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PARABOLIC EQUATIONS WITH DEVIATING ARGUMENT

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1. Introduction

In the present paper we shall consider the parabolic functional differential equation with deviating argument with infinite memory:

$$\frac{dx}{dt} + Ax = f(t, x_{\omega(t)}), \qquad x_0 = h, \tag{E}$$

where A is a sectorial operator on the Banach space X, f is a continuous function $\langle 0, \infty) \times C \to X$ satisfying a special lipschitz like condition. We prove the existence of a solution and some properties of this solution. The obtained results and the methods of proofs are analogous to those in [2], where the problem without deviation had been treated. Therefore we omit in §1 the proofs of the corresponding results (Lemma 1, Lemma 3), which can be proved by the same technique. In §2 we can consider the stability and the instability of the zero solution for such a problem, following the [2] ideas.

In §3, we analogously [1] define the Ljapunov function for such a problem and investigate the stability.

§0. Assumptions and denotations

X is a real Banach space with the norm I.I.

$A: D(A) \subset X \to X$ is a sectorial operator with $Re\, \alpha(A) \rangle$ a (For the definition, see [He]).

$$\alpha \in (0, 1), \qquad t_0 \geq 0.$$

X^α denotes a power space (following the [21 terminology) with the norm $\mathrm{I.I}_\alpha$.

We shall also need the following two estimates, see [2]:

(A1) $\quad |e^{-At}x|_\alpha \leq C_\alpha e^{-at} t^{-\alpha}|x|$ for each $t > 0$ and for each $x \in X$.

(A2) $\quad |(e^{-At} - I)x| \leq C(a, A, \alpha, T)\, t^\alpha |x|_\alpha$ for each $x \in X^\alpha$, $t \in \langle 0, T \rangle$.

$C = \{u; u \in C((-\infty, 0\rangle, X^\alpha); \sup_{\Theta \in (-\infty, 0\rangle} |u(\Theta)|_\alpha < \infty\}$ is a Banach space with the norm $\|u| = \sup_{\Theta \in (-\infty, 0\rangle} |u(\Theta)|_\alpha$.

$O_\varepsilon(x)$ denotes $\{u \in C;\ \|x - u\| < \varepsilon\}$ for each $x \in C$.

$f\colon \langle t_0, \infty) \times C \to X$ is a continuous function which satisfies the assumptions (P1) and (P2):

(P1) f is a locally Lipschitz continuous in both variables $(t_0, \infty) \times C \to X$

(P2) For each $x_1^* \in C$ and $T \in (t_0, \infty)$ there exists $L(x_1^*, T) > 0$ and $\varepsilon(x_1^*, T)$ such that if $\|x_1 - x_1^*\| < \varepsilon$, $\|x_2 - x_1^*\| < \varepsilon$ and $t \in \langle t_0, T\rangle$, then

$$|f(t, x_1) - f(t, x_2)|_X \leq L(x_1^*, T)\|x_1 - x_2\|$$

$h \in C$; h is a uniformly continuous function $(-\infty, 0\rangle \to X^\alpha$,

$\omega(t)\colon \langle t_0, \infty) \to R$ is a continuous function such that

$$\omega(t) \leq t \qquad \text{for each } t \in \langle t_0, \infty).$$

If $x \in C((-\infty, u\rangle, X^\alpha)$, we denote $x_u(s) = x(u + s)$ for each $s \in (-\infty, 0\rangle$.

We shall consider the problem (E) together with the initial condition

$$x_{t_0} = h \tag{Co}$$

§1. Definition of a mild solution

We consider the problem (E, C_0). *Let* $T \geq t_0$. *Any solution* $x \in C((-\infty, T), X^\alpha)$ *of the integral equation*

$$x(t) = \mathrm{e}^{-A(t - t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{-A(t-s)}f(s, x_{\omega(s)})\, ds \,\forall\, t \in \langle t_0, T) \tag{E1}$$

which satisfies (C_0), *is said to be a mild solution to the initial problem* (E, C_0) *on the interval* $(-\infty, T)$.

Existence theorem 1. *There exists a unique mild solution of the problem* (E, C_0) *on the interval* $(-\infty, T)$ *for some* $T > t_0$.

Proof. Choose an arbitrary $T_0 > t_0$ and $L(x_{\omega(t_0)}, T_0)$, $\varepsilon(x_{\omega(t_0)}, T_0)$ from the assumption (P2) such that

$|f(s, x_1) - f(s, x_2)| \leq L\|x_1 - x_2\|$ for each $s \in \langle t_0, T_0\rangle$ and for each x_1, $x_2 \in O_\varepsilon(x_{\omega(t)})$.

Let $S = \{x; x \in C((-\infty, T\rangle, X^\alpha); x_{t_0} = h; |x(t) - h(t_0)|_\alpha \leq \delta \forall t \in \langle t_0, T)\}$ and let $Gx(t) = \mathrm{e}^{-A(t-t_0)}h(0) + \int_{t_0}^{t} \mathrm{e}^{-A(t-s)}f(s, x_{\omega(s)})\,ds$ for each $t \in \langle t_0, T)$, $x \in S$, $Gx(t) = h(t - t_0)$ for $t < t_0$.

The proof of this theorem is based on the Banach fixed point theorem.

We shall show that there exists $\delta > 0$ and $t_0 < T \leq T_0$ such that G maps S into S and G is a contraction.

First we take $T_1 < T_0$ such that

$$L \int_{t_0}^{t} C_\alpha (t-s)^{-\alpha} \mathrm{e}^{-a(t-s)}\,ds < \frac{1}{4} \text{ for each } t \in \langle t_0, T_1 \rangle .$$

Then for $x_1, x_2 \in C((-\infty, T_1), X^\alpha)$, $x_{1,t_0} = x_{2,t_0} = h$ and $x_{1,\omega(s)}, x_{2,\omega(s)} \in O_\varepsilon(x, \omega(t_0))$ the following estimate takes place:

$$\left| \int_{t_0}^{t} \mathrm{e}^{-A(t-s)}(f(s, x_{1,\omega(s)}) - f(s, x_{2,\omega(s)})\,ds \right|_\alpha \leq \frac{1}{2} \|x_1 - x_2\| . \tag{1}$$

Let $\delta = \dfrac{\varepsilon}{2}$.

We shall find such a small $T_1^* < T_1$ (we denote it again by T_1) that for each $x \in S$ and $s \in (t_0, T_1)$ $x_{\omega(s)} \in O_\varepsilon(x_{\omega(t_0)})$.

1. First we consider the case $\omega(t_0) < t_0$. In this case there exists because of the uniform continuity of the function h $\gamma > 0$ such that

$$|z_1 - z_2| < \gamma \text{ implies } |h(z_1) - h(z_2)|_\alpha < \varepsilon \text{ for each } z_1, z_2 \in (-\infty, 0\rangle$$

Put T_1^*, $T_1 > T_1^* > t_0$ such that $|\omega(s) - \omega(t_0)| < \gamma$ and $\omega(s) < t_0$ for each $s \in \langle t_0, T_1)$. Then

$$|x_{1,\omega(s)} - x_{\omega(t_0)}|_\alpha \leq \sup_{u \in (-\infty, 0\rangle} |h(u) - h(u + (\omega(s) - \omega(t_0))|_\alpha < \varepsilon$$

for each $x_1 \in S$.

2. In the case of $\omega(t_0) = t_0$ there exists such a $\beta > 0$, $\gamma > 0$ that $\sup_{u \in \langle -\beta, 0\rangle} |x(t_0) - x(u + t_0)|_\alpha < \dfrac{\varepsilon}{3}$ and $|z_1 - z_2| < \gamma$ implies $|h(z_1) - h(z_2)| < \dfrac{\varepsilon}{3}$ for each $z_1, z_2 \in (-\infty, 0\rangle$. Next we put $T_1^* > t_0$ such that the following statements hold:

(i) $t_0 < T_1^* < T_1$

(ii) $|\omega(s) - \omega(t_0)| < \min\{\beta, \gamma\} \, \forall s \in \langle t_0, T_1^* \rangle$

After some calculations we get that

$$|x_{1,\omega(s)} - x_{\omega(t_0)}|_\alpha < \varepsilon \, \forall x_1 \in S .$$

We denote T_1^* again by T_1.
Finally, we choose T_1^{**} such that $T_1^{**} < T_1$,

$$|(e^{-A(t-t_0)} - I)\,h(0)|_\alpha = |(e^{-A(t-t_0)} - I)\,A^\alpha h(0)| < \frac{\delta}{4}$$

and

$$\int_{t_0}^{t} C_\alpha e^{-a(t-s)}(t-s)^{-\alpha}|f(s, x_{\omega(t_0)})|_X\, ds < \frac{\delta}{4}$$

for each $t \in \langle t_0, T_1\rangle$. Then

$$|Gx(t) - Gx(t_0)|_\alpha \leq (e^{-A(t-t_0)} - I)\,x_0|_\alpha +$$

$$+ \int_{t_0}^{t} C_\alpha e^{-a(t-s)}(t-s)^{-\alpha} L\,\|x_{\omega(s)} - x_{\omega(t_0)}\|\, ds +$$

$$+ \int_{t_0}^{t} C_\alpha e^{-a(t-s)}(t-s)^{-\alpha}|f(s, x_{\omega(t_0)}|\, ds < \delta$$

for each $x \in S$.
Thus G maps S into S and G is a contraction.

Corollary 1. *On the basis of Theorem* 1. *there exists a unique mild solution of the problem* (E, C_0) *on the maximal interval* $\langle t_0, T)$, *(eventually* $T = \infty$*). We shall call this solution a maximal solution.*

We shall follow under what assumptions the interval of the maximal solution is (t_0, ∞). *In the next theorems and lemmas the solution is always a mild solution.*

Lemma 1. *Assume that the imago* $f(B)$ *of every closed and bounded set* $B \subset \subset \langle 0, \infty) \times C$ *is bounded in* X.
If x_1 *is a maximal solution of* (E, C_0) *on* (t_0, t_1), *then either* $t_1 = +\infty$ *or else there exists a sequence* $t_n \to T_1^-$ *as* $n \to \infty$ *such that* $|x(t_n)|_\alpha \to \infty$.

Lemma 2. *Let* $K(t)$: $\langle 0, \infty) \to R$ *be a continuous function. Let.* $|f(t, x)| \leq K(t)(1 + \|x\|)$ for each $x \in C$, $t \geq 0$. *Then the maximal interval of the existence of the solution of* (E, C_0) *is* (t_0, ∞).

Proof. We shall show it by contradiction. Under the assumptions above let $\langle t_0, t_1)$ be the maximal interval of the existence of the solution of (E, C_0), where $t_1 < \infty$.

Cleatrly, by Lemma 1., there exists a sequence $t_n \to t_1$ such that $|x(t_n)|_\alpha \to \infty$. Now

$$|x(t)|_\alpha \leq C_\alpha e^{-at}|x_0|_\alpha + \int_{t_0}^{t} C_\alpha (t-s)^{-\alpha} e^{-a(t-s)}|\,K(s)|(1 + \|x_s\|)\, ds\,.$$

Since the functions e^{-at}, $e^{-a(t-s)}$, $K(t)$, $\int_{t_0}^{t} (t-s)^a ds$ are bounded on the interval $\langle t_0, t_1)$, we have:

$$|x(t)|_\alpha \le C_1 + c_1 \int_{t_0}^{t} (t-s)^{-\alpha} \|x_s\| ds$$

for each $t \in \langle t_0, t_1)$.

After simple calculations it is clear that the function $t \to \int_{t_0}^{t} (t-s)^{-\alpha} \|x_s\| ds$ is nondecreasing on the interval $\langle t_0, t_1)$. Hence

$$\sup_{u \in \langle 0, t)} |x(u)|_\alpha \le C_1 + C_1 \int_{t_0}^{t} (t-s)^{-\alpha} \|x_s\| \, ds\,,$$

which implies that

$$\|x_t\| \le C_1 + \|h\| + C \int_{t_0}^{t} (t-s)^{-\alpha} \|x_s\| \, ds\,.$$

From Gronwall-like Theorem, see [10, p. 188] it follows that $\|x_t\|$ is bounded function on $\langle t_0, t_1\rangle$, which is a contradiction.

Corollary 2. *Let $x(t)$ be a maximal solution on $\langle t_0, t_1)$ and let $\dfrac{|f(t, x_t)|}{1 + \|x_t\|}$ be a bounded function. Then $t_1 = +\infty$.*

Lemma 3. *Assume f maps all sets $R^+ \times B \subset R \times C$, with B closed and bounded into bounded sets of X. Assume A has a compact resolvent. If $x(t)$ is a bounded solution to the problem (E, C_0) on (t_0, ∞), then $\{x(t);\, t > t_0\}$ lies in a compact set in X^α.*

Lemma 4. (Continuous dependence on the initial condition). *Let $h, h_n \in C$ be uniformly continuous functions on $(-\infty, 0\rangle \to X^\alpha$. Suppose that h_n converges to h for $n \to \infty$. Denote the solutions of the problem (E, C_0) as $x[h]$, similarly $x[h_n]$ and let $\langle t_0, T_n)$, $\langle t_0, T_0)$ be a maximal interval of existence of the solution $x[h_n]$, $x[h]$, respectively. Then* $\liminf T_n \ge T_0$ *and $x_t[h]$ converge to $x_t[h]$ uniformly on the compact subintervals of $\langle t_0, T_0)$.*

Proof. Let T be an arbitrary such that $\langle t_0, T\rangle \subset \langle t_0, T_0)$. By the compactness of the set $\{x_{\omega(t)};\, t \in \langle t_0, T\rangle\} \subset C$ and by (P2) it follows that there exists $\gamma > 0$ and $L > 0$ such that $|x_1 - x_{\omega(t)}| < \gamma$ for some $t \in \langle t_0, T\rangle$ implies

$$|f(s, x_1) - f(s, x_{\omega(t)}| \le L \|x_1 - x_{\omega(t)}\| \text{ for each } s \in \langle t_0, T\rangle\,.$$

Now the following estimate holds on each subinterval $\langle t_0, T^*\rangle \subset \langle t_0, T\rangle$ such that $x_t[h_n]$ exists and $\|x_t[h_n] - x_t[h]\| < \gamma$ for each $t \in \langle t_0, T^*\rangle$:

$$|x[h_n](t) - x[h](t)|_\alpha \leq C_\alpha e^{-a(t-t_0)} |h_n(0) - h(0)|_\alpha + \\ + C \int_{t_0}^{t} (t-s)^{-\alpha} \|x_s[h_n] - x_s[h]\| \, ds .$$

The function $t \to \int_0^t (t-s)^{-\alpha} \|x_s[h_n] - x_s[h]\| \, ds$ is nondecreasing.

Hence

$$\|x_t[h_n] - x_t[h]\| \leq C_1 \|h_n - h\| + C \int_{t_0}^{t} (t-s)^{-\alpha} \|x_s[h_n] - x_s[h]\| \, ds$$

for each $t \in \langle t_0, T^*\rangle$. From the Gronwall-like Theorem, see [He, p. 188] it follows that

$$\|x_t[h_n] - x_t[h]\| \leq C^{**} \|h_n - h\| \tag{3}$$

We have shown that for each $T^* < T$ such that $x_t[h_n]$ exists on $\langle t_0, T^*\rangle$ and $\|x_t[h_n] - x_t[h]\| < \gamma$ the estimate (3) holds. As well as $\|h_n - h\| \to 0$ for $n \to \infty$, we have that $\|x_t[h_n] - x_t[h]\| < \gamma$ on $\langle t_0, T\rangle$ from certain n_0 for each $n > n_0$. This completes the proof.

We shall deal with the problem, under what assumptions on f the strong and the mild solution are the same.

Definition *of a strong solution to the problem* (E, C_0). *We consider the problem* (E, C_0). *Let* $T \geq t_0$. *Any solution* $x \in C((-\infty, T), X^\alpha)$ *of the equation* (E) *for each* $t \in \langle t_0, T)$ *which satisfies* (C_0) *such that*

(i) $x(t) \in D(A)$ *for each* $t \in (t_0, T)$

(ii) x *is differentiable on* (t_0, T) *and*

(iii) *the function* $t \to f\,t, x_{\omega(t)})$ *is a locally Hölder continuous one on* (t_0, T) *into* X.

is said to be a strong solution to the problem (E, C_0).

Remark 1. If x is a strong solution to (E, C_0), it is also a mild solution. This follows from the theorem 3.2.2. [2].

Lemma 5. *Suppose that* ω: $\langle 0, \infty) \to R$ *is a locally Hölder continuous function and* h *is Hölder continuous. Let* $h(0) \in X^{\alpha+\varepsilon}$ *for some small* $\varepsilon > 0$, $\alpha + \varepsilon < 1$. *Then the mild solution to the problem* (E, C_0) *is also a strong one.*

Proof. We apply Theorem 3.2.2 in [2]. We must show that the function $t \to f(t, x_{\omega(t)})$ is under the assumptions above locally Hölder continuous on (t_0, T).

One can easy show that the mild solution x is locally Hölder continuous on $\langle t_0, T)$ into X^α (on the basis of the Variation of constants formula and the fact that $x(0) \in X^{\alpha+\varepsilon}$). Hence x is Hölder continuous on each interval $(-\infty, t\rangle$, where $t < T$. From that and from the local Hölder continuity of f on $(t_0, \infty) \times C$ it follows that for each $t_1 \in (t_0, T)$ there exists $\delta_1 > 0$, $L > 0$ such that

$$|t_1 - t_2| < \delta_1 \text{ implies } |f(t_1, x_{\omega(t_1)}) - f(t_2, x_{\omega(t_2)})| \leq L(|t_1 - t_2| +$$

$$+ \|x_{\omega(t_1)} - x_{\omega(t_2)}\| \leq L(|t_1 - t_2| + H(\omega(t_1) - \omega(t_2))^r) \leq$$

$$\leq L(|t_1 - t_2| + H_1(t_1 - t_2)^{r\omega_1}).$$

This completes the proof.

§2. Stability.

Let $x(., t_1, h)$ denote a mild solution of the problem (E), with an initial condition $x_{t_1} = h$. Suppose that in the assumptions $t_0 = 0$.

Definition of the stability. *Let $f(t, 0) = 0$ for each $t \geq 0$ and hence 0 is a solution of the equation (E) with the initial condition $x_0 = 0$. Then 0 is stable if, for any $t_1 > 0$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|h\| < \delta$ and $h \in C$ is a uniformly continuous function implies $|x(t, \,, t_1)|_\alpha < \varepsilon$ for each $t \geq t_1$.*

0 is uniformly stable if it is stable and ε is independent of t_1.

0 is asymptotically stable if it is stable and $x(t, h, t_1)$ converges to 0 when $t - t_1$ converges to $+\infty$.

0 is unstable iff it is not stable.

Theorem 2. *Let $f(t, 0) = 0$ for each $t \geq 0$. Let $Re\,\sigma(A) > a > 0$. If $f(t, u) = o(\|u\|)$ uniformly when $\|u\| \to 0$, then the null solution is stable.*

Proof. The proof is very similar to the proof of Th 5.1.1. in [2]. We can take a $\sigma > 0$ such that.

$\sigma C_\alpha \int_0^\infty s^{-\alpha} e^{-as} ds < \frac{1}{2}$ and we can choose $\varrho > 0$ such that the following statements take place:

$$\|u\| < \varrho \text{ implies } |f(s, u)| \leq \sigma\|u\| \text{ for each } s \in \langle 0, \infty).$$

Let $\delta < \min\left(\frac{\varrho}{2}, \frac{\varrho}{C_0}\right)$. Suppose that $\|h\| < \delta$

If $|z(t, t_1, h)|_\alpha \leq \varrho$ on $\langle t_1, T^*\rangle$, the following estimate is true:

$$|z(t)|_\alpha \leq C_0 e^{-a(t-t_1)}|z(t_1)|_\alpha + \sigma C_\alpha \varrho \int_{t_1}^t (t-s)^{-\alpha} e^{-a(t-s)} ds \leq$$

$$\leq \frac{\varrho}{2} e^{-a(t-t_1)} + \frac{\varrho}{2}.$$

Hence $|z(t)|_\alpha < \varrho$ for each $t \geq t_1$ and 0 is uniformly stabile.

In the next example we shall see that under the assumptions of the above theorem the trivial solution is not asymptotically stable. This differs from the theory in [2] for the parabolic equations without delay.

Example 1. We shall consider $X = L_2(0, \pi)$, $A = -u''$, $D(A) = W_2^2 \cap \cap \mathring{W}_2^1 = X^1$, $X^{1/2} = \mathring{W}_2^1 \to C(0, \pi)$.

Let $f(t, u) = u^2(-1)$ and let $\omega(t) = 0$. Let $h_n(t, x) = \frac{1}{n} \sin x$ for each $t \leq -1$, $n \in \{1, 2, \ldots\}$, $h_n(t, x) = \frac{-t}{n} \sin x + u_n(x)(1 + t)$ for each $t \in \langle -1, 0\rangle$, where

$$u_n(x, t) = \frac{1}{n^2} \frac{x}{\pi} \int_0^\pi \int_0^r \sin^2 z \, dz - \frac{1}{n^2} \int_0^x \int_0^r \sin^2 z \, dz \, dr \qquad \text{for } t \geq 0$$

$$u_n(x, t) = h_n(x, t) \qquad \text{for } t \leq 0 .$$

It is easy to verify that $u_n(x, t)$ is the solution of the problem (E) with initial condition $u_0(t) = h_n(t)$. Of course, $\frac{|f(u)|}{\|u\|} = \frac{|u^2(-1)|}{\|u\|} L_2$ converges to 0, if $\|u\| \to 0$. Since u_n is independent of t and different from 0 for each n, 0 is not asymptotically stable.

Example 2. We shall consider two problems:

$$\text{(j)} \qquad \frac{du}{dt} - u_{xx} = \frac{1}{t} \int_0^t u^2(r) \, dr$$

$$u(t, 0) = u(t, \pi) = 0$$

$$\text{(jj)} \qquad \frac{du}{dt} - u_{xx} = \frac{1}{t} \int_{-t}^t u^2(r, x) \, dr$$

$$u(t, 0) = u(t, \pi) = 0$$

Both can be represented in the form of the equations with the delayed argument, since

$$\text{(j)} \qquad f(t, u) = \frac{1}{t} \int_{-t}^0 u^2(r) \, dr, \qquad \omega(t) = t$$

$$\text{(jj)} \qquad f(t, u) = \frac{1}{t} \int_{-2t}^0 u^2(r) \, dr, \qquad \omega(t) = t,$$

where A, X, X^a are as in the previous example. Also $\lim\limits_{\|u\| \to x} \frac{|f(u)|}{\|u\|} = 0$. Hence 0 is stable in both cases.

In the rest of this paragraph we shall deal with the case when $Re\sigma(A) \cap \{x; x < 0\} \neq 0$ and we shall see that under this assumption 0 will be an unstable solution. First we shall need some assertion, which will be needed in the proof.

Assertion 1. *Suppose X is a real Banach space, $M: z \to e^{-A}z$ is a continuous linear operator on X^a with spectral radius $r > 0$. Given any $\delta > 0$ and $N_0 \geq 0$, there exists an integer $N \geq N_0$ and $u \in X$, $\|u\| = 1$ such that $|M^n u| \leq (2^{1/2} + \delta) r^n$* for n integer $n \in \langle 0, N\rangle$, $|M^N u| \geq (1 - \delta) r^n$ and $|M^{n+t}u| \leq C_1(2^{1/2} + \delta) r^n$ *for each $n + t \in \langle 0, N\rangle$, $t \in \langle 0, 1\rangle$.*

Proof. The assertion is the same as Lemma 5.1.4. in [2], only the last extimate follows from the estimate

$$|M^{n+t}u| = |M^t M^n u| \leq |M^t| |M^n u| \leq C_1(2^{1/2} + \delta) r^n.$$

Theorem 3. *Let all assumptions from §0. hold, except the one about $\sigma(A)$. Let $\sigma(A) \cap \{x;\ Rex < 0\} \neq \emptyset$. Let $|f(s, u)| = O(\|u\|^p)$ when $\|u\| \to 0$* and $p > 1$, *uniformly with respect to s. Let O be a solution of (E). Then O is unstable.*

Proof. The idea of the proof is due to [2]. Put $M: X^a \to X^a$, $Mz = e^{-A}z$. Clearly M is a continuous linear operator with spectral radius $r > 1$. Hence we can apply Assertion 1.

We can define operator T_{n-1+t}: $C \cap \{u,\ u$ is uniformly continuous$\} \to X^a$ for each $t \in (0, 1\rangle$ such that for each $h \in D(T_{n-1+t})$: $T_{n-1+t}(h) = x(t + n - 1; n - 1, h)$.

We shall proceed by contradiction. Suppose that 0 is stable. Then there exist such $\varepsilon > 0$, $\varrho > 0$ and $\delta > 0$, that if $\|h\| \leq \delta$, then $\|x(t, 0, h)\| < \varepsilon$ for each $t \geq 0$ and $|f(s, x_{\omega(s)})| \leq \varrho(\varepsilon)\|x_s\|^p$ for each $s \geq 0$.

First we shall show the implication

(I) If $\|h\| < \delta$, $\|x_s(., n - 1, h)\| \leq \varepsilon$ for each $s \geq n - 1$, then $\|x_s(., n - 1, h)\| \leq C_2\|h\|$ for each $s \in \langle n - 1, n\rangle$ and C_2 is independent on n. To prove this we use the Variation of constants formula and the following estimate

$$|x(t_1)|_a \leq C^*|h(0)|_a + \int_{n-1}^{t_1} C^*(t_1 - s)^{-a} \varrho(\|h\| + \sup_{r \in \langle n-1, s\rangle} |x(r)|_a)\, ds$$

for each $t_1 \in \langle n - 1, n\rangle$.

In the same way as in the previous proofs, by using the Gronwalls lemma we get (I).

Further we prove the statement

(II) There exist a, b such that

if $\|h\| < a$ and $\|x_{\omega(s)}\| \leq \varepsilon$ for each $s \geq n - 1$, $h \in D(T_{n-1+t})$, then $|T_{n-1+t}(h) - M^t h(0)|_a \leq b|h|^p$ for each $t \in \langle 0, 1\rangle$.

This statement follows easily from the fact that

$$|T_{n-1+t}(h) - M^t h(0)|_\alpha = \left| \int_{n-1}^{n-1+t} e^{-A(t+n-1-s)} f(s, x_{\omega(s)})\, ds \right|_\alpha \leq$$

$$\leq \int_{n-1}^{n-1+t} C(t+n-1-s)^{-\alpha} \|x_s\|^p\, ds \leq b\|h\|^p.$$

Now we shall proceed as in [2] in Thm.5.5.1. We choose $0 < \delta < \frac{1}{2}$, $R = 2C_1(2^{1/2} + \delta)$ (where C_1 is a constant from Assertion 1.) and σ is so small that $\sigma \leq \frac{a}{R}$, $\frac{KbR^p\sigma^{p-1}}{r^p - r - \eta} \leq \frac{1}{2}$; where η and K are such constants that

$$|M^t| \leq K(r+\eta)^t \quad \text{for each } t \geq 0 \text{ and } r^p - r > \eta > 0.$$

We show that to each $N_0 > 0$ there exist $x_0 \in X^\alpha$ and $N \geq N_0$ such that $|x_0| = \frac{\sigma}{r^N}$ and the initial function $h(t) = x_0$ for each $t \leq 0$ such that if $x_n = T_n(x_{n-1})$, $n \geq 1$, then $\|x_n\| \leq a$ for each $1 \leq n \leq N$ and $\|x_N\| \geq \left(\frac{1}{2} - \delta\right)\sigma$. But this will be a contradiction. We take on arbitrary integer $N_0 \geq 0$. In agreement with the assertion 1 there exists a $u \in X^\alpha$, $N \geq N_0$ such that $|u|_\alpha = 1$ and u satisfies the assumptions of the assertion 1. Put $x_0 = \varepsilon_1 u$, where $\varepsilon_1 = \frac{\sigma}{r^N}$. Clearly,

(III) $x(n-1+t) = T_{n-1+t}x_{n-1} - M^t x(n-1) + \sum_{k=0}^{n-2} M^{n-2-k+t}(T_{k+1}x_k - Mx(k)) + M^{n-1+t}x(0)$, where x is the mild solution of the problem (E, C_0).

Now

$$|M^{n-1+t}x_0|_\alpha \leq \varepsilon_1 |M^{n-1+t}u|_\alpha \leq \frac{\varepsilon_1 R r^n}{2} \tag{4}$$

and for each n

$$\left|\left(\left(\sum_{k=0}^{n-2} M^{n-2-k+t}(T_{k+1}x_k - Mx(k)\right)\right) + T_{n-1+t}x_{n-1} - M^t x(n-1)\right|_\alpha \leq$$

$$\leq \sum_{k=0}^{n-2} K(r+\eta)^{n-2-k+t} |T_{k+1}x_k - Mx(k)|_\alpha + |T_{n-1+t}x_{n-1} - M^t x(n-1)|_\alpha \leq$$

$$\leq \sum_{k=0}^{n-1} bK(r+\eta)^{n-1-k} \|x_k\|^p. \tag{5}$$

In the same way as in [2], one can show by induction that $\|x_k\| \le \varepsilon_1 Rr^k$ and

$$\|x_{k+t}\| \le \varepsilon Rr^{k+1} \text{ for each } t \in \langle 0, 1\rangle,\ k+t \in \langle 0, N\rangle,\ 0 \le k \text{ integer.} \tag{6}$$

Since

$$\sum_{k=0}^{n-1} bK(r+\eta)^{n-1-k}(\varepsilon_1 Rr^k)^p = bK(\varepsilon_1 R)^p r^{p(n-1)} \sum_{k=0}^{n-1}\left(\frac{r+\eta}{r^p}\right)^{n-k-1} \le$$

$$\le \frac{1}{2}\varepsilon_1 r^n \le \frac{R}{2}\varepsilon_1 r^n$$

Thus $\sup_{t\in\langle 0,n\rangle} |x(t)|_a \le \varepsilon_1 Rr^n \le R\sigma \le a \forall n \in \langle 0, N\rangle$ and $|x_N|_a \ge |M^N x_0| -$
$- \left|\sum_{k=0}^{n-1} M^{N-k-1}(T_{k+1}x_k - Mx(k))\right|_a \ge (1-\delta)r^N\varepsilon_1 - \frac{\varepsilon_1 r^n}{2} = \left(\frac{1}{2}-\delta\right)\sigma.$

Remark. If $f(t, 0) = 0$, $\omega(t) \le k < 0$ for each $t \in \emptyset, \infty)$ and A is a sectorial operator such that $\sigma(A) \cap \{x; \operatorname{Re} x < 0\} = \sigma_1$ is a nonempty spectral set, then 0 is unstable.

Proof. We shall need a standard decomposition $X = X_1 \oplus X_2$, where $A_1 = A|_{X_1}$ with the spectrum σ_1 and hence, A_1 is a continuous linear operator and clearly $X_1 \ne \emptyset$.

For some $x_0 \in X_1$, $x_0 \ne 0$ we define the initial function

$$h_{x_0}(t) = 0 \qquad \text{for each } t \in \left(-\infty, \frac{k}{2}\right\rangle$$

$$h_{x_0}(t) = \left(1 - \frac{2t}{k}\right)x_0 \qquad \text{for each } t \in \left(\frac{k}{2}, 0\right\rangle.$$

The problem (E, C_0) is equivalent to the

$$\frac{du}{dt} + Au = 0$$

$$u(0) = x_0,$$

which has a unique solution $e^{-A_1 t}x_0$. Since σ_1 is a spectral set and hence closed, there exists $\beta > 0$ such that $\sigma(A_1) < -\beta$. Thus $|e^{A_1 t}| \le M e^{-\beta t}$ for each $t > 0$ and $|e^{-A_1 t}x_0|_a \ge M e^{\beta t}|x_0|_a$ for each $t \ge 0$. This means 0 is unstable.

§3. Ljapunov functions.

We proceed as in [1], where the Ljapunov functions of the functional ordinary differential equations with the delayed argument have been defined, whereby the delay is of bounded length. We again consider the problem (E, C_0). Let $x[t, \omega(t), \Phi]$ denote the solution of the problem (E) with the condition $x_{\omega(t)} = \Phi$.

We suppose that $\langle 0, \infty) \subset \{\omega(t);\, t \geq 0\}$ and $\omega(t) \leq t$ for each $t \geq 0$. Let f be a function such that there exists a $\delta_0 > 0$ with the property that for each Φ, $\|\Phi\| < \delta_0$, $\Phi \in C$ is uniformly continuous, the existence interval of the solution (E), $x_0 = \Phi$ is $\langle 0, \infty)$.

Definition. *Let $V\colon R_0^+ \times C \to R$ be a continuous function and let there exist continuous functions $u, v, w\colon R_0^+$ such that $u(s) > 0$, $v(s) > 0$, if $s > 0$ and $u(0) = v(0) = 0$. Let the following estimates*

$$|u(|\Phi(0)|_a)| \leq V(t, \Phi) \leq v(\|\Phi\|)$$

$$\dot{V}(t, \Phi) \leq -w(|\Phi(0)|_a)$$

for each $\Phi \in C$, $t > 0$ hold.
Here $\dot{V}(t, \Phi)$ *is defined as follows*:

$$\dot{V}(t, \Phi) = \limsup_{h \to 0+} \frac{1}{h}(V(t+h, x_{\omega(t+h)}[., \omega(t), \Phi]) - V(t, x_{\omega(t)}[., \omega(t), \Phi]).$$

Then $V(t, \Phi)$ *is said to be a Ljapunov function for the equation* (E).

Theorem 4. *If V is a Ljapunov function for the equation (E), then the following statements hold*:

(i) *To each $\varepsilon > 0$ there exists such* a $\delta > 0$, $\delta < \varepsilon$ *that $\|\Phi\| < \delta$ implies $|x(t, 0, \Phi)|_a < \varepsilon$ for each $t > 0$.*

(ii) *If $\lim_{s \to \infty} u(s) = \infty$, then each mild solution of our problem is uniformly bounded with respect to the initial condition in the following sense: To each $\alpha > 0$ there exists a $\beta > 0$ such that $\|\Phi\| < \alpha$ implies $|x[t, 0, \Phi]|_a < \beta$ for each $t > 0$.*

(iii) *Suppose that $\lim_{|x| \to 0} f(t, x) = 0$ uniformly with respect to t* or *f maps $R \times B$ into bounded sets in X, where B is a bounded set in C. Further $Re\, \sigma(A) > 0$, $\omega\colon \langle 0, \infty) \to R$ is uniformly continuous, $w(s) > 0$ for each $s > 0$. Then there exists a $\delta_0 > 0$ such that $\|\Phi\| < \delta_0$ implies $|x[t, 0, \Phi]|_a$ converges to if $t \to \infty$.*

Proof. (i) We prove similarly as in [1]:
To each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\delta < \varepsilon$ and $v(\delta) \langle u(\varepsilon)$. If $\Phi \in C$, $\|\Phi\| < \delta$, then

$$u(|x(0, \Phi])(\omega(t))|_a) \leq V(t, x_{\omega(t)}[0, \Phi]) \leq V(0, x_{\omega(0)}[0, \Phi]) \leq$$
$$\leq v(\|\Phi\|) \langle v(\delta) \langle u(\varepsilon).$$

Hence $|x[0, \Phi](\omega(t)|_a < \varepsilon$ for each $t \geq 0$. Thus $|x(0, \Phi](t)|_a < \varepsilon$ for each $t \geq 0$.
(ii) In the same way as in [1]: To each $\alpha > 0$ there exists a $\beta > 0$ that $u(\beta) = v(\alpha)$. Let $\|\Phi\| < \alpha$, then $|x(t, 0, \Phi)|_a \leq \beta$ for each $t \geq 0$.
(iii) Let $\delta_0 > 0$, $\varepsilon_0 > 0$ and $L > 0$ be such that $\|\Phi\| < \delta_0$ implies $|x[t, 0, \Phi]|_a <$

$< \varepsilon_0$ and $|f(t, x(., 0, \Phi)| < L$ for each $t \geq 0$. Let $\alpha + z < 1$, $z > 0$. It is easy to show that by Variation of constants formula there exists a $K > 0$ such that $\|\Phi\| < \delta_0$ implies $|x[t, 0, \Phi]|_{\alpha+z} < K$ for each $t > 1$ and $x[t, 0, \Phi]$ is a Hölder continuous in t on $\langle 1, \infty)$, i.e.

$$|x(t+h, 0, \Phi) - x(t, 0, \Phi)|_\alpha \leq K^* h^\gamma \text{ for each } t,\ t+h \in \langle 1, \infty),\ \gamma < 1\,. \tag{6}$$

We suppose by contradiction that there exists an initial condition $\phi \in C$ such that $\|\Phi\| < \delta_0$ and $\lim_{t \to \infty} |x(t, 0, \Phi)|_\alpha \neq 0$. Then there exists a $\delta > 0$ and a sequence $t_K \to \infty$ such that $|x(t_k, 0, \Phi)|_\alpha > \delta$. Now we construct a subsequence $\{t_{k_n}\}$ such that there exist a sequence $t^*_{k_n}$, which converges to $+\infty$ and $\omega(t^*_{k_n}) = t_{k_n}$. With the help of this sequence $t^*_{k_n}$ we can construct such a subsequence (we denote it again by t^*_k) that $t^*_k \nearrow \infty$ and $t^*_{k+1} - t^*_k > 1$ for each k. Thus, we have a sequence t_k with the property $|x(t_k, 0, \Phi)|_\alpha > \delta$ and $t_k = \omega(t^*_k)$, $t^*_k \nearrow \infty$ and $t^*_{k+1} - t^*_k > 1$. Now by (6) and by the uniform continuity of ω on $\langle 0, \infty)$ there exists an $h_1 > 0$ such that $t \in \langle t^*_k, t^*_k + h_1\rangle$ implies

$$\dot V(t, x_{\omega(t)}) \leq -w(|x(\omega(t))|_\alpha) \leq -w\left(\frac{\delta}{2}\right).$$

From this it follows that

$$V(t^*_k, x_{\omega(t^*_k)}) - V(0, x_{\omega(0)}) \leq -w\left(\frac{\delta}{2}\right) . h_1 . (k-1)\,.$$

We take a limit in the last estimate and we get $V(t^*_k, x_{\omega(t^*_k)}) < 0$ for some $k > 0$. But this is a contradiction.

Example. We shall consider $X = L_2(0, \pi)$, $A = -u''$, $\alpha = 1/2$, $D(A) = W_2^2 \cap \mathring{W}_2^1 (0, \pi)$ and hence $X^\alpha = \mathring{W}_2^1 (0, \pi)$.

Let $|u|_\alpha^2 = \int_0^\pi |u'|^2 dx$ and $|u|_{L_2} \leq \pi |u|_\alpha$

Let $f(t, u) = au(0) + bu(-r)$ for $u \in \{z, z \in C((-\infty, 0\rangle, X^\alpha)$ and z is a bounded function$\}$.

Let $\omega(t) = t$ and hence $f(t, u_t) = au(t) + bu(t-r)$.

So we can consider the problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u_{xx} + au(t) + bu(t-r)$$

$$u(t, 0) = u(t, \pi) = 0$$

$$u(0, x) = \Phi(x),$$

where $\Phi \in C((-\infty, 0\rangle, X^\alpha)$ is a Hölder continuous function with $\Phi(0) \in X^{\alpha+\beta}$ for some $\beta > 0$, $\alpha + \beta < 1$.

We define a function V (for some $p > 0$)

$$V(t, \Phi) = \int_0^\pi \left(\frac{1}{2} \Phi_x^2(0, x) + p \int_{-r}^0 \Phi^2(z, x)\, dz\right) \mathrm{d}x.$$

(Hence V is independent of t).
We can show that V satisfies the conditions for the Ljapunov function

$$V(t, \Phi) \geq \frac{1}{2} |\Phi(0)|_a^2 \text{ with } u(s) = \frac{1}{2} s^2$$

and

$$V(\Phi) \leq \frac{1}{2} |\Phi(0)|_a^2 + p \int_0^\pi \int_{-r}^0 \Phi^2(z, x)\, dz\, dx \leq \frac{1}{2} |\Phi(0)|_a^2 +$$

$$+ pr \sup_{z \in \langle 0, r\rangle} |\Phi(z)|_{L_2}^2 \leq \left(\frac{1}{2} + pr\pi^2\right) \| \Phi \|^2.$$

Hence $v(s) = \left(\frac{1}{2} + \pi^2 pr\right)$.

We see that $u(s) \to \infty$ if $s \to \infty$. Now we shall calculate $\dot{V}(t, \Phi)$ for Φ. Let ϕ_1 be a strong solution of the problem (E) with an initial condition $u_0 = \Phi$. Clearly $x_{t+h}[t, \Phi] = \Phi_{1,h}$.

Hence

$$\dot{V}(t, \Phi) = \limsup_{h \to 0+} \frac{V(t + h, \Phi_{1,h}) - V(t, \Phi_{1,0})}{h} =$$

$$\limsup_{h \to 0+} \frac{V(h, \Phi_{1,h}) - V(0, \Phi_{1,0})}{h} \leq \limsup_{t \to 0+} \frac{\mathrm{d}}{\mathrm{d}t} V(t, \Phi_{1,t}).$$

Now we shall find $\frac{\mathrm{d}}{\mathrm{d}t} V(t, \phi_{1,t})$. We have that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t, \Phi_{1,t}) = \int_0^\pi \Phi_{1,x} \Phi_{1,xt} + p\Phi_1^2(t) - p\Phi_1^2(t - r)\, \mathrm{d}x =$$

$$= \int_0^\pi - \Phi_{1,xx} \Phi_t + p\Phi_1^2(t) - p\Phi_1^2(t - r)\, \mathrm{d}x = - \int_0^\pi \Phi_{1,xx}(\Phi_{1,xx} + a\Phi_1(t) +$$

$$+ b\Phi_1(t - r))\, \mathrm{d}x + \int_0^\pi p\Phi_1^2(t) - p\Phi_1^2(t - r)\, \mathrm{d}x = \int_0^\pi \Phi_{1,xx}^2 - a\Phi_{1,x}^2 +$$

$$+ b\Phi_1(t - r)\, \Phi_{1,xx} - p\Phi_1^2(t) + p\Phi_1^2(t - r)\, \mathrm{d}x \leq \int_0^\pi \left(\frac{b\varepsilon^2}{2} - 1\right) \Phi_{1,xx}^2 +$$

$$+\left(\frac{b}{2\varepsilon^2}-p\right)\Phi_1^2(t-r)+p\left(\Phi_1^2(t)+\frac{a}{p}\Phi_{1,x}^2\right)\mathrm{d}x.$$

Clearly, if $\frac{a}{p\pi^2}<-1$ and $b\leq 2_p^{1/2}$, then there exists an $\varepsilon>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,\Phi_{1,t})\leq p\int_0^\pi \Phi_1^2(t)-\pi^2\Phi_{1,x}^2\,dx+$$

$$+p\int_0^\pi\left(\frac{a}{p}+\pi^2\right)\Phi_{1,x}^2\,dx\leq(a+p\pi^2)|\Phi_1(t)|_a^2.$$

Thus $\limsup_{t\to 0+}\frac{\mathrm{d}}{\mathrm{d}t}V(t,\Phi_{1,t})\leq(a+p\pi^2)|\Phi_1(0)|_a^2$, $\dot V(t,\Phi)\leq(a+p\pi^2)\,|\Phi(0)|_a^2$ and we can put $w(s)=-(a+p\pi^2)\,s^2$. Hence, if $b>0$ and $a<\frac{-\pi^2b^2}{4}$, then for each solution of our problem with an initial condition $\|\Phi\|\leq\sigma$ the statements (i), (ii) and (iii) take place.

REFERENCES

[1] HALE, J.: Theory of Functional Differential Equations. Applied Mathematical Sciences v. 3., Springer Verlag 1977.

[2] HENRY, D.: Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics No 840, Springer 1981

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ПАРАБОЛИЧЕСКИЕ УРАВНЕНИЯ С ОПОЗДЫВАНИЕМ

Ľubica Šedová

Резюме

Пусть A-секториальный оператор в пространстве Банаха X. В работе доказано локальное существование решения задачи

$$\frac{du}{dt}+Au=f(t,u_{\omega(t)})$$

$$u_0=h$$

Дальше исследована функция Ляпунова и устойчивость нулевого решения по Ляпунову.