \in \mathcal{I}(A)$, then conditions* $1-10$ *from Theorem* 6 *and the preceding Remark are equivalent.*

Remark. The equivalence of 1 and 4 for MV-algebras is also proved in [11; Theorem 4.14].

Now we can prove a generalization of [8; Theorem **3.3**].

THEOREM 8. *Let A be a representable DRl-monoid and I a prime ideal in A. Then $I^{\perp\perp} = A$ or $I^{\perp\perp} = I$.*

Proof. Let $I^{\perp} \neq \{0\}$ and let $x \in I^{\perp\perp} \setminus I$. Then also $x * 0 \in I^{\perp\perp} \setminus I$. If $y \in I^{\perp}$, then $(x*0) \wedge (y*0) = 0$, hence, because $x * 0 \notin I$, we get $y * 0 \in I$, and thus $y \in I$. Since $I^{\perp\perp}$ is the complement of $I^{\perp}$ in $\mathcal{P}(A)$, $I^{\perp} \cap I^{\perp\perp} = \{0\}$, and so we also have $I^{\perp} \cap I = \{0\}$, therefore $y = 0$. That means $I^{\perp} = \{0\}$, a contradiction. Thus $I^{\perp\perp} = I$. $\square$

COROLLARY 9. *Let A be a representable DRl-monoid and I a prime ideal in A such that $I^\perp \neq \{0\}$. Then*

a) *$I^\perp$ is a maximal proper linearly ordered ideal in A.*
b) *I is a minimal prime ideal in A.*

Proof.

a) By Theorem 8, $I = I^{\perp\perp}$. We have $I^\perp \in \mathcal{I}(A)$ and $I^\perp \neq \{0\}$, hence $(I^\perp)^\perp = I^{\perp\perp} = I$ is a prime ideal, thus by Theorem 6, $I^\perp$ is linearly ordered. Moreover $(I^\perp)^{\perp\perp} = I^\perp$, therefore, again by Theorem 6, $I^\perp$ is maximal among linearly ordered ideals in A.

b) By Theorem 6 it is now obvious that $I = (I^\perp)^\perp$ is a maximal prime ideal.

□

Remark.

a) We have shown that if I is a prime ideal in A, then either $I^{\perp\perp} = A$ or I is minimal prime.

b) Using the same arguments as before Corollary 7, one can reformulate Theorem 8 and Corollary 9 for prime ideals in MV-algebras. Then [8; Theorem 3.4] is a special case of Corollary 9a).

Now we shall study annihilators in DRl-monoids and MV-algebras, which generalize the polars of those algebras. (Annihilators in normal autometrized l-algebras were introduced in [4].)

DEFINITION.

a) If A is a DRl-monoid and $a, b \in A$, then the set

$$\langle a, b\rangle = \{x \in A : (a * 0) \wedge (x * 0) \leq n(b * 0) \text{ for some } n \in \mathbb{N}\}$$

is called a *relative annihilator of the element a with respect to the element b.*

b) A subset $B \subseteq A$ is called a *relative annihilator in A* if $B = \langle a, b\rangle$ for some $a, b \in A$.

Remark.

a) If a is an element in a DRl-monoid A, then, by [29], the principal ideal $I(a)$ is

$$I(a) = \{x \in A : x * 0 \leq n(a * 0) \text{ for some } n \in \mathbb{N}\}.$$

Therefore for any $a, b \in A$, $\langle a, b\rangle = \{x \in A : (a * 0) \wedge (x * 0) \in I(b)\}$.

b) For any $a \in A$, the polar $a^\perp$ is a relative annihilator in A because $a^\perp = \langle a, 0\rangle$. In particular, $A = 0^\perp$ is a relative annihilator in A.

c) Since each DRl-monoid A is a normal autometrized l-algebra, by [4; Theorem 1], we have that every relative annihilator in A is an ideal of A.

d) As is shown in [4], the relative annihilators in a (representable) DRl-monoid need not form a complete lattice with respect to set inclusion, so we will use the following concept.

DEFINITION. If A is a DRl-monoid and $B \subseteq A$, then B is called an *annihilator in* A if $B = \bigcap\{B_\gamma : \gamma \in \Gamma\}$ for a system of relative annihilators B_γ $(\gamma \in \Gamma)$ in A.

The set of all annihilators in A will be denoted by $\mathrm{Ann}(A)$.

THEOREM 10. *Let A be a DRl-monoid. Then $\mathcal{P}(A) \subseteq \mathrm{Ann}(A) \subseteq \mathcal{I}(A)$ and $\mathrm{Ann}(A)$ is a complete lattice with respect to set inclusion which is a complete inf-subsemilattice of the lattice $\mathcal{I}(A)$.*

P r o o f . By the definition of an annihilator and parts b) and c) of the preceding remark it is obvious, firstly that $(\mathrm{Ann}(A), \subseteq)$ is a complete lattice with least element $\{0\}$ and greatest element A in which infima coincide with intersections, and secondly that $\mathrm{Ann}(A) \subseteq \mathcal{I}(A)$.

If $B \in \mathcal{P}(A)$, then $B = \bigcap\limits_{c \in B^\perp} c^\perp = \bigcap\limits_{c \in B^\perp} \langle c, 0\rangle$, therefore $B \in \mathrm{Ann}(A)$. □

Let us denote by $A(M)$ the least annihilator in A containing $M \subseteq A$. In particular, set $A(c) = A(\{c\})$ for any $c \in A$.

The following theorem shows some further connections between ideals and annihilators.

THEOREM 11.

a) *If A is a DRl-monoid, then every principal ideal of A belongs to* $\mathrm{Ann}(A)$.

b) *If A is representable and I is a prime ideal in A such that $I^\perp \neq \{0\}$, then $I \in \mathrm{Ann}(A)$.*

P r o o f .

a) By [4; Theorem 4], $A(c) = A(I(c)) = I(c)$ for any $c \in A$, hence $I(c) \in \mathrm{Ann}(A)$.

b) By Theorem 10 it is obvious that $I \subseteq A(I) \subseteq I^{\perp\perp}$. If A is representable, then by Theorem 8, $I = I^{\perp\perp}$, and thus $I = A(I)$. □

COROLLARY 12. *Theorems* 10 *and* 11 *are true for arbitrary MV-algebra A.*

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Department of Algebra and Geometry
Palacký University
Tomkova 40
CZ–779 00 Olomouc
CZECH REPUBLIC
E-mail: rachunek@risc.upol.cz