Ágnes Szendrei

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Mathematica Slovaca, Vol. 46 (1996), No. 4, 391--403

Persistent URL: http://dml.cz/dmlcz/129412

Math. Slovaca, **46** (1996), No. 4, 391–403

Mathematica Slovaca

Dedicated to the memory of Professor Milan Kolibiar

NEARLY-IDEMPOTENT PLAIN ALGEBRAS ARE INDEED NEARLY IDEMPOTENT PLAIN ALGEBRAS

Ágnes Szendrei

(*Communicated by Tibor Katriňák*)

ABSTRACT. It is proved that for every nearly idempotent plain algebra **A** with at least two idempotent elements there exists an idempotent plain algebra **B** such that the varieties $\mathcal{V}(\mathbf{A})$ and $\mathcal{V}(\mathbf{B})$ are categorically equivalent; furthermore, except for some cases when **A** has three or four elements, the full idempotent reduct of **A** is plain. These facts lead also to a classification, up to term equivalence, of nearly idempotent plain algebras with at least two idempotent elements.

Introduction

An algebra **A** is called *plain* (or *strictly simple*) if **A** is finite, simple, and **A** has no nontrivial proper subalgebras. Recently K. A. Kearnes [3] has found a short, elementary proof for the result in [10] that plain idempotent algebras generate minimal varieties. In fact, the proof is given for a more general class of algebras that are called in [3] nearly idempotent plain algebras. By definition, a plain algebra **A** is *nearly idempotent* if **A** has at least one idempotent element, and the automorphism group of **A** acts transitively on the set of non-idempotent elements of **A**.

Our aim is to study how far nearly idempotent plain algebras are from idempotent plain algebras. If **A** is a nearly idempotent plain algebra with a single idempotent element 0, then clearly 0 is the only fixed point of each nonidentity

AMS Subject Classification (1991): Primary 08A40, 08A05.
Key words: plain algebra, idempotent, invertible unary term, categorical equivalence.

Research supported by a fellowship from the Alexander von Humboldt Stiftung and by the Hungarian National Foundation for Scientific Research Grant No. 17005.

automorphism of $\mathbf{A}$. Furthermore, since the automorphisms of $\mathbf{A}$ act transitively on $A \setminus \{0\}$, therefore every unary term operation of $\mathbf{A}$ is either the constant with value 0 or a permutation of A fixing 0. Hence $\mathbf{A}$ is one kind of so called term minimal algebras discussed in detail in [11].

In this paper, we restrict our attention to nearly idempotent plain algebras with at least two idempotent elements. We show that these algebras are indeed very close to idempotent plain algebras. Firstly, it turns out that for every nearly idempotent plain algebra $\mathbf{A}$ with at least two idempotent elements there exists an idempotent plain algebra $\mathbf{B}$ such that the varieties $\mathcal{V}(\mathbf{A})$ and $\mathcal{V}(\mathbf{B})$ generated by these algebras are categorically equivalent. Secondly, we prove that, except for some cases when $\mathbf{A}$ has three or four elements, the full idempotent reduct of $\mathbf{A}$ is plain; that is, up to term equivalence, $\mathbf{A}$ arises from a plain idempotent algebra by expanding it with some new operations.

Both of these results point out a reason for the fact that the property of generating minimal varieties carries over naturally from idempotent plain algebras to nearly idempotent plain algebras. This is obvious for the result on category equivalence. To see it for the other result, one would need to know

(•) to what extent the property of generating minimal varieties is hereditary for expansions among plain algebras.

It is well known (see, e.g., [2; Theorems 12.1, 12.4]) that if for a plain algebra $\mathbf{A}$ the variety $\mathcal{V}(\mathbf{A})$ is congruence modular, then $\mathcal{V}(\mathbf{A})$ is minimal unless $\mathbf{A}$ is an affine algebra having no idempotent elements. Since every idempotent plain algebra $\mathbf{A}$ with at least three elements generates a congruence modular variety (cf. [10]) – and this property is inherited by the expansions of $\mathbf{A}$ – it follows immediately that all nearly idempotent expansions of $\mathbf{A}$ will generate a minimal variety. We remark at this point that hereditariness in the sense of (•) holds true also under much weaker assumptions than congruence modularity, see [12].

Comparing nearly idempotent plain algebras to idempotent plain algebras

We follow the convention that algebras are denoted by boldface capitals and their base sets by the corresponding letters in italics.

Let $\mathbf{A}$ be an arbitrary algebra. By an *idempotent element* of $\mathbf{A}$, we mean an element $u \in A$ such that $\{u\}$ is a subuniverse of $\mathbf{A}$. The set of idempotent elements of $\mathbf{A}$ will be denoted by $U_{\mathbf{A}}$. We will say that $\mathbf{A}$ is an *idempotent algebra* if $U_{\mathbf{A}} = A$. By an *idempotent operation* on A, we mean an operation f on A such that $f(x, \dots, x) = x$ for all $x \in A$. Clearly, the algebra $\mathbf{A}$ is idempotent if and only if every fundamental operation (hence every term operation) of $\mathbf{A}$ is idempotent. The *full idempotent reduct* of $\mathbf{A}$ is defined to be the algebra with base set A whose operations are all idempotent term operations of $\mathbf{A}$.

Observe that by the definition above, a unary operation is idempotent if and only if it is the identity mapping, denoted id; this is the meaning involved in the concept of an idempotent algebra or full idempotent reduct of an algebra. However, among unary operations (that is, among transformations) a more customary meaning attached to the word "idempotent" – which we will adopt from now on – is the following: a unary operation f on A is called *idempotent* if $f^2(x) = f(x)$ holds for all $x \in A$.

For an algebra $\mathbf{A}$, $\operatorname{Clo}\mathbf{A}$ will denote the clone of term operations of $\mathbf{A}$; $\operatorname{Clo}_n\mathbf{A}$ the set of n-ary term operations of $\mathbf{A}$ $(n \geq 1)$; $\operatorname{Aut}\mathbf{A}$ the automorphism group of $\mathbf{A}$; and $\operatorname{Sub}\mathbf{A}$ the set of subuniverses of $\mathbf{A}$. One-element subuniverses [subalgebras] of $\mathbf{A}$ are usually referred to as *trivial* subuniverses [subalgebras]. Two algebras $\mathbf{C}$, $\mathbf{D}$ on the same base set $C = D$ are called *term equivalent* if $\operatorname{Clo}\mathbf{C} = \operatorname{Clo}\mathbf{D}$. We will find it convenient to extend this notion to algebras $\mathbf{C}$, $\mathbf{D}$ on arbitrary base sets as follows: $\mathbf{C}$, $\mathbf{D}$ will be called *term equivalent* – in symbols: $\mathbf{C} \equiv \mathbf{D}$ – if $\mathbf{C}$ is isomorphic to an algebra $\mathbf{C}'$ on D such that $\operatorname{Clo}\mathbf{C}' = \operatorname{Clo}\mathbf{D}$.

For an n-tuple x the coordinates of x will be denoted by $x^0, \ldots, x^{n-1}$. If C is a set, $V \subseteq C$, and g is an operation on C with $g(V, \ldots, V) \subseteq V$, then $g|_V$ stands for the restriction of g to V. Similarly, for a subset S of C^n $(n \geq 1)$, $S|_V$ is used as an abbreviation for $S \cap V^n$. The *diagonal* of $C \times C$ is the set $\Delta_C = \{(c, c) : c \in C\}$. For $S, T \subseteq C^2$ the relational product of S and T is the set

$$S \circ T = \{(a, c) \in C^2 : \text{ there exists } b \in C \text{ with } (a, b) \in S, \ (b, c) \in T\},$$

and the *converse* of S is

$$S^{\cup} = \{(b, a) \in C^2 : (a, b) \in S\}.$$

Let $\mathbf{C} = (C; F)$ be an arbitrary algebra. For a positive integer m the *mth matrix power* $\mathbf{C}^{[m]}$ of $\mathbf{C}$ is the algebra with base set C^m, whose operations are, for all k, all k-ary operations $g\colon (C^m)^k \to C^m$ of the form

$$g(x_0, \ldots, x_{k-1}) = (\ldots, \overbrace{g_i(x_0, \ldots, x_{k-1})}^{i\text{th}}, \ldots),$$

where $g_i \in \operatorname{Clo}_{mk}\mathbf{C}$ for all i $(0 \leq i \leq m-1)$.

Let $e \in \operatorname{Clo}_1\mathbf{C}$ be an idempotent unary term operation of $\mathbf{C}$. Then $e(\mathbf{C})$ is defined to be the algebra with base set $e(C) = \{x \in C : e(x) = x\}$, whose operations are all operations of the form $eg|_{e(C)}$ with $g \in \operatorname{Clo}\mathbf{C}$. Notice that the algebra $e(\mathbf{C})$ is essentially independent of e as soon as the range of e is fixed. That is, if $\bar{e} \in \mathrm{Clo}_1 \mathbf{C}$ is idempotent and $\bar{e}(C) = e(C)$, then $\bar{e}(\mathbf{C})$ and $e(\mathbf{C})$ have the same set of operations, namely the restrictions to $\bar{e}(C) = e(C)$ of all term operations of $\mathbf{C}$ whose range is contained in $\bar{e}(C) = e(C)$.

It is easy to see that $\mathbf{C}^{[m]}$ and $e(\mathbf{C})$ have no other term operations than their fundamental operations described in the preceding paragraphs. Since we are interested in algebras only up to term equivalence, we need not be very rigorous on the similarity types; the similarity types of $\mathbf{C}^{[m]}$ and $e(\mathbf{C})$ could be selected arbitrarily so that the sets of term operations are those described.

Following [6] we call $e \in \mathrm{Clo}_1 \mathbf{C}$ *invertible* if for some integer $m \geq 1$ there exist $t_0, \ldots, t_{m-1} \in \mathrm{Clo}_1 \mathbf{C}$ and $t \in \mathrm{Clo}_m \mathbf{C}$ such that

$$t\big(e(t_0(x)), \ldots, e(t_{m-1}(x))\big) = x \qquad \text{for all} \quad x \in C.$$

The importance of the constructions $\mathbf{C} \mapsto \mathbf{C}^{[m]}$ and $\mathbf{C} \mapsto e(\mathbf{C})$ for an invertible idempotent $e \in \mathrm{Clo}_1 \mathbf{C}$ lies in the fact that they yield category equivalences $\mathcal{V}(\mathbf{C}) \to \mathcal{V}(\mathbf{C}^{[m]})$ and $\mathcal{V}(\mathbf{C}) \to \mathcal{V}(e(\mathbf{C}))$ between the varieties generated by these algebras. Moreover, as R. McKenzie proved in [6], if for two algebras $\mathbf{C}$, $\mathbf{D}$ there is a category equivalence $\mathcal{V}(\mathbf{C}) \to \mathcal{V}(\mathbf{D})$ carrying $\mathbf{C}$ to $\mathbf{D}$, then $\mathbf{D} \equiv \varepsilon(\mathbf{C}^{[m]})$ for some $m \geq 1$ and some invertible idempotent $\varepsilon \in \mathrm{Clo}_1 \mathbf{C}^{[m]}$. The latter deep result will not be applied in this paper. What we will need are merely the natural bijections between the subuniverses of the squares of the algebras $\mathbf{C}$, $\mathbf{C}^{[m]}$ and $\mathbf{C}$, $e(\mathbf{C})$. It is straightforward to check (see also [13]) that

$$\mathrm{Sub}(\mathbf{C} \times \mathbf{C}) \to \mathrm{Sub}\big(\mathbf{C}^{[m]} \times \mathbf{C}^{[m]}\big), \qquad S \mapsto S^{[m]} \tag{†}$$

with

$$S^{[m]} = \big\{(x, y) \in C^m \times C^m : \ (x^i, y^i) \in S \ \text{for} \ i = 0, \ldots, m-1\big\}.$$

is a bijection. Suppose now that $e \in \mathrm{Clo}_1 \mathbf{C}$ is invertible and idempotent, and invertibility is witnessed by the term operations $t_0, \ldots, t_{m-1}$ and t as above. For every subuniverse S of $\mathbf{C} \times \mathbf{C}$, each pair $\sigma \in S$ can be canonically represented as

$$\sigma = t\big(et_0(\sigma), \ldots, et_{m-1}(\sigma)\big) \qquad \text{with} \quad et_0(\sigma), \ldots, et_{m-1}(\sigma) \in e(S) \ \ (\sigma \in S), \tag{••}$$

implying that S is generated by its subset $e(S)$ (throughout, the operations are applied coordinatewise). Furthermore, the definition of the operations of $e(\mathbf{C})$ combined with the idempotence of e yields that $e(S)$ is a subuniverse of $e(\mathbf{C}) \times e(\mathbf{C})$, and $e(S) = S|_{e(C)}$. Thus

$$\mathrm{Sub}(\mathbf{C} \times \mathbf{C}) \to \mathrm{Sub}\big(e(\mathbf{C}) \times e(\mathbf{C})\big), \qquad S \mapsto e(S) = S|_{e(C)}, \tag{‡}$$

is a bijection. In particular, it follows that for a subuniverse S of $\mathbf{C} \times \mathbf{C}$

$(\dagger)'$ S is a congruence of $\mathbf{C}$ if and only if $S^{[m]}$ is a congruence of $\mathbf{C}^{[m]}$,
$(\dagger)''$ S is an automorphism of $\mathbf{C}$ if and only if $S^{[m]}$ is an automorphism of $\mathbf{C}^{[m]}$;

and similarly,

$(\ddagger)'$ S is a congruence of $\mathbf{C}$ if and only if $e(S)$ is a congruence of $e(\mathbf{C})$,
$(\ddagger)''$ S is an automorphism of $\mathbf{C}$ if and only if $e(S)$ is an automorphism of $e(\mathbf{C})$.

The only nontrivial claims that are involved in these statements are that

in $(\ddagger)'$, $e(S)\circ e(S) \subseteq e(S)$ implies $S\circ S \subseteq S$, that is, if $e(S)$ is transitive, then so is S,

while

in $(\ddagger)''$, $e(S) \circ e(S^{\sqcup}) = e(S) \circ \big(e(S)\big)^{\sqcup} \subseteq \Delta_{e(C)}$ implies $S \circ S^{\sqcup} \subseteq \Delta_C$; and the same with the roles of S, $S^{\sqcup}$ interchanged.

Both claims follow easily if one applies the canonical representation (••) for the pairs in S.

Clearly, the algebra $\mathbf{C}$ can essentially be recovered from $\mathbf{C}^{[m]}$ as follows: if $\varepsilon_0 \in \mathrm{Clo}_1\,\mathbf{C}^{[m]}$ is defined by $\varepsilon_0(y) = (y^0,\dots,y^0)$, then ε_0 is idempotent and invertible, furthermore, $\mathbf{C} \equiv \varepsilon_0\big(\mathbf{C}^{[m]}\big)$. The following lemma – a variation of Remark 2 in [6; Section 2] – the proof of which is straightforward, shows how the construction $\mathbf{C} \mapsto e(\mathbf{C})$ for an invertible idempotent $e \in \mathrm{Clo}_1\,\mathbf{C}$ can be 'inverted' to recover $\mathbf{C}$ from $e(\mathbf{C})$.

Lemma 1. ([6]) *Let $\mathbf{C}$ be an arbitrary algebra with an invertible idempotent unary term operation e, and let $\mathbf{D} = e(\mathbf{C})$. If the invertibility of e is witnessed by $t_0,\dots,t_{m-1} \in \mathrm{Clo}_1\,\mathbf{C}$ and $t \in \mathrm{Clo}_m\,\mathbf{C}$, then $\varepsilon(y) = \big(et_0(t(y)),\dots$ $\dots,et_{m-1}(t(y))\big)$ is an invertible idempotent unary term operation of $\mathbf{D}^{[m]}$, and the mapping*

$$C \to \varepsilon(D^m)\,, \qquad c \mapsto \big(et_0(c),\dots,et_{m-1}(c)\big)\,,$$

is an isomorphism between $\mathbf{C}$ and an algebra term equivalent to $\varepsilon\big(\mathbf{D}^{[m]}\big)$.

One explanation why nearly idempotent plain algebras $\mathbf{A}$ with at least two idempotent elements behave so similarly to plain idempotent algebras is given in the next proposition, which implies that the variety $\mathcal{V}(\mathbf{A})$ is categorically equivalent to a variety generated by a plain idempotent algebra.

Proposition 2. *Let $\mathbf{A}$ be a nearly idempotent plain algebra with at least two idempotent elements.*

(1) *$\mathbf{A}$ has an invertible idempotent unary term operation e with $e(A) = U_{\mathbf{A}}$,*
(2) *$\mathbf{B} = e(\mathbf{A})$ is a plain idempotent algebra, and*
(3) *$\mathbf{A} \equiv \varepsilon\big(\mathbf{B}^{[2]}\big)$ for some invertible idempotent unary term operation ε of $\mathbf{B}^{[2]}$.*

P r o o f. For brevity we will write U instead of $U_{\mathbf{A}}$. If $\mathbf{A}$ is idempotent, then the claims of the proposition are obvious (with e the identity operation and $\varepsilon = \varepsilon_0$, where ε_0 is the operation described in the paragraph preceding Lemma 1). Therefore, we assume that $U \neq A$. Select arbitrary distinct elements $u_0, u_1 \in U$, and let $c \in A \setminus U$. The sets $\{c\}$ and $\{u_0, u_1\}$ generate $\mathbf{A}$, hence $\mathbf{A}$ has unary term operations e_i for $i = 0, 1$ and a binary term operation h such that $e_i(c) = u_i$ and $h(u_0, u_1) = c$. Thus the equality

$$h\big(e_0(x), e_1(x)\big) = x \tag{$*$}$$

holds for $x = c$. If $x \in U$, then the equalities $e_i(x) = x$ and $(*)$ are obvious. Finally, since $\operatorname{Aut}\mathbf{A}$ is transitive on $A \setminus U$, we conclude that $(*)$ holds for all $x \in A \setminus U$ and $e_i(A \setminus U) \subseteq U$ $(i = 0, 1)$. This implies that $e = e_0$ is idempotent, $e_0\big(e_1(x)\big) = e_1(x)$ for all $x \in A$, and e is invertible (as witnessed by e_0, e_1 and h). This completes the proof of (1).

The claim in (2) is now an easy consequence of the properties of the bijection in (‡) (cf. also (‡)′), and (3) follows immediately from Lemma 1. □

We note that the weaker versions of the claims (1)–(2) in Proposition 2, where the invertibility of e is not required, is true for any plain algebra with at least two idempotent elements (cf. [11], [3]).

Making use of Proposition 2 we now prove a structure theorem for nearly idempotent plain algebras. A 2-element algebra $\mathbf{B}$ will be called *orderable* if $\mathbf{B}$ has a nontrivial compatible order.

THEOREM 3. *If $\mathbf{A}$ is a nearly idempotent plain algebra with at least two idempotent elements, then exactly one of the following conditions holds for $\mathbf{A}$:*

- (a) *the full idempotent reduct of $\mathbf{A}$ is plain;*
- (b) $\mathbf{A} \equiv \mathbf{B}^{[2]}$ *for a two-element orderable idempotent algebra $\mathbf{B}$ with $|\operatorname{Aut}\mathbf{B}| = 2$;*
- (c) $\mathbf{A} \equiv \varepsilon\big(\mathbf{B}^{[2]}\big)$ *for a two-element orderable idempotent algebra $\mathbf{B}$ with $|\operatorname{Aut}\mathbf{B}| = 1$, and for an invertible idempotent unary term operation ε of $\mathbf{B}^{[2]}$ with three-element range containing the diagonal.*

P r o o f. Throughout the proof we write U instead of $U_{\mathbf{A}}$. If the algebra $\mathbf{A}$ is idempotent, then (a) obviously holds, so we assume that $A \setminus U \neq \emptyset$. We use the claims and the notation of Proposition 2, including the term operations $e_0 = e$, e_1 and h constructed in its proof to witness the invertibility of e.

By Lemma 1 and Proposition 2, the mapping $\varphi\colon A \to \varepsilon(B^2)$, $a \mapsto \big(e_0(a), e_1(a)\big)$, is an isomorphism between $\mathbf{A}$ and an algebra term equivalent to $\varepsilon(\mathbf{B}^{[2]})$. For any $u \in U$ we have $\varphi(u) = \big(e_0(u), e_1(u)\big) = (u, u)$, therefore the image of U under φ is the diagonal Δ_B of B^2. The term operation of $\varepsilon(\mathbf{B}^{[2]})$ corresponding to e under φ is

$$e' = \varphi \circ e \circ \varphi^{-1} : \varepsilon(B^2) \to \varepsilon(B^2),$$
$$\big(e_0(a), e_1(a)\big) = \varphi(a) \mapsto \varphi(e(a)) = \big(e_0(e_0(a)),\, e_1(e_0(a))\big) = \big(e_0(a), e_0(a)\big).$$

That is, e' is the term operation $\varepsilon_0(y) = (y^0, y^0)$ of $\mathbf{B}^{[2]}$ restricted to $\varepsilon(B^2)$. Clearly, $e'\big(\varepsilon(\mathbf{B}^{[2]})\big) \equiv \varepsilon_0\big(\mathbf{B}^{[2]}\big) \equiv \mathbf{B}$.

Later on, we will need the following claim, which is an easy consequence of the assertion in $(\ddagger)''$.

Claim 1. *Restriction to* $U = e(A) = B$ *yields an isomorphism* $|_U\colon \operatorname{Aut}\mathbf{A} \to \operatorname{Aut}\mathbf{B}$.

It is worth noting that the invertibility of e is not a crucial assumption in this claim, though the proof presented here makes use of it. In fact, as was observed in [4], Claim 1 is true for arbitrary plain algebra $\mathbf{A}$ with at least two idempotent elements and for any idempotent $e \in \operatorname{Clo}_1 \mathbf{A}$ with $e(A) = U_{\mathbf{A}}$, even if e is not invertible.

In order to prove Theorem 3, we consider first the case when $\mathbf{B}$ is not a two-element orderable algebra. Our aim is to show that the full idempotent reduct of $\mathbf{A}$ is plain. To this end, we need to investigate the subuniverses of $\mathbf{A} \times \mathbf{A}$. First we establish the required facts for the subuniverses of $\mathbf{B} \times \mathbf{B}$, and then, using Proposition 2, we lift the results to $\mathbf{A}$.

For a set C and for $c \in C$, $X^c(C)$ will denote the subset $\big(\{c\}\times C\big)\cup\big(C\times\{c\}\big)$ of $C \times C$.

Claim 2. (see [9])

(1) *Every proper subdirect subuniverse of* $\mathbf{B} \times \mathbf{B}$ *is either an automorphism of* $\mathbf{B}$ *or of the form* $X^0(B)$ *for some element* $0 \in B$.

(2) *If* $X^0(B)$ *is a subuniverse of* $\mathbf{B}$ *for some* $0 \in B$, *then* 0 *is a fixed point of each automorphism of* $\mathbf{B}$.

Proof. Let S be a proper subdirect subuniverse of $\mathbf{B} \times \mathbf{B}$, which is not an automorphism of $\mathbf{B}$. The sets ${}_uS = \big\{b \in B : (u,b) \in S\big\}$ and $S_u = \big\{b \in B : (b,u) \in S\big\}$ are nonempty subuniverses of $\mathbf{B}$ for all $u \in B$. Consequently, each of them is either a one-element set or equals B, as $\mathbf{B}$ is plain. By the assumptions on S, not all of them are one-element sets. Therefore a short analysis of the possible cases yields that there exist elements $0, 1 \in B$ such that $S = \big(\{0\} \times B\big) \cup \big(B \times \{1\}\big)$. If $0 = 1$, then we get that $S = X^0(B)$. Suppose therefore that $0 \neq 1$. Then $\big\{b \in B : (b,b) \in S\big\} = \{0,1\}$ is a subuniverse of $\mathbf{B}$, hence $B = \{0,1\}$ and S is the order $0 \leq 1$. This contradicts our assumption that $\mathbf{B}$ is not orderable, so the proof of (1) is complete.

To verify (2), assume $X^0(B)$ is a subuniverse of $\mathbf{B}$ and consider an automorphism π of $\mathbf{B}$. Then the relational product $T = X^0(B) \circ \pi$ is a subuniverse of $\mathbf{B} \times \mathbf{B}$, and $T = (\{0\} \times B) \cup (B \times \{\pi(0)\})$. Thus the claim proved in (1) implies that $\pi(0) = 0$. $\square$

Claim 3. *Every proper subdirect subuniverse of* $\mathbf{A} \times \mathbf{A}$ *is either an automorphism of* $\mathbf{A}$ *or of the form* $X^0(A)$ *for some element* $0 \in U$.

P r o o f. Let S be a proper subdirect subuniverse of $\mathbf{A} \times \mathbf{A}$ such that S is not an automorphism. Then $\varphi(S)$ is a proper subdirect subuniverse of $\varepsilon(\mathbf{B}^{[2]}) \times \varepsilon(\mathbf{B}^{[2]})$ which is not an automorphism. In view of the bijections described in (†) and (‡), $\mathbf{B} \times \mathbf{B}$ has a subuniverse T such that $\varphi(S) = T^{[2]}|_{\varepsilon(B^2)}$. It is easy to see that T must be a proper subdirect subuniverse of $\mathbf{B}$ which is not an automorphism. Thus by Claim 2, $T = X^0(B)$ for some element $0 \in B = e(A) = U$. Hence

$$\begin{aligned}\varphi(S) &= \left(X^0(B)\right)^{[2]}|_{\varepsilon(B^2)}\\ &= X^{(0,0)}(\varepsilon(B^2)) \cup (E_0 \times E_1) \cup (E_1 \times E_0),\end{aligned}$$

where $E_i = \{(b_0, b_1) \in \varepsilon(B^2) : \ b_i = 0, \ b_{1-i} \neq 0\}$ $(i = 0, 1)$. Applying φ^{-1} we get that

$$S = X^0(A) \cup (D_0 \times D_1) \cup (D_1 \times D_0) \qquad \text{with} \quad D_i = \varphi^{-1}(E_i) \ \ (i = 0, 1).$$

Here, the sets D_0, D_1 are disjoint from U, because $E_0 = \varphi(D_0)$, $E_1 = \varphi(D_1)$ are disjoint from $\Delta_B = \varphi(U)$.

We show that $D_0 \times D_1 = \emptyset$. Otherwise we would have a pair (c, d) in S such that $c, d \in A \setminus U$. Since $\operatorname{Aut} \mathbf{A}$ acts transitively on $A \setminus U$, there exists $\pi \in \operatorname{Aut} \mathbf{A}$ with $d = \pi(c)$. By Claim 1 and Claim 2 (2), we have $\pi(0) = 0$. It follows now that $V = \{a \in A : \ (a, \pi(a)) \in S\}$ is a subuniverse of $\mathbf{A}$ such that $c \in V$ and $V \cap U = \{0\}$. Thus V is a proper subuniverse of $\mathbf{A}$ containing a nonidempotent element of $\mathbf{A}$, contradicting the plainness of $\mathbf{A}$. This completes the proof of Claim 3. $\square$

Now we are in a position to prove that the full idempotent reduct of $\mathbf{A}$ is plain. Since each block of a congruence of an idempotent algebra is a subuniverse, it suffices to show that the full idempotent reduct of $\mathbf{A}$ has no nontrivial proper subalgebras.

Let a, b_0, b_1 be arbitrary elements of $\mathbf{A}$ such that $b_0 \neq b_1$. We are done if we show that $\mathbf{A}$ has an idempotent term operation g with $g(b_0, b_1) = a$. Let us select and fix an element $c \in A \setminus U$, and consider the subuniverse S of $\mathbf{A} \times \mathbf{A}$ generated by the pairs (b_i, c) $(i = 0, 1)$. Then S is a subdirect subuniverse, because $\{b_0, b_1\}$ and $\{c\}$ generate $\mathbf{A}$. Clearly, S is not an automorphism of $\mathbf{A}$

and is not of the form $X^0(A)$ for any $0 \in U$. Thus, by Claim 3, $S = A \times A$. Hence $\mathbf{A}$ has a binary term operation g such that $g\big((b_0, c), (b_1, c)\big) = (a, c)$, that is, $g(b_0, b_1) = a$ and $g(c, c) = c$. The transitivity of $\operatorname{Aut} \mathbf{A}$ on $A \setminus U$ implies that $g(x, x) = x$ for all $x \in A \setminus U$, while for the elements $x \in U$ this equality is trivial. Thus g is idempotent. This completes the proof that (a) holds whenever $\mathbf{B}$ is not a two-element orderable algebra.

Remark. Notice that – except for one step in the proof of Claim 3, where near idempotence is made use of – most of the argument above is based solely on the representation $\mathbf{A} \equiv \varepsilon\big(\mathbf{B}^{[2]}\big)$ of $\mathbf{A}$ from Proposition 2 and on the corresponding natural bijections (†), (‡). Nevertheless, the assumption that $\mathbf{A}$ is nearly idempotent cannot be omitted if we want to conclude that the full idempotent reduct of $\mathbf{A}$ is plain. It is easy to see, for instance, that if $\mathbf{A} \equiv \mathbf{B}^{[2]}$ for a plain idempotent algebra $\mathbf{B}$ with $X^0(B) \in \operatorname{Sub}(\mathbf{B}^2)$ (cf. Theorem 4), then the full idempotent reduct of $\mathbf{A}$ is not plain.

Returning to the proof of Theorem 3 we now consider the case when $\mathbf{B}$ is a two-element orderable algebra. By Claim 1, we have $|\operatorname{Aut} \mathbf{A}| = |\operatorname{Aut} \mathbf{B}|$. Since $\operatorname{Aut} \mathbf{A}$ acts transitively on $A \setminus U$, and no nonidentity automorphism has a fixed point in $A \setminus U$, therefore we get that $|\operatorname{Aut} \mathbf{A}| = |A \setminus U|$. In the case $|\operatorname{Aut} \mathbf{B}| = 2$, these considerations yield that $|A| = 4$, hence ε has to be the identity operation. Thus (b) follows immediately from Proposition 2. Similarly, if $|\operatorname{Aut} \mathbf{B}| = 1$, then $|A| = 3$, and (c) is an immediate consequence of Proposition 2 and the inclusion $\Delta_B \subseteq \varepsilon(B^2)$ established at the beginning of the proof.

In case (b), $\mathbf{A}$ has a compatible Boolean lattice order with bounds in U, while in case (c), $\mathbf{A}$ has a compatible total order with bounds in U. Therefore in both cases, the intervals of these lattices are subalgebras of the full idempotent reduct of $\mathbf{A}$, hence the full idempotent reduct of $\mathbf{A}$ is not plain. This implies that no two of conditions (a) – (c) can hold simultaneously for a nearly idempotent plain algebra. □

Nearly idempotent plain algebras, up to term equivalence

Plain idempotent algebras are known, up to term equivalence (see [9], and for the two-element case [7]). This description, combined with Theorem 3, allows one to determine up to term equivalence all nearly idempotent plain algebras with at least two idempotent elements. For comparison, we note that according to the results in [11], for each finite set A with $|A| \geq 3$ there are continuously many pairwise non-equivalent nearly idempotent plain algebras $\mathbf{A}$ with base set A such that $\mathbf{A}$ has a single idempotent element.

We introduce some notation. Let A be a finite set. For a set $U \subseteq A$ and for a permutation group G acting on A, let $\mathcal{R}_U(G)$ denote the clone of all operations f on A such that $f(u,\dots,u)=u$ for all $u \in U$ and f admits each member of G as an automorphism. If $G=\{\mathrm{id}\}$, then we write $\mathcal{R}_U$ instead of $\mathcal{R}_U(G)$.

For an element $0 \in A$ and for $n \geq 2$, let $\mathcal{F}_n^0$ denote the clone of all operations f on A preserving the relation

$$X_n^0(A)=\{(a_0,\dots,a_{n-1}) \in A^n : \ a_i=0 \text{ for at least one } i,\ 0 \leq i \leq n-1\}.$$

Furthermore, we put $\mathcal{F}_\omega^0 = \bigcap_{k=2}^{\infty} \mathcal{F}_k^0$. Clearly, $X_2^0(A)=X^0(A)$ is the subset of $A \times A$ that played an important role in the proof of Theorem 3.

For a partial order $\leq$ on A, $\mathcal{P}_\leq$ denotes the clone of all operations on A that are monotone with respect to $\leq$.

In addition, we will use the notation $\mathbf{2}$ for the two-element set $\{0,1\}$; $\wedge$ for the semilattice operation on $\mathbf{2}$ with absorbing element 0; $\bar{0}=(0,0)$, $\bar{1}=(1,1)$ and $a=(0,1)$ for the elements of $\mathbf{2}\times\mathbf{2}$; and $\mathbf{3}$ for the subset $\{\bar{0},a,\bar{1}\}$ of $\mathbf{2}\times\mathbf{2}$.

For the readers' convenience, we recall first the description of idempotent plain algebras.

THEOREM 4. ([9], [7], cf. [11]) *Up to term equivalence, the idempotent plain algebras are the following:*

(i) $(A;\mathcal{R}_A(G))$ *for a permutation group G on A such that every nonidentity permutation in G has at most one fixed point;*

(ii) $(A;\mathcal{R}_A(G)\cap\mathcal{F}_k^0)$ *for some k $(2 \leq k \leq \omega)$, for some element $0 \in A$, and some permutation group G on A such that 0 is the unique fixed point of each nonidentity permutation in G;*

(iii) $(A;\ x-y+z, \{rx+(1-r)y:\ r \in \mathrm{End}\,{}_K\widehat{A}\})$ *where ${}_K\widehat{A}$ is a finite vector space over a finite field K;*

(iv) $(\mathbf{2};\mathcal{R}_\mathbf{2}(G)\cap\mathcal{P}_\leq)$ *where G is a permutation group, and $\leq$ is the order $0 \leq 1$ on $\mathbf{2}$;*

(v) $(\mathbf{2};\mathcal{R}_\mathbf{2}\cap\mathcal{F}_k^0\cap\mathcal{P}_\leq)$ *for some k $(2 \leq k \leq \omega)$, where $\leq$ is the order $0 \leq 1$;*

(vi) $(\mathbf{2};\wedge)$;

(vii) $(\mathbf{2};\mathrm{id})$.

Now we are in a position to describe all nearly idempotent plain algebras $\mathbf{A}$ with $2 \leq |U_\mathbf{A}| < |A|$.

THEOREM 5. *Up to term equivalence, the nearly idempotent plain algebras that are not idempotent, but have at least two idempotent elements are the following:*

(a.i) $(A;\mathcal{R}_U(G))$ *for a proper subset U of A with $|U| \geq 2$ and for a permutation group G on A such that $A\setminus U$ is an orbit of G, every nonidentity* *permutation in* G *has at most one fixed point, and that fixed point is in* U;

(a.ii) $(A; \mathcal{R}_U(G) \cap \mathcal{F}_k^0)$ *for a proper subset* U *of* A *with* $|U| \geq 2$, *for some* k $(2 \leq k \leq \omega)$, *for some element* $0 \in U$, *and some permutation group* G *on* A *such that* $A \setminus U$ *is an orbit of* G, *and* 0 *is the unique fixed point of each nonidentity permutation in* G;

(a.iii) $(A; x-y+z, \{rx+(1-r)y : r \in \mathrm{End}_K \widehat{A}\}, e)$ *where* ${}_K\widehat{A}$ *is an at least two-dimensional finite vector space over a finite field* K, *and* e *is a projection onto a subspace of* ${}_K\widehat{A}$ *of codimension* 1;

(b.iv) $(\mathbf{2} \times \mathbf{2}; \mathcal{R}_{\{\bar{0},\bar{1}\}}(\{\mathrm{id},'\}) \cap \mathcal{P}_{\leq})$ *where* $\leq$ *is the Boolean lattice order on* $\mathbf{2} \times \mathbf{2}$, *and* $'$ *is complementation;*

(b.vii) $(\mathbf{2} \times \mathbf{2}; \circ)$ *with* $(x^0, x^1) \circ (y^0, y^1) = (x^1, y^0)$;

(c.iv) $(\mathbf{3}; \mathcal{R}_{\{\bar{0},\bar{1}\}} \cap \mathcal{P}_{\leq})$ *where* $\leq$ *is the order* $\bar{0} \leq a \leq \bar{1}$;

(c.v) $(\mathbf{3}; \mathcal{R}_{\{\bar{0},\bar{1}\}} \cap \mathcal{F}_k^{\bar{0}} \cap \mathcal{P}_{\leq})$ *for some* k $(2 \leq k \leq \omega)$, *where* $\leq$ *is the order* $\bar{0} \leq a \leq \bar{1}$;

(c.vi) $(\mathbf{3}; *, e_0, e_1)$ *where* $(x^0, x^1) * (y^0, y^1) = (x^0 \wedge y^1, y^1)$ *and* $e_i((x^0, x^1)) = (x^i, x^i)$ $(i = 0, 1)$.

P r o o f . It is straightforward to check that the algebras listed in the theorem are indeed nearly idempotent plain algebras which are not idempotent and have at least two idempotent elements. In fact, the sets of idempotent elements are the following: U in cases (a.i) – (a.ii), $e(A)$ in case (a.iii), and $\{\bar{0}, \bar{1}\}$ in the remaining cases; furthermore, the automorphism groups are the following: G in cases (a.i) – (a.ii),

$$\{\kappa x + a : \kappa \in K \setminus \{0\},\ a \in e(A)\} \qquad (**)$$

in case (a.iii), $\{\mathrm{id},'\}$ in cases (b.iv), (b.vii), and the one-element group in cases (c.iv) – (c.vi).

Conversely, let $\mathbf{A}$ be a nearly idempotent plain algebra with $2 \leq |U_{\mathbf{A}}| < |A|$, and, for simplicity, let us write U instead of $U_{\mathbf{A}}$. If Theorem 3 (b) or (c) holds for $\mathbf{A}$, then obviously $\mathbf{A}$ is uniquely determined (up to term equivalence) by the two-element orderable idempotent algebra $\mathbf{B}$. To see this for (c), recall that $\varepsilon(\mathbf{B}^{[2]})$ depends on the range of ε only, and the two possible three-element ranges containing the diagonal yield term equivalent algebras. Furthermore, we know from the proof of Theorem 3 and from Proposition 2 that $\mathbf{B} = e(\mathbf{A})$, where e is a unary term operation of $\mathbf{A}$ with range U.

It is easy to check that for the algebras $\mathbf{A}$ listed in (b.iv) – (c.vi) the corresponding algebras $\mathbf{B} = e(\mathbf{A})$ exhaust all two-element orderable idempotent algebras; namely, the algebra (iv) with $|G| = 2$ if $\mathbf{A}$ is of type (b.iv), (vii) if $\mathbf{A}$ is of type (b.vii), and (iv) with $|G| = 1$, (v), (vi), respectively, if $\mathbf{A}$ is of type (c.iv), (c.v), (c.vi). Thus the uniqueness established in the previous paragraph proves the claim of Theorem 5 for all nearly idempotent plain algebras satisfying condition (b) or (c) in Theorem 3.

Suppose now that condition (a) in Theorem 3 holds for $\mathbf{A}$, and let $\mathbf{A}_{\mathrm{id}}$ denote the full idempotent reduct of $\mathbf{A}$. Clearly, $|A| \geq 3$, so $\mathbf{A}_{\mathrm{id}}$ is term equivalent to one of the algebras (i)–(iii) in Theorem 4. In case (i), $\mathbf{A}_{\mathrm{id}}$, and hence $\mathbf{A}$, too, is quasiprimal. Since $\mathbf{A}$ is plain, it follows easily that $G = \operatorname{Aut} \mathbf{A}$ satisfies the conditions required in (a.i), and $\mathbf{A}$ is term equivalent to $(A; \mathcal{R}_{U}(G))$.

In case (iii), $\mathbf{A}_{\mathrm{id}}$ is a plain algebra generating a congruence permutable variety, therefore $\mathbf{A}$ also has these properties. By McKenzie's theorem [5] $\mathbf{A}$ is quasiprimal or affine. But $\mathbf{A}$ cannot be quasiprimal, as $\mathbf{A}_{\mathrm{id}}$ is not quasiprimal. Hence $\mathbf{A}$ is affine. From the description of plain affine algebras up to term equivalence (see, e.g., [1] or [8]) and from the assumption that $2 \leq |U| < |A|$, we conclude that $\mathbf{A}$ is term equivalent to an algebra of the form described in (a.iii) with ${}_K\widehat{A}$ an at least two-dimensional vector space over a finite field K and e a projection onto a nontrivial proper subspace of ${}_K\widehat{A}$. Since the automorphism group of this algebra is the group in $(**)$, one can easily see that the automorphism group acts transitively on $A \setminus U = A \setminus e(A)$ if and only if $e(A)$ has codimension 1.

Finally, consider the case when $\mathbf{A}_{\mathrm{id}}$ is (term equivalent to) the algebra in (ii). Clearly, $G = \operatorname{Aut} \mathbf{A}_{\mathrm{id}}$. Let us call a subuniverse S of $\mathbf{A}^n$ or of $\mathbf{A}_{\mathrm{id}}^n$ $(n \geq 1)$ *irredundant* if S is a subdirect subuniverse, and $S_{ij} = \{(x^i, x^j) \in A^2 : x \in S\}$ is not a permutation for any $0 \leq i < j \leq n-1$. Since $\mathbf{A}_{\mathrm{id}}$ is not quasiprimal, $\mathbf{A}$ is not quasiprimal, either. Hence, for some $m \geq 2$, $\mathbf{A}^m$ has an irredundant subuniverse T distinct from A^m. Obviously, all irredundant subuniverses of finite powers of $\mathbf{A}$ are among the irredundant subuniverses of finite powers of $\mathbf{A}_{\mathrm{id}}$. Thus the description of irredundant subuniverses of finite powers of $\mathbf{A}_{\mathrm{id}}$ (see [9; Proposition 2.3 and its application on p. 263]) yields that

— $\{x \in A : (x, \ldots, x) \in T\} = \{0\}$, whence $0 \in U$,

moreover,

— $\mathbf{A}$ is term equivalent to $(A; \mathcal{R}_U(G') \cap \mathcal{F}_l^0)$, where $G' = \operatorname{Aut} \mathbf{A}$, and $l = \omega$ if $X_n^0(A)$ is a subuniverse of $\mathbf{A}^n$ for all $n \geq 2$, while l is the largest such n otherwise. (Observe that if $X_n^0(A)$ is a subuniverse of $\mathbf{A}^n$, then $X_{n-1}^0(A) = \{x \in A^{n-1} : (x^0, x^0, x^1, \ldots, x^{n-1}) \in X_n^0(A)\}$ is a subuniverse of $\mathbf{A}^{n-1}$.)

Since $G' = \operatorname{Aut} \mathbf{A} \subseteq \operatorname{Aut} \mathbf{A}_{\mathrm{id}} = G$ and $\mathbf{A}$ is nearly idempotent, all other requirements in (a.ii) hold for G'. (It can be verified that, in fact, we must have $G = G'$ and $k = l$.) □

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Received August 7, 1995
Revised November 11, 1995

Bolyai Institute
József Attila University
Aradi vértanúk tere 1
H–6720 Szeged
HUNGARY