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THE ASYMPTOTIC PROPERTIES OF SOLUTIONS OF LINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. The paper deals with the asymptotic behaviour of solutions of the delay differential equation

$$\dot{x}(t) = a(t)x(t) + b(t)x(\tau(t)) + f(t), \qquad t \in [t_0, \infty),$$

where a is a positive function and the ratio $|b|/a$ can be estimated by means of a continuous and nonincreasing function. Some known asymptotic formulae concerning this equation are improved and extended to a more general case.

1. Introduction

This paper is concerned with the nonhomogeneous linear differential equation

$$\dot{x}(t) = a(t)x(t) + b(t)x(\tau(t)) + f(t), \qquad t \in I := [t_0, \infty), \tag{1}$$

where a, b, τ, f are continuous functions. Throughout this paper we assume that functions a and b are not identically zero on I.

The asymptotic behaviour of (1) has been analysed, under various assumptions, by many authors. From papers, related to our further investigation, we can mention those by F. V. Atkinson and J. Haddock [1], H. Beretoglu and M. Pituk [2], J. Diblík [6], T. Krisztin [8], [9] and others. Particularly, our aim is to generalize the asymptotic results derived by T. Kato and J. B. McLeod [7], E. B. Lim [10] and J. Čermák [3], [4], where special cases of (1) have been considered. Papers [7] and [10] discussed the asymptotic behaviour of

$$\dot{x}(t) = ax(t) + bx(\lambda t) + f(t), \qquad 0 < \lambda < 1, \ \ t \geq 0.$$

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It was shown that if $a > 0$, $b \neq 0$ and $f(t) = O\big(\exp\{a\tau(t)\}\big)$ as $t \to \infty$, then the solution x either grows exponentially or is polynomial. These results have been generalized in [4], [3] to equations

$$\dot{x}(t) = ax(t) + bx\big(\tau(t)\big) + f(t)\,, \qquad a > 0\,,\ \ b \neq 0\,,\ \ t \in I\,, \tag{2}$$

resp.

$$\dot{x}(t) = a(t)\big[x(t) - kx\big(\tau(t)\big)\big] + f(t)\,, \qquad k \neq 0\,,\ \ t \in I\,,$$

where $a \in C^0(I)$ is positive and nondecreasing on I and $\tau \in C^1(I)$, $\tau(t) < t$ for all $t \in I$, $\tau(t) \to \infty$ as $t \to \infty$, $0 < \dot{\tau} \leq \lambda < 1$ on I. It follows from the assumptions on τ that considered equations have an unbounded lag. Our wish is to derive the relevant asymptotic formulae also provided a is decreasing and we do not suppose explicitly that the lag is unbounded. Moreover, we show that in some cases these formulae can be given in a more precise form.

Let us denote $I_0 := \big[\tau(t_0), t_0\big]$. As it is customary, the function x is called a solution of (1) if $x \in C^0\big(I_0\big) \cap C^1(I)$ and satisfies (1) for all $t \geq t_0$. If we are given a continuous function x_0 defined on I_0, then there is a unique solution x of (1) such that $x \equiv x_0$ on I_0.

2. Preliminary statements

In this section we formulate some auxiliary results, which we utilize in the proof of the main result.

PROPOSITION 2.1. ([11; Corollary 1]) *Consider the functional differential equation*

$$\dot{x}(t) = g\big(t, x\big(\tau(t)\big)\big)\,, \qquad t \in I = [t_0, \infty)\,, \tag{3}$$

where $\tau \in C^0(I)$, $\tau(t) \leq t$ on I and $\inf\{\tau(t) :\ t \in I\} > -\infty$. Further, let $g(t,x)$ be a continuous function for which there exist a real C and a continuous function r fulfilling the relations

$$|g(t, x_1) - g(t, x_2)| \leq r(t)|x_1 - x_2| \tag{4}$$

and

$$|g(t, 0)| \leq Cr(t) \tag{5}$$

for any $t \in I$ and $x_1, x_2 \in \mathbb{R}$. If

$$\int_{t_0}^{\infty} r(t)\ \mathrm{d}t < \infty\,, \tag{6}$$

then for every solution x of (3) *exists a real L such that*

$$x(t) \to L \qquad \text{as} \quad t \to \infty \,.$$

Moreover, if

$$\int_{t_0}^{\infty} r(t)\ \mathrm{d}t < 1\,,$$

then for every $L \in \mathbb{R}$ there exists a solution x_L of (3) *defined on I such that*

$$x_L(t) \to L \qquad \text{as} \quad t \to \infty \,.$$

Lemma 2.2. *Consider the functional differential equation* (1), *where $a, b, \tau, f \in C^0(I)$, τ is subject to the same assumptions as in Proposition* 2.1, $f(t) = O\left(|b(t)| \exp\left\{ \int_{t_0}^{\tau(t)} a(s)\ \mathrm{d}s \right\}\right)$ *as $t \to \infty$ and the integral*

$$\int_{t_0}^{\infty} |b(t)| \exp\left\{ - \int_{\tau(t)}^{t} a(s)\ \mathrm{d}s \right\}\ \mathrm{d}t$$

converges. If x is a solution of (1), *then there exists a real constant L such that*

$$\exp\left\{ - \int_{t_0}^{t} a(s)\ \mathrm{d}s \right\} x(t) \to L \qquad \text{as} \quad t \to \infty \,. \tag{7}$$

Conversely, for every $L \in \mathbb{R}$ there exists solution x_L of (1) *such that*

$$\exp\left\{ - \int_{t_0}^{t} a(s)\ \mathrm{d}s \right\} x_L(t) \to L \qquad \text{as} \quad t \to \infty \,. \tag{8}$$

Proof. Setting

$$z(t) = \exp\left\{ - \int_{t_0}^{t} a(s)\ \mathrm{d}s \right\} x(t) \tag{9}$$

into the equation (1) we get

$$\dot{z}(t) = b(t) \exp\left\{ - \int_{\tau(t)}^{t} a(s)\ \mathrm{d}s \right\} z\bigl(\tau(t)\bigr) + f(t) \exp\left\{ - \int_{t_0}^{t} a(s)\ \mathrm{d}s \right\}. \tag{10}$$

Equation (10) is equation (3) with

$$g\big(t, z(\tau(t))\big) = b(t)z\big(\tau(t)\big)\exp\Big\{-\int\limits_{\tau(t)}^{t} a(s)\ \mathrm{d}s\Big\} + f(t)\exp\Big\{-\int\limits_{t_0}^{t} a(s)\ \mathrm{d}s\Big\}.$$

Further, put $r(t) = |b(t)|\exp\Big\{-\int\limits_{\tau(t)}^{t} a(s)\ \mathrm{d}s\Big\}$. Then it is easy to check that the relations (4), (5) and (6) are satisfied. Hence, using the backward substitution in (9) our assertion follows from Proposition 2.1. □

PROPOSITION 2.3. *Let $a \in C^0(I)$, $\tau \in C^1(I)$, $\tau(t) < t$, $\tau(t) \to \infty$ as $t \to \infty$ and let the relation*

$$0 < a\big(\tau(t)\big)\dot{\tau}(t) \le \delta a(t) \tag{11}$$

hold for a suitable real $0 < \delta < 1$ and all $t \in I$. Then

$$\lim_{t\to\infty} \int\limits_{\tau(t)}^{t} a(s)\ \mathrm{d}s = \infty\,.$$

P r o o f . First we show that $a(t) \ge m\dot{\rho}(t)/\big(\rho(t)\big)^{\alpha}$ for all $t \in I$, where $m > 0$ and $0 < \alpha < 1$ are suitable reals and $\rho \in C^1(I)$ is fulfilling the functional relation $\rho\big(\tau(t)\big) = \lambda\rho(t)$ with a suitable real $0 < \lambda < \delta$. Let $a(t) \ge m\dot{\rho}(t)/\big(\rho(t)\big)^{\alpha}$ for all $t \in \big[\tau(t_0), t_0\big]$ and

$$\alpha = \big(\log(\lambda/\delta)\big)/\log\lambda < 1\,.$$

Then for all $t \in \big[t_0, \tau^{-1}(t_0)\big]$ we get

$$a(t) \ge \frac{a\big(\tau(t)\big)\dot{\tau}(t)}{\delta} \ge m\frac{\dot{\rho}\big(\tau(t)\big)\dot{\tau}(t)}{\big(\rho(\tau(t))\big)^{\alpha}\delta} = m\frac{\lambda\dot{\rho}(t)}{\lambda^{\alpha}\big(\rho(t)\big)^{\alpha}\delta} = m\frac{\dot{\rho}(t)}{\big(\rho(t)\big)^{\alpha}}\,.$$

Repeating this procedure we get

$$a(t) \ge m\frac{\dot{\rho}(t)}{\big(\rho(t)\big)^{\alpha}} \qquad \text{for all} \quad t \in I\,.$$

Using this we have

$$\int\limits_{\tau(t)}^{t} a(s)\ \mathrm{d}s \ge m\int\limits_{\tau(t)}^{t} \frac{\dot{\rho}(s)}{\big(\rho(s)\big)^{\alpha}}\ \mathrm{d}s = \frac{1-\lambda^{1-\alpha}}{1-\alpha}\big(\rho(t)\big)^{1-\alpha}\,.$$

Since $\rho(t) \to \infty$ as $t \to \infty$ (this fact follows from assumptions on τ) and $\alpha < 1$, we have

$$\lim_{t\to\infty} \int\limits_{\tau(t)}^{t} a(s)\, \mathrm{d}s = \infty\,.$$

□

The next functional equation plays the key role in our further asymptotic investigations. Let us consider

$$\varphi(t) = c(t)\varphi\big(\tau(t)\big)\,, \qquad t \in I\,, \tag{12}$$

where c is a continuous function depending on a and b. The following simple statement ensures the existence of a solution of (12) with required properties.

PROPOSITION 2.4. *Consider the functional equation* (12) *where* $c, \tau \in C^0(I)$, $c(t) > 0$, $\tau(t) < t$ *for all* $t \in I$ *and* τ *is increasing on* I. *Let* $\varphi_0 \in C^0(I_0)$, *where* $I_0 := \big[\tau(t_0), t_0\big]$, *be a positive function satisfying*

$$\varphi_0(t_0) = c(t_0)\varphi\big(\tau(t_0)\big)\,.$$

Then there exists a unique positive solution $\varphi \in C^0(I)$ *of* (12) *such that* $\varphi(t) = \varphi_0(t)$ *for every* $t \in I_0$.

Proof. The statement can be proved by the step method. □

LEMMA 2.5. *Consider the functional differential equation*

$$\dot{x}(t) = a(t)x(t) + b(t)x\big(\tau(t)\big)\,, \qquad t \in I = [t_0, \infty)\,, \tag{13}$$

where $a, b \in C^0(I)$, $\tau \in C^1(I)$, $a(t) > 0$, $\tau(t) < t$ *for all* $t \in I$, $\tau(t) \to \infty$ *as* $t \to \infty$ *and let there exist a nonincreasing function* $c \in C^0(I)$ *such that* $|b(t)| \le c(t)a(t)$ *for all* $t \in I$. *Further assume that the relation* (11) *is valid for a suitable real* $0 < \delta < 1$ *and arbitrary* $t \in I$ *and let* $\varphi \in C^0(I)$ *be a positive solution of* (12). *If* x *is a solution of* (13) *fulfilling*

$$\exp\left\{-\int\limits_{t_0}^{t} a(s)\, \mathrm{d}s\right\} x(t) \to 0 \qquad \text{as} \quad t \to \infty\,, \tag{14}$$

then $x(t) = O\big(\varphi(t)\big)$ *as* $t \to \infty$.

Proof. First, let us denote

$$t_n := \tau^{-n}(t_0) \quad \text{and} \quad I_{n+1} := [t_n, t_{n+1}]\,, \qquad n = -1, 0, 1, \dots\,.$$

From (12) we get

$$\begin{aligned} \varphi(t) &= c(t)\varphi\big(\tau(t)\big) = c(t)c\big(\tau(t)\big)\varphi\big(\tau^2(t)\big) = \cdots \\ &= c(t)c\big(\tau(t)\big)c\big(\tau^2(t)\big)\cdots c\big(\tau^n(t)\big)\varphi_0\big(\tau^{n+1}(t)\big) \\ &\ge M_0 c(t)c\big(\tau(t)\big)\cdots c\big(\tau^n(t)\big) \qquad \text{for any} \quad t \in I_{n+1}\,, \end{aligned} \tag{15}$$

where $M_0 = \min_{t \in I_0} \varphi(t)$. The relation (15) will be used in the end of the proof.

Now let us multiply the equation (13) by the term $\exp\left\{-\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left\{-\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}x(t)\right] = b(t)\exp\left\{-\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}x\bigl(\tau(t)\bigr).$$

Since the relation (14) is assumed, integrating the previous equality over $[t, \infty)$ we obtain

$$-\exp\left\{-\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}x(t) = \int_{t}^{\infty} b(u)\exp\left\{-\int_{t_0}^{u} a(s)\,\mathrm{d}s\right\}x\bigl(\tau(u)\bigr)\,\mathrm{d}u.$$

Hence, the following inequality is valid for all $t \ge t_0$:

$$|x(t)| \le \exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\int_{t}^{\infty} |b(u)|\exp\left\{-\int_{t_0}^{u} a(s)\,\mathrm{d}s\right\}\bigl|x\bigl(\tau(u)\bigr)\bigr|\,\mathrm{d}u. \tag{16}$$

Now let M be a suitable positive real such that

$$|x(t)| \le M\exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\} \qquad \text{for all} \quad t \ge t_0.$$

Using (16) we derive the following estimate of $x(t)$ valid for all $t \ge t_1$:

$$\begin{aligned}
|x(t)| &\le \exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\int_{t}^{\infty} |b(u)|\exp\left\{-\int_{t_0}^{u} a(s)\,\mathrm{d}s\right\}\bigl|x\bigl(\tau(u)\bigr)\bigr|\,\mathrm{d}u \\
&\le \exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\int_{t}^{\infty} |b(u)|\exp\left\{-\int_{t_0}^{u} a(s)\,\mathrm{d}s\right\}M\exp\left\{\int_{t_0}^{\tau(t)} a(s)\,\mathrm{d}s\right\}\mathrm{d}u \\
&= M\exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\int_{t}^{\infty} c(u)a(u)\exp\left\{-\int_{\tau(u)}^{u} a(s)\,\mathrm{d}s\right\}\mathrm{d}u \\
&= M\exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\times \\
&\quad\times\int_{t}^{\infty}\frac{c(u)a(u)}{-a(u)+a\bigl(\tau(u)\bigr)\dot{\tau}(u)}\frac{\mathrm{d}}{\mathrm{d}u}\left(\exp\left\{-\int_{\tau(u)}^{u} a(s)\,\mathrm{d}s\right\}\right)\mathrm{d}u.
\end{aligned}$$

Using Proposition 2.3 and the fact that c is nonincreasing we get

$$\begin{aligned}|x(t)| &\le \frac{Mc(t)}{1-\delta}\exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\int_t^\infty -\frac{\mathrm{d}}{\mathrm{d}u}\left(\exp\left\{-\int_{\tau(u)}^{u} a(s)\,\mathrm{d}s\right\}\right)\mathrm{d}u\\ &= \frac{Mc(t)}{1-\delta}\exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\exp\left\{-\int_{\tau(t)}^{t} a(s)\,\mathrm{d}s\right\}.\end{aligned}$$

Hence

$$|x(t)| \le \frac{Mc(t)}{1-\delta}\exp\left\{\int_{t_0}^{\tau(t)} a(s)\,\mathrm{d}s\right\}, \qquad t \ge t_1\,. \tag{17}$$

Now we derive the analogous estimate for all $t \ge t_2$. Substituting the estimate (17) in (16) we get

$$\begin{aligned}|x(t)| &\le \exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\times\\ &\quad\times\int_t^\infty |b(u)|\exp\left\{-\int_{t_0}^{u} a(s)\,\mathrm{d}s\right\}\frac{Mc\big(\tau(u)\big)}{1-\delta}\exp\left\{\int_{t_0}^{\tau(\tau(u))} a(s)\,\mathrm{d}s\right\}\mathrm{d}u\\ &\le \frac{Mc\big(\tau(t)\big)}{1-\delta}\exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\int_t^\infty c(u)a(u)\exp\left\{-\int_{\tau^2(u)}^{u} a(s)\,\mathrm{d}s\right\}\mathrm{d}u\\ &\le \frac{Mc\big(\tau(t)\big)}{1-\delta}\exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\times\\ &\quad\times\int_t^\infty \frac{c(u)a(u)}{-a(u)+a\big(\tau^2(u)\big)\dot\tau\big(\tau(u)\big)\dot\tau(u)}\frac{\mathrm{d}}{\mathrm{d}u}\left(\exp\left\{-\int_{\tau^2(u)}^{u} a(s)\,\mathrm{d}s\right\}\right)\mathrm{d}u\\ &\le \frac{Mc\big(\tau(t)\big)c(t)}{(1-\delta)(1-\delta^2)}\exp\left\{\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\exp\left\{-\int_{\tau^2(t)}^{t} a(s)\,\mathrm{d}s\right\}\\ &= \frac{Mc\big(\tau(t)\big)c(t)}{(1-\delta)(1-\delta^2)}\exp\left\{\int_{t_0}^{\tau^2(t)} a(s)\,\mathrm{d}s\right\}.\end{aligned}$$

Using the same procedure as above we can derive the following estimate of $x(t)$ for all $t \geq t_n$, $n = 1, 2, \dots$:

$$|x(t)| \leq \frac{Mc(t)c(\tau(t)) \cdots c(\tau^{n-1}(t))}{(1-\delta)(1-\delta^2)\cdots(1-\delta^n)} \exp\left\{ \int_{t_0}^{\tau^n(t)} a(s)\, \mathrm{d}s \right\}.$$

Now, it is easy to see that for $n = 1, 2, 3 \dots$ it holds that

$$\exp\left\{ \int_{t_0}^{\tau^n} a(s)\, \mathrm{d}s \right\} \leq \exp\left\{ \int_{t_0}^{t_1} a(s)\, \mathrm{d}s \right\} \leq M_1 \qquad \text{for all} \quad t \leq t_{n+1},$$

where $M_1 > 0$ is a real constant. If we put

$$M^* := \frac{M_1 M}{\prod\limits_{j=1}^{n} (1-\delta^j)},$$

then we can write

$$|x(t)| \leq M^* c(t) c(\tau(t)) \cdots c(\tau^{n-1}(t)) \qquad \text{for all} \quad t \in I_n, \quad n = 1, 2, 3, \dots .$$

Now using the relation (15) we have

$$|x(t)| \leq \frac{M^*}{M_0 c(\tau^n(t))} \varphi(t) \qquad \text{for all} \quad t \in I_n, \quad n = 1, 2, 3, \dots .$$

Since c is nonincreasing, we get

$$|x(t)| \leq \frac{M^*}{M_0 c(t_1)} \varphi(t) \qquad \text{for all} \quad t \in I_n, \quad n = 1, 2, 3, \dots$$

and the lemma is proved. □

3. The main result and some consequences

Summarizing the preliminary results we can formulate the asymptotic description of all solutions of (1). In accordance with Lemma 2.2 we denote by x_L a particular solution of (1) possessing the property (8) for a given arbitrary real L.

THEOREM 3.1. *Consider equation* (1), *where* $a, b, f \in C^0(I)$, $\tau \in C^1(I)$, $a(t) > 0$, $\tau(t) < t$ *for all* $t \in I$, $\tau(t) \to \infty$ *as* $t \to \infty$,

$$f(t) = O\left(|b(t)| \exp\left\{ \int_{t_0}^{\tau(t)} a(s)\, \mathrm{d}s \right\} \right) \qquad \text{as} \quad t \to \infty$$

and let there exist a nonincreasing function $c \in C^0(I)$ such that $|b(t)| \le c(t)a(t)$ for all $t \in I$. Further assume that the relation (11) *is valid for a suitable real $0 < \delta < 1$ and arbitrary $t \in I$ and let $\varphi \in C^0(I)$ be a positive solution of* (12). *If x is a solution of* (1), *then there exists a real constant L such that*

$$x(t) = x_L(t) + O\big(\varphi(t)\big) \qquad \text{as} \qquad t \to \infty .$$

Proof. First we verify that all the assumptions of Lemma 2.2 are valid. Indeed,

$$\begin{aligned}
&\int_{t_0}^{\infty} |b(t)| \exp\left\{ - \int_{\tau(t)}^{t} a(s)\, \mathrm{d}s \right\} \mathrm{d}t \\
&\qquad \le c(t_0) \int_{t_0}^{\infty} a(t) \exp\left\{ - \int_{\tau(t)}^{t} a(s)\, \mathrm{d}s \right\} \mathrm{d}t \\
&\qquad = c(t_0) \int_{t_0}^{\infty} \frac{a(t)}{-a(t) + a\big(\tau(t)\big)\dot\tau(t)} \frac{\mathrm{d}}{\mathrm{d}t} \left[\exp\left\{ - \int_{\tau(t)}^{t} a(s)\, \mathrm{d}s \right\} \right] \mathrm{d}t \\
&\qquad \le c(t_0) \frac{1}{1-\delta} \exp\left\{ - \int_{\tau(t_0)}^{t_0} a(s)\, \mathrm{d}s \right\} < \infty
\end{aligned}$$

by use of Proposition 2.3.

Now let x be a solution of (1). In accordance with Lemma 2.2 there exists a constant L such that (7) holds. Then $y(t) = x(t) - x_L(t)$ is a solution of the homogeneous equation

$$\dot y(t) = a(t)y(t) + b(t)y\big(\tau(t)\big) , \qquad t \in I ,$$

fulfilling the relation

$$\exp\left\{ - \int_{t_0}^{t} a(s)\, \mathrm{d}s \right\} y(t) \to 0 \qquad \text{as} \quad t \to \infty .$$

Hence, using Lemma 2.5, we have $y(t) = O\big(\varphi(t)\big)$ as $t \to \infty$, i.e., $x(t) = x_L(t) + y(t) = x_L(t) + O\big(\varphi(t)\big)$ as $t \to \infty$. □

Remark 3.2. In a more general way, any solution φ^* of the inequality

$$\varphi^*(t) \ge c(t)\varphi^*\big(\tau(t)\big) , \qquad t \in I , \tag{18}$$

can be considered instead of a solution φ of equation (12) occurring in Theorem 3.1. Then the assertion of Theorem 3.1 can be proved using the same line of arguments as it has been done previously.

In the remaining part of this paper we show that Theorem 3.1 generalizes some known asymptotic results. In these particular cases we consider also delays intersecting the identity function at the initial points. It is easy to see that the result of Theorem 3.1 holds also for this case.

If we put $a(t) \equiv a > 0$, $b(t) \equiv b \neq 0$ and $\tau(t) = \lambda t$, $0 < \lambda < 1$, then we have:

COROLLARY 3.3. *Consider the equation*

$$\dot{x}(t) = ax(t) + bx(\lambda t) + f(t)\,, \qquad 0 < \lambda < 1\,, \quad t \geq 0\,, \tag{19}$$

where $a > 0$, $b \neq 0$ are constants, $f \in C^0\bigl([0,\infty)\bigr)$ and let $f(t) = O\bigl(\exp\{a\lambda t\}\bigr)$ as $t \to \infty$. Then for any $L \in \mathbb{R}$ there exists a solution x_L of (19) such that

$$\exp\{-at\}x_L(t) \to L \qquad \text{as} \quad t \to \infty\,.$$

Moreover, for any solution x of (19) there exists a suitable $L \in \mathbb{R}$ such that

$$x(t) = x_L(t) + O(t^{\alpha}) \qquad \text{as} \quad t \to \infty\,, \quad \alpha = \frac{\ln\bigl(a/|b|\bigr)}{\ln \lambda}\,.$$

P r o o f . The proof follows immediately from Theorem 3.1, where $c(t) \equiv c = |b|/a$ and $\varphi(t) = t^{\alpha}$, $\alpha = \frac{\ln(a/|b|)}{\ln \lambda}$. □

Remark 3.4. This statement was published in a weaker form in [10], resp. [7]. Its generalization to equation (2) with a more general form of a delay has been done in [4] and also in [5], where equation (1) with $b(t) = ka(t)$, $k \neq 0$, has been considered. We emphasize, that the key assumption in [3] is a nondecreasing. The following example illustrates that Theorem 3.1 works also for the case a decreasing.

EXAMPLE 3.5. Consider the equation

$$\dot{x}(t) = \frac{1}{t}\bigl[x(t) - x\bigl(\sqrt{t}\,\bigr)\bigr] + f(t)\,, \qquad t \geq 1\,, \tag{20}$$

where $f \in C^0\bigl([1,\infty)\bigr)$ and $f(t) = O\bigl(1/\sqrt{t}\,\bigr)$ as $t \to \infty$. It is easy to verify that all assumptions of Theorem 3.1 are fulfilled. Then for any $L \in \mathbb{R}$ there exists a solution x_L of (20) such that

$$x_L(t)/t \to L \qquad \text{as} \quad t \to \infty\,.$$

Moreover, for any solution x of (20) there exists $L \in \mathbb{R}$ such that

$$x(t) = x_L(t) + O(1) \qquad \text{as} \quad t \to \infty\,.$$

The following example demonstrates the case when solving of the corresponding functional equation (12) is not trivial.

EXAMPLE 3.6. Consider the equation

$$\dot{x}(t) = ax(t) + \frac{b}{t}x(\lambda t) + f(t)\,, \qquad 0 < \lambda < 1\,,\ \ t \ge 1\,, \tag{21}$$

where $a > 0$, $b \neq 0$ are reals, $f \in C^0([1,\infty))$ and let $f(t) = O\left(\frac{1}{t}\exp\{a\lambda t\}\right)$ as $t \to \infty$. Then in accordance with Theorem 3.1, for any $L \in \mathbb{R}$ there exists a solution x_L of (21) such that

$$\exp\{-at\}x_L(t) \to L \qquad \text{as} \quad t \to \infty\,.$$

Now it is easy to check that functional equation (12) has the form

$$\varphi(t) = \frac{|b|}{at}\varphi(\lambda t)$$

and admits the solution

$$\varphi(t) = t^{\frac{\log|b|/a}{\log\lambda^{-1}} - \frac{1}{2}\left(\frac{\log t}{\log\lambda^{-1}}+1\right)}\,,$$

which is the function tending to zero as $t \to \infty$. Hence, by Theorem 3.1, for any solution x of (21) there exists $L \in \mathbb{R}$ such that

$$x(t) = x_L(t) + O\Big(t^{\frac{\log|b|/a}{\log\lambda^{-1}} - \frac{1}{2}\left(\frac{\log t}{\log\lambda^{-1}}+1\right)}\Big) \qquad \text{as} \quad t \to \infty\,.$$

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