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ON THE SUM OF DIVISORS OF THE MERSENNE NUMBERS

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ABSTRACT. For any positive integer n, let $M_n = 2^n - 1$ be the nth Mersenne number. In this paper, we show that the set $\{\sigma(M_n)/M_n\}_n$ is dense in $[1, \infty]$, where for a positive integer k we use $\sigma(k)$ for the sum of divisors function of k.

For a positive integer $n \geq 1$, write $M_n := 2^n - 1$ for the *nth Mersenne number*. For a positive integer m, we write $\phi(m)$, $\sigma(m)$, $\tau(m)$, $\Omega(m)$, and $\omega(m)$ for the Euler function of m, the sum of divisors function of m, the number of divisors function of m, and the number of prime factors of m, counted with or without multiplicities, respectively. Throughout this paper, for a large positive real number x and any positive integer k, we write $\log_k x$ for the recursively defined function $\log_k x := \max\{\log\log_{k-1} x, 1\}$, where $\log x$ is the natural logarithm of x. We omit the subscript k when $k = 1$ and simply write $\log x$ with the understanding that this number is larger than or equal to 1. We also use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbols O and o with their usual meaning.

There are several papers in the literature dealing with arithmetic functions of the Mersenne numbers. Erdős (see [1]) showed that

$$\frac{\sigma(M_n)}{M_n} = O(\log_2 n). \tag{1}$$

Further results of the same type as estimate (1) above can be found in the papers [2] and [8]. Shparlinski (see [11]) showed that there exists a constant Γ so that for large x the asymptotic formula

$$\frac{1}{x} \cdot \sum_{n<x} \frac{\phi(M_n)}{M_n} = \Gamma + O\left(\frac{\log x}{x}\right) \tag{2}$$

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holds. In fact, the asymptotic formula (2) holds also when the sequence of numbers $(M_n)_{n\geq 0}$ is replaced by any *nondegenerate linearly recurrent sequence of positive integers*, and when the function $f(m) := \phi(m)/m$ is replaced by any arithmetic function $f(m)$ having the property that there exists some other function $g(m)$ with $g(m) = O(1)$ and so that

$$f(m) = \sum_{d|m} \frac{g(d)}{d} \tag{3}$$

holds (see [3]). The function $f(m) := \phi(m)/m$ satisfies the above condition with $g(m) := \mu(m)$, where μ is the Möbius function. Notice that the function $f(m) := \sigma(m)/m$ satisfies also this condition with $g(m) := 1$. In particular, a similar formula as (2) holds with ϕ replaced by σ. A more general result of this type can be found in [6], where it is shown that when $f(m)$ is a function satisfying the above conditions, then for every positive integer $k \geq 1$ there exists a constant Γ_k so that for large x we have

$$\frac{1}{x}\sum_{n<x}(f(M_n))^k = \Gamma_k + O_k\left(\frac{(\log x)^k}{x}\right). \tag{4}$$

In particular, all the moments of both $\phi(M_n)/M_n$ and $\sigma(M_n)/M_n$ can be computed. In [5], it is shown that if f is a multiplicative function that satisfies the above conditions, then $f(M_n)$ has a distribution function. That is, for every value of the real number z, the asymptotic density of the set of positive integers n so that $f(M_n) < z$ exists. In particular, both functions $\phi(M_n)/M_n$ and $\sigma(M_n)/M_n$ have distribution functions. Finally, some results pertaining to $\omega(M_n)$ and $\Omega(M_n)$ can be found in [9] and [10].

The above results seem to indicate that most of the interesting questions that can be answered for the function $\sigma(n)/n$ can also be answered when n is allowed to run only in the thinner set of all Mersenne numbers. That is, estimate (1) deals with the maximal order of such function, estimates such as (2) and (4) show that the moments of such functions can be computed, while the result from [5] shows that this function has a limiting distribution, and the same is true when σ is replaced by ϕ.

It is a well-known fact, and an easy exercise in elementary number theory, to prove that the set $\{\sigma(n)/n\}_n$ is dense in $[1,\infty]$. Similarly, the set $\{\phi(n)/n\}_n$ is dense in $[0,1]$. In this note, we show that the same is true when n is allowed to run only in the subset of Mersenne numbers. That is, we have the following theorem:

THEOREM.

i) *The set* $\{\sigma(M_n)/M_n\}_n$ *is dense in* $[1,\infty]$.
ii) *The set* $\{\phi(M_n)/M_n\}_n$ *is dense in* $[0,1]$.

We point out that the method of proof of our Theorem can be adapted to show that our Theorem is valid once one replaces the sequence of Mersenne numbers $(M_n)_n$ by some other *Lucas sequence* satisfying some technical conditions, such as the sequence $(F_n)_n$ of all the *Fibonacci numbers*. Our method of proof can also be adapted to show that the conclusion of our Theorem remains valid when the functions $\sigma(n)/n$ and $\phi(n)/n$ are replaced by any multiplicative function $f(n)$ such that there exist two constants $c \neq 0$ and $\lambda > 1$ so that $f(p^a) = 1 + \frac{c}{p} + O\big(\frac{1}{p^\lambda}\big)$ holds for all prime numbers p and all positive integers a. The two functions $\sigma(n)/n$ and $\phi(n)/n$ satisfy the above conditions with $(c, \lambda) = (1, 2)$ and $(-1, 2)$, respectively. For such functions, the method of proof of our Theorem yields that the set $\{f(M_n)\}_n$ is dense in the interval $\Big[\liminf\limits_n f(M_n), \limsup\limits_n f(M_n)\Big]$. An example of such a function $f(n)$ is the function $\alpha(n)/n$ studied in [4], where for a positive integer n the number $\alpha(n)$ is the average order of the elements in the cyclic group of order n (see [4; Lemma 1]). Thus, the set $\{\alpha(M_n)/M_n\}_n$ is dense in $[0, 1]$, which says that every number between zero and one can be approximated arbitrarily well by ratios of the average multiplicative order of elements in some finite field of characteristic two to the number of invertible elements in such field. We give no further details and proceed to the proof of our Theorem.

The proof of Theorem

Throughout this paper, we use $c_1, c_2, \dots$ for absolute constants which are computable. For positive integers $k < l$ with k and l coprime we write $\pi(x, k, l)$ for the number of prime numbers $p < x$ with $p \equiv k \pmod{l}$. We start with a couple of lemmas.

Lemma 1. *Let $\mathcal{P}$ be the set of all prime numbers p which fulfill the following conditions:*

1) $p \equiv 7 \pmod{8}$;
2) *the smallest prime factor of $(p-1)/2$ is larger than $\log_2 p$;*
3) *$p - 1$ is squarefree.*

For any positive real number x let $\mathcal{P}(x)$ be the cardinality of the set $\mathcal{P} \cap (1, x)$. Then, the estimate

$$\mathcal{P}(x) \gg \frac{x}{\log x \log_3 x} \tag{5}$$

holds for large values of the positive real number x.

Proof. We let x be large, set $y := \log_2 x$, and let $R := 8 \cdot \prod_{3 \le r \le y} r$, where the index r above is allowed to run only through odd prime numbers. For every odd prime number $r \le y$, we let a_r be some integer in the interval $[2, r-1]$. Then, a prime number $p < x$ is counted in $\mathcal{P}(x)$ if $p \equiv 7 \pmod 8$, $p-1$ is squarefree, and there exists a vector $\boldsymbol{a} := (a_r)_{3 \le r \le y}$ so that $p \equiv a_r \pmod r$ holds for all odd prime $r \le y$. Fix such a vector $\boldsymbol{a} := (a_r)_{3 \le r \le y}$. By the Chinese remainder lemma, if p is a prime so that $p - 1 \equiv 7 \pmod 8$ and $p \equiv a_r \pmod r$ holds for each odd prime $r \le y$, then there exists a number b depending on $\boldsymbol{a}$ so that $p \equiv b \pmod R$. Thus, with the fixed vector $\boldsymbol{a}$, the number of such primes is $\pi(x, b, R)$. The inequality

$$R = 8 \cdot \prod_{3 \le r \le y} r = \exp\big((1 + o(1))y\big) < \exp(2y) = \log^2 x \tag{6}$$

holds for large values of x, and now the Siegel-Walfitz theorem (see [12; p. 255]) tells us that there exists an absolute constant c_1 so that

$$\pi(x, b, R) = \frac{\pi(x)}{\phi(R)} + O\bigg(\frac{x}{\exp\big(c_1\sqrt{\log x}\,\big)}\bigg). \tag{7}$$

Since there are $\prod_{3 \le r \le y} (r-2)$ choices for the vector $\boldsymbol{a}$, it follows that up to x, there are at least

$$\frac{\pi(x)}{4} \cdot \prod_{3 \le r \le y} \Big(1 - \frac{1}{r-1}\Big) + O\bigg(\frac{Rx}{\exp\big(c_1\sqrt{\log x}\,\big)}\bigg) \tag{8}$$

prime numbers $p \equiv 7 \pmod 8$ so that the smallest prime factor of $(p-1)/2$ is at least as large as $\log_2 x > \log_2 p$. The main term in (8) is

$$\begin{aligned}
&\gg \frac{x}{\log x} \cdot \prod_{3 \le r \le y} \Big(1 - \frac{1}{r-1}\Big) \\
&= \frac{x}{\log x} \exp\bigg(\sum_{3 \le r \le y} \log\Big(1 - \frac{1}{r-1}\Big)\bigg) \\
&= \frac{x}{\log x} \cdot \exp\bigg(-\sum_{3 \le r \le y} \frac{1}{r-1} + O\bigg(\sum_{\substack{r \ge 3 \\ \beta \ge 2}} \frac{1}{r^\beta}\bigg)\bigg) \\
&= \frac{x}{\log x} \cdot \exp\big(-\log_2 y + O(1)\big) \\
&\gg \frac{x}{\log x \log y} = \frac{x}{\log x \log_3 x},
\end{aligned} \tag{9}$$

while the error term in (8) is

$$\begin{aligned} &\ll \frac{Rx}{\exp(c_1\sqrt{\log x})} < \frac{x\log^2 x}{\exp(c_1\sqrt{\log x})} = \frac{x}{\exp(c_1\sqrt{\log x} - 2\log_2 x)} \\ &= o\left(\frac{x}{\log x \log_3 x}\right). \end{aligned} \tag{10}$$

From (8), (9), and (10), it follows that there are prime numbers p

$$p \gg \frac{x}{\log x \log_3 x}, \tag{11}$$

$p < x$ with $p \equiv 7 \pmod 8$ so that the smallest prime factor of $(p-1)/2$ is at least y. Let us show now that most of these have the property that $p-1$ is squarefree. Clearly, $p-1$ is not a multiple of 4. If there exists a prime number q so that $q^2 \mid p-1$, it follows that $q > y$. Fix such a prime number q in the interval $(y, x^{1/2})$. Since $p \equiv 1 \pmod{q^2}$, it follows that there can be at most $\pi(x, 1, q^2)$ such prime numbers $p < x$. When $q < x^{1/4}$, we use the fact that

$$\pi(x,1,q^2) \le \frac{2x}{\phi(q^2)\log(x/q^2)} \tag{12}$$

(see [7]), while when $x^{1/4} \le q < x^{1/2}$ we use the trivial fact that

$$\pi(x,1,q^2) \le \left\lfloor \frac{x}{q^2} \right\rfloor + 1 \le \frac{2x}{q^2}. \tag{13}$$

From (12) and (13), we get that the number of prime numbers $p < x$ so that $p-1$ is divisible with the square of some prime number $q > y$ is

$$\ll \sum_{y<q<x^{1/4}} \frac{x}{\phi(q^2)\log(x/q^2)} + \sum_{q \ge x^{1/4}} \frac{x}{q^2} =: S_1 + S_2. \tag{14}$$

Since when $q < x^{1/4}$ we have $x/q^2 > x^{1/2}$, it follows that

$$\begin{aligned} S_1 = \sum_{y<q<x^{1/4}} \frac{x}{\phi(q^2)\log(x/q^2)} &\ll \frac{x}{\log x}\sum_{y<q}\frac{1}{q^2} \\ &\ll \frac{x}{y\log y\log x} = \frac{x}{\log x\log_2 x\log_3 x} \\ &= o\left(\frac{x}{\log x\log_3 x}\right), \end{aligned} \tag{15}$$

while

$$S_2 = \sum_{q\ge x^{1/4}} \frac{x}{q^2} \ll \frac{x}{x^{1/4}\log x} = \frac{x^{3/4}}{\log x} = o\left(\frac{x}{\log x\log_3 x}\right). \tag{16}$$

Comparing (14)–(16) with (11), we get the assertion of Lemma 1. □

LEMMA 2. *Let m be a squarefree number. Then,*

$$\sum_{d|m} \frac{\log d}{d} \ll \log_2^2 m\,. \tag{17}$$

P r o o f . Throughout this proof, we use the fact that both estimates $\sigma(n) \ll n\log_2 n$ and $\omega(n) \ll \log n$ hold. Since m is squarefree, it follows that the formula

$$\log d = \sum_{p|d} \log p \tag{18}$$

holds for all divisors d of m. Thus, we may use (18) in (17) and change the order of summation to get that

$$\begin{aligned}\sum_{d|m} \frac{\log d}{d} &= \sum_{d|m}\sum_{p|d} \frac{\log p}{d} = \sum_{p|m} \frac{\log p}{p} \sum_{d|m/p} \frac{1}{d} = \sum_{p|m} \frac{\log p}{p} \cdot \frac{\sigma(m/p)}{m/p} \\ &\ll \log_2 m \sum_{p|m} \frac{\log p}{p}\,.\end{aligned} \tag{19}$$

To estimate the remaining sum in (19), let $p_1 < p_2 < \dots$ be the increasing sequence of all prime numbers. Then, since the function $\log x/x$ is decreasing for $x \geq 3$, it follows that

$$\sum_{p|m} \frac{\log p}{p} \leq \sum_{i=1}^{\omega(m)+1} \frac{\log p_i}{p_i} \ll \log\bigl(\omega(m)+1\bigr) \ll \log_2 m\,, \tag{20}$$

and now the assertion of Lemma 2 follows from (19) and (20). □

P r o o f o f T h e o r e m . We shall prove only part i) of the theorem because the proof of part ii) is entirely similar.

We let x be a large positive real number. We write $z := \exp\Bigl((\log x)^{\log_3 x}\Bigr)$. Let P be the set of all prime numbers in $\mathcal{P} \cap (x,z)$, and for a number $t \in (x,z)$ we write $P(t)$ for the cardinality of the set $\mathcal{P} \cap (x,t)$. We first claim that the estimate

$$S := \sum_{p\in P} \frac{1}{p} > 2\log_4 x \tag{21}$$

holds for large values of x. Indeed, by partial integration, we have

$$S = \sum_{p\in P} \frac{1}{p} = \int_x^z \frac{\mathrm{d}P(t)}{t} = \frac{P(t)}{t}\Bigg|_{t=x}^{t=z} + \int_x^z \frac{P(t)\,\mathrm{d}t}{t^2}\,. \tag{22}$$

Clearly,

$$\frac{P(t)}{t} \le \frac{\pi(t)}{t} \ll \frac{1}{\log t},$$

therefore the first term in (22) above is $O(1/\log x)$. By Lemma 1, it follows that for large values of t, the inequality

$$P(t) > \frac{3t}{\log t \log_2 t}$$

holds. Thus, if x is large, then

$$\begin{aligned} S &> 3\int_x^z \frac{\mathrm{d}t}{t\log t \log_2 t} + o(1) = 3\log_3 t\Big|_{t=x}^{t=z} + o(1) \\ &= 3(\log_3 z - \log_3 x) + o(1) = 3\log_2\Big((\log x)^{\log_3 x}\Big) - 3\log_3 x + o(1) \\ &= 3\log\big(\log_3 x \log_2 x\big) - 3\log_3 x + o(1) = 3\log_4 x + o(1) > 2\log_4 x\,. \end{aligned} \tag{23}$$

For every odd positive integer m let $t(m)$ be the *order of apparition of* m *in the sequence* $(M_n)_n$. That is, $t(m)$ is the multiplicative order of 2 modulo m. Write $N := \prod_{p\in P} p$, and $T := t(N)$. We notice that if $p \in P$, then $t(p) \mid (p-1)/2$. Indeed, by Fermat's little theorem, we certainly have that $t(p) \mid p-1$. Thus, $p \mid 2^{p-1}-1$, therefore $p \mid \big(2^{(p-1)/2}-1\big)\big(2^{(p-1)/2}+1\big)$. It is now easy to see that p cannot divide the factor $2^{(p-1)/2}+1$, for if this were so, then, since $p+1$ is a multiple of 4, it would follow that $\big(2^{(p+1)/4}\big)^2 \equiv -2 \pmod p$, meaning that -2 is a quadratic residue modulo p, which is not possible because $p \equiv 7 \pmod 8$. Thus, $t(p) \mid (p-1)/2$, and therefore

$$T \mid \operatorname{lcm}\big[(p-1)/2,\, p \in P\big]\,.$$

In particular, we learn that T is squarefree and that its smallest prime factor is $> y := \log_2 x$. Write $T = q_1 \cdot q_2 \cdots q_l$, where $l = \omega(T)$, and $q_1 < q_2 < \cdots < q_l$ are prime numbers with $q_1 > y$.

We write $(n_i)_{i=1}^{l}$ for the finite sequence given by $n_i = \prod_{j\le i} q_j$. Clearly, $n_1 = q_1$, $n_{i+1} = n_i q_{i+1}$ holds for all $i = 1, 2, \ldots, l-1$, and $n_l = T$. Let

$$s_i := \frac{\sigma(M_{n_i})}{M_{n_i}} \qquad \text{for} \quad i = 1, \ldots, l\,. \tag{24}$$

Since $n_i \mid n_{i+1}$, it follows that $s_i < s_{i+1}$ holds for all $i = 1, \ldots, l-1$. We first give a lower bound on s_l. Since $n_l = T$, it follows that M_{n_l} is divisible by the number N. In particular,

$$s_l \geq \prod_{p\in P}\Big(1+\frac{1}{p}\Big) = \exp\Big(\sum_{p\in P}\log\Big(1+\frac{1}{p}\Big)\Big) = \exp\Bigg(\sum_{p\in P}\frac{1}{p} + O\Big(\sum_{p\in P}\frac{1}{p^2}\Big)\Bigg)$$
$$> \exp\big(2\log_4 x + o(1)\big) > \exp(\log_4 x) = \log_3 x\,, \tag{25}$$

with the last inequalities above following from (21).

We now show that there exists a constant c_2 so that the inequality

$$\frac{s_{i+1}}{s_i} < 1 + c_2\cdot\frac{\log_3^2 x}{\log_2 x} \tag{26}$$

holds for all $i = 0, 1, \ldots, l{-}1$, where we set $s_0 := 1$. Let us assume that we have proved (26) for the moment, and let us see how the combination of (25) with (26) proves part i) of the theorem. Choose any number $\alpha > 1$, and let $\varepsilon > 0$ be arbitrarily small. Choose x sufficiently large so that both inequalities

$$\log_3 x > \alpha + 1 \qquad \text{and} \qquad c_2\cdot\frac{\log_3^2 x}{\log_2 x} < \min\{\alpha{-}1,\ \varepsilon/\alpha\} \tag{27}$$

hold. Inequalities (25), (26) and (27) show that $s_1 < \alpha$ and $s_l > \alpha$. Thus, there must exist an index $k < l$ so that $s_k \leq \alpha$, but $s_{k+1} > \alpha$. However, by (26) and (27), we know that

$$\alpha < s_{k+1} < s_k\left(1 + c_2\cdot\frac{\log_3^2 x}{\log x}\right) < \alpha\Big(1+\frac{\varepsilon}{\alpha}\Big) = \alpha + \varepsilon\,.$$

Thus, the interval $(\alpha,\, \alpha{+}\varepsilon)$ contains a number of the form $\sigma(M_n)/M_n$ for some n. Since $\alpha > 1$ and $\varepsilon > 0$ were arbitrary, part i) of Theorem follows.

It remains to prove (26). For $i = 0, \ldots, l{-}1$, we write $M_i' = M_{n_{i+1}}/M_{n_i}$ with the convention that $n_0 = 1$ (and so, $M_{n_0} = 1$). Since the inequalities $\sigma(ab) \leq \sigma(a)\sigma(b)$ and $\sigma(c)/c \leq c/\phi(c)$ hold for all positive integers a, b and c, it follows that

$$s_{i+1} = \frac{\sigma(M_{n_{i+1}})}{M_{n_{i+1}}} \leq \frac{\sigma(M_{n_i})\sigma(M_i')}{M_{n_i}M_i'} = s_i\cdot\frac{\sigma(M_i')}{M_i'} < s_i\frac{M_i'}{\phi(M_i')}\,,$$

therefore

$$\frac{s_{i+1}}{s_i} < \frac{M_i'}{\phi(M_i')} = \prod_{p\mid M_i'}\Big(1+\frac{1}{p-1}\Big) < \exp\Big(\sum_{p\mid M_i'}\frac{1}{p-1}\Big)\,. \tag{28}$$

To estimate (28), we argue as follows. Let p be a prime divisor of M_i'. It follows that $p \mid 2^{n_i q_{i+1}} - 1$ but $p \nmid 2^{n_i} - 1$. It then follows that there exists a divisor d

of n_i so that $t(p) = q_{i+1}d$. Since p is a *primitive divisor* for $M_{t(p)}$, it follows, from the well-known properties of the primitive divisors of Lucas sequences, that $p \equiv 1 \pmod{q_{i+1}d}$. We now fix d, and let $j(d)$ be the number of primes p having $t(p) = dq_{i+1}$. Clearly,

$$2^{dq_{i+1}} > 2^{dq_{i+1}} - 1 \geq (2dq_{i+1} + 1)^{j(d)},$$

therefore we certainly have $j(d) < dq_{i+1}$. Thus,

$$\sum_{t(p)=dq_{i+1}} \frac{1}{p-1} \leq \frac{1}{dq_{i+1}} \sum_{k=1}^{j(d)} \frac{1}{k} \ll \frac{\log(dq_{i+1})}{dq_{i+1}}. \tag{29}$$

Hence,

$$\begin{aligned} \sum_{p \mid M_i'} \frac{1}{p-1} &\leq \sum_{d \mid n_i} \sum_{t(p)=dq_{i+1}} \frac{1}{p-1} \ll \sum_{d \mid n_i} \frac{\log dq_{i+1}}{dq_{i+1}} \\ &= \frac{\log q_{i+1}}{q_{i+1}} \cdot \sum_{d \mid n_i} \frac{1}{d} + \frac{1}{q_{i+1}} \cdot \sum_{d \mid n_i} \frac{\log d}{d} \\ &\ll \frac{\log q_{i+1}}{q_{i+1}} \cdot \frac{\sigma(n_i)}{n_i} + \frac{\log_2^2 n_i}{q_{i+1}} \\ &\ll \frac{1}{q_{i+1}} \cdot \bigl(\log q_{i+1} + \log_2 n_i\bigr) \log_2 n_i, \end{aligned} \tag{30}$$

where in the above estimates we used the fact that $\sigma(m)/m \ll \log_2 m$ together with Lemma 2. However, from the way we arranged our numbers, we have that $n_i = q_1 \cdots q_i$ can be at most the product of all the prime numbers up to q_{i+1} (in fact, it is smaller than this product because $q_1 > y$). Thus, for x large, we get that $\log n_i < \bigl(1 + o(1)\bigr) q_{i+1} < 2q_{i+1}$, therefore $\log_2 n_i \ll \log q_{i+1}$. And so, we have shown that

$$\sum_{p \mid M_i'} \frac{1}{p-1} \ll \frac{\log^2 q_{i+1}}{q_{i+1}}. \tag{31}$$

The function $\log^2 t/t$ is decreasing for large values of t, and since $q_{i+1} \geq q_1 > y$ holds for all $i = 0, \dots, l-1$, it follows that

$$\sum_{p \mid M_i'} \frac{1}{p-1} \ll \frac{\log^2 y}{y} = \frac{\log_3^2 x}{\log_2 x}. \tag{32}$$

Thus, with (28) and (32), we have

$$\frac{s_{i+1}}{s_i} < \exp\left(O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right) = 1 + O\left(\frac{\log_3^2 x}{\log x}\right),$$

which proves (26) and completes the proof of Theorem. □

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