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CONCRETE REPRESENTATION OF SOME EQUIVALENCE LATTICES

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1. Notation and introduction

The present paper generalizes some results of [5] and [4] which are given below in a slightly modified form (see 1.5—1.8). Two representation theorems are given for some distributive equivalence lattices with permutable elements.

The cardinality of a set X is denoted by card (X). Every ordinal number α is considered as the set of all ordinal numbers less than α, hence we can speak about card (α). The signs $\cap$, $\cup$, $\bigcap$, $\bigcup$ are used for set-theoretical operations, and the sings $\wedge$, $\vee$, $\bigwedge$, $\bigvee$ for lattice (or complete lattice) operations; $\wedge$, $\vee$ are also used as logical connectives. Throughout the whole paper A is a nonempty set. The ordered n-tuple $(a_1, \ldots, a_n) \in A^n$ is denoted by $\bar{a}_n$, and analogously for $\bar{x}_n$, $\bar{y}_n$, ... An n-ary (partial) function on the set A is a mapping of (a subset of) the set A^n into A; we always consider only the partial functions of finite arity. Dom (g) and Rng (g) denote the domain and the range of the partial function g, respectively. If f, g are partial functions on A and $f \subseteq g$, g is said to be an extension of f (and f a restriction of g). If, moreover, g is a function, g is called a completion of f.

The set of all equivalence relations on A is denoted by Eq (A); it is considered as a complete lattice with respect to $\subseteq$. Elements of Eq (A) are denoted by the Greek letters ϑ, η, ξ, ..., and we write $a\xi b$ instead of $(a, b) \in \xi$. Further we denote $\xi(x) = \{y \in A\,;\, y\xi x\}$ for all $\xi \in \mathrm{Eq}\,(A)$, $x \in A$. For $\bar{x}_n, \bar{y}_n \in A^n$ and $\xi \in \mathrm{Eq}\,(A)$ we write $\bar{x}_n \xi \bar{y}_n$ instead of $x_1 \xi y_1 \wedge \ldots \wedge x_n \xi y_n$.

Let $m \geqq \aleph_0$. An m-complete sublattice of Eq (A) is a (nonempty) subset L of Eq (A) such that $\bigwedge X \in L$, $\bigvee X \in L$ for all $X \subseteq L$, card $(X) < m$. Since $\emptyset \subseteq L$, we have $\mathrm{id}_A = \{(x, x)\,;\, x \in A\} \in L$ and $A \times A \in L$ for every m-complete sublattice of Eq (A).

1.1. Definition. *Let f be an n-ary partial function on A, $L \subseteq \mathrm{Eq}\,(A)$ and $\vartheta \in \mathrm{Eq}\,(A)$.*

1. *We shall say that f is compatible with the equivalence ϑ (or briefly: ϑ-compatible) if for all $\bar{x}_n, \bar{y}_n \in \mathrm{Dom}\,(f)$ such that $\bar{x}_n \vartheta \bar{y}_n$ there holds $f(\bar{x}_n)\vartheta f(\bar{y}_n)$.*

2. *We shall say that f is compatible with the set L (or briefly: L-compatible) if f is ϑ-compatible for all $\vartheta \in L$.*

The congruence lattice Con $(\mathscr{A})$ of an algebra $\mathscr{A} = (A\,;\ (f_i\,;\, i \in I))$ can be described as the greatest set $L \subseteq \mathrm{Eq}\,(A)$ such that all f_i, $i \in I$ are L compatible. In what follows we study mainly the case when $L \subseteq \mathrm{Eq}\,(A)$ is an m-arithmetical sublattice of Eq (A) (see 1.3) or an equivalence lattice of type 0 (see 1.4).

1.2. Definition. 1. *For every* $\vartheta \in \mathrm{Eq}\,(A)$ *we denote by* Part (ϑ) *the partition belonging to* ϑ, *i.e.* Part $(\vartheta) = \{\vartheta(x)\,;\, x \in A\}$, *where* $\vartheta(x) = \{y \in A\,;\, y\vartheta x\}$.

2. *For every* $L \subseteq \mathrm{Eq}\,(A)$ *we denote* Part $(L) = \bigcup\{\mathrm{Part}\,(\vartheta)\,;\ \vartheta \in L\}$.

1.3. Definition. *Let m be an infinite cardinal. A sublattice L of* Eq (A) *will be called m-arithmetical if*

1. *L is distributive and all its elements are pairwise permutable;*
2. *every chain* $B \subseteq \mathrm{Part}\,(L)$, card $(B) < m$, *has a nonempty intersection.*

For $m = \aleph_0$ the second condition is trivial, because the chains considered in it are finite, hence they have the least element. The same holds if Part (L) fulfils the descending chain condition, e.g. when L is finite.

1.4. Definition. *A sublattice L of Eq (A) is said to be of type 0 if* $\xi \vee \eta = \xi \cup \eta$ *for all* $\xi, \eta \in L$.

This definition is to a certain extent analogous to the definition of a representation by equivalence relations of type n in [1]. (The exact analogy would be $\xi \vee \eta = \xi$ instead of $\xi \vee \eta = \xi \cup \eta$; it is useless, because it implies card $(L) = 1$.) The lattices $L \subseteq \mathrm{Eq}\,(A)$ of type 0 will be considered in the fourth section of the present paper.

Using Definition 1.3 we can formulate Lemma 3.1 and Theorem 3.2 of [5] as follows.

1.5. Lemma. *Let L be a finite $\aleph_0$-arithmetical sublattice of* Eq (A). *Then there is a ternary L-compatible function f satisfying*

$$f(x, x, z) = f(z, x, x) = f(z, x, z) = z \tag{1.5.1}$$

for all $x, z \in A$.

1.6. Theorem. *Let L be a finite complete sublattice of* Eq (A). *Then the following conditions are equivalent:*

(i) *L is $\aleph_0$-artithmetical;*

(ii) *there is a ternary L-compatible function f satisfying* (1.5.1) *for all* $x, z \in A$.

In the proof of (ii)$\rightarrow$(i) the finiteness of L was not used; it was used only in the proof of (i)$\rightarrow$(ii), i.e. essentially in the proof of Lemma 1.5. A. F. Pixley stated the problem whether the finiteness of L in Lemma 1.5 can be omitted. We shall not omit this condition, but we shall replace it by a weaker one, namely the card (A)-arithmeticity. In [4] this condition was replaced as follows.

1.7. Theorem. *Let A be a countable set and L be an $\aleph_0$-arithmetical sublattice of* Eq (A). *Then there is a ternary L-compatible function f which satisfies* (1.5.1) *for all* $x, z \in A$.

1.8. Theorem. *Let A be a countable set and let L be a complete sublattice of* Eq (A). *Then the following conditions are equivalent:*

(i) *L is $\aleph_0$-arithmetical;*

(ii) *there is a ternary L-compatible function f which satisfies* (1.5.1) *for all* $x, z \in A$;

(iii) *L is the congruence lattice of an algebra, among the fundamental operations of which there is a ternary function f satisfying* (1.5.1) *for all* $x, z \in A$.

We shall not repeat the proof of the above theorems. They will follow from the results below.

2. Infinite Chinese remainder theorem

In this section we shall generalize the following Chinese remainder theorem [5, Lemma 2.1; 2, Exercise 68, page 211].

2.1. Theorem *For every sublattice L of* Eq (A) *such that* $\mathrm{id}_A, A \times A \in L$ *the following conditions are equivalent:*

1. *L is $\aleph_0$-arithmetical;*
2. *for every finite sequence $\vartheta_1, \ldots, \vartheta_n$ of elements of L and every finite sequence $x_1, \ldots, x_n$ of elements of A satisfying*

$$x_i(\vartheta_i \vee \vartheta_j)x_j \quad \text{for all } i, j \in \{1, \ldots, n\} \tag{2.1.1}$$

there is an $x \in A$ which satisfies

$$x\vartheta_i x_i \quad \text{for all } i = 1, \ldots, n. \tag{2.1.2}$$

To generalize Theorem 2.1 we shall need the following lemma.

2.2. Lemma. *Let L be an m-arithmetical sublattice of* Eq (A) *and let τ be an ordinal number,* card $\tau < m$, *Let $(\vartheta_\alpha\,; 0 \leqq \alpha < \tau)$ be such a transfinite sequence of elements of L, that*

$$\bigcap\{\vartheta_\beta\,; \beta < \gamma\} \in L \quad \text{for all } \gamma \leqq \tau \tag{2.2.1}$$

and let $(a_\alpha\,; 0 \leqq \alpha < \tau)$ be a transfinite sequence of elements of A. Then the following conditions are equivalent:

$$a_\alpha \vartheta_\alpha \vartheta_\beta a_\beta \quad \text{for all } \alpha, \beta, 0 \leqq \alpha, \beta < \tau \tag{2.2.2}$$

(2.2.3) *there is an $x \in A$ such that $x\vartheta_\alpha a_\alpha$ for all α, $0 \leqq \alpha < \tau$.*

Proof. Since (2.2.3) $\rightarrow$ (2.2.2) is obvious, we shall prove only the direct implication; let (2.2.2) hold. For every $\beta<\tau$ denote $B_\beta=\vartheta_\beta(a_\beta)$.

Then (2.2.2), (2.2.3) can be formulated as

(2.2.4) $$B_\alpha\cap B_\beta\neq\emptyset \quad \text{for all } \alpha,\beta, 0\leqq\alpha,\beta<\tau$$

(2.2.5) $$\bigcap\{B_\beta\,;\beta<\tau\}\neq\emptyset,$$

respectively; in (2.2.3) x can be an arbitrary element of the left-hand side of (2.2.5). Assume now that (2.2.5) does not hold. Consider the set Y of all ordinal numbers $\alpha\leqq\tau$ with the following property:

there is a finite set $M=\{\alpha_1, \ldots, \alpha_n\}$ *of ordinal numbers less than* τ *such that*

(2.2.6) $$\bigcap\{B_\beta\,;\beta<\alpha\vee\beta\in M\}=\emptyset.$$

The set Y is nonempty because it contains the ordinal τ. Let α be the least element of the set Y. The ordinal number α must be zero or a limit ordinal. For $\alpha=0$ we have $\bigcap\{B_\beta\,;\beta\in M\}=\emptyset$, which contradicts Theorem 2.1 (used for $n=\text{card}\,(M)$, $x_i=a_{\alpha_i}$, $\vartheta_i=\vartheta_{\alpha_i}$). Now let α be a limit number. For every $\gamma\leqq\alpha$ denote

$$A_\gamma=\bigcap\{B_\beta\,;\beta<\gamma\vee\beta\in M\}.$$

Then (2.2.6) obviously implies $\bigcap\{A_\gamma\,;\gamma<\alpha\}=\emptyset$. However, every set A_γ, $\gamma<\alpha$ is nonempty, hence it is a class of the equivalence relation

$$\bigcap\{\vartheta_\beta\,;\beta<\gamma\vee\beta\in M\}=\bigcap\{\vartheta_\beta\,;\beta<\gamma\}\cap\bigcap\{\vartheta_\beta\,;\beta\in M\},$$

which belongs to L by the assumption of Lemma 2.2. The set $\{A_\gamma\,;\gamma<\alpha\}\subseteq$ Part (L) is obviously a chain of cardinality less than m, hence $\bigcap\{A_\gamma\,;\gamma<\alpha\}\neq\emptyset$, which is a contradiction.

2.3. Theorem. (Infinite Chinese remainder theorem.) *For every m-complete sublattice L of* Eq (A) *the following conditions are equivalent:*

1) *L is m-arithmetical;*

2) *for every set I,* card $(I)<m$ *and every two systems* $(x_i\,;i\in I)$, $(\vartheta_i\,;i\in I)$ of elements of A, resp. L, satisfying

(2.3.1) $$x_i(\vartheta_i\vee\vartheta_j)x_k \quad \text{for all } i,j\in I$$

there is an $x\in A$ *such that*

(2.3.2) $$x\vartheta_i x_i \quad \text{for all } i\in I.$$

Proof. The implication 1)$\rightarrow$2) follows from Lemma 2.2. Let now 2) hold; then Theorem 2.1 implies that L is $\aleph_0$-arithmetical. It remains to show that every chain $B\subseteq$ Part (L), card $(B)<m$ has a nonempty intersection. Let $B=\{\vartheta_i(x_i)\,;i\in I\}$ for some I, card (I) = card $(B)<m$. The systems $(x_i\,;i\in I)$, $(\vartheta_i\,;i\in I)$ fulfil the

condition (2.3.1) and hence also (2.3.2). The element x from (2.3.2) belongs to $\bigcap B$, hence $\bigcap B \neq \emptyset$, q. e. d.

3. Representation of m-arithmetical lattices

In this section we shall generalize Theorems 1.7 and 1.8 in such a way that the cardinal $\aleph_0$ will be replaced by an infinite cardinal m.

3.1. Definition. 1) *For every subset L of* Eq (A) *and arbitrary B, $C \subseteq A^n$ we denote*

$$L(B, C) = \{\vartheta \in L\,;\ (\exists \bar{x}_n \in B)(\exists \bar{y}_n \in C)(\bar{x}_n \vartheta \bar{x}_n)\}.$$

2) *We shall write $L(x_n, C)$ instead of $L(\{\bar{x}_n\}, C)$ and $L(B, \bar{y}_n)$ instead of $L(B, \{\bar{y}_n\})$.*

3.2. Definition. 1) *Let m be a cardinal and $L \subseteq$* Eq (A). *A set $B \subseteq A^n$ is said to be (m, L)-determining if for every $\bar{x}_n \in A^n$ there is a set $C \subseteq B$,* card $(C) < m$ *such that $L(\bar{x}_n, B) = L(\bar{x}_n, C)$.*

2) *Instead of "$(\aleph_0, L)$-determining" we shall write simply L-determining.*

If $B \subseteq A^n$ is L-determining, it is obviously (m, L)-determining for every infinite cardinal m. The set $B^* = \{\vartheta(\bar{x}_n);\ \in L(\bar{x}_n, B)\} \subseteq$ Part (L) determines in some sense (not necessarily uniquely) the n-tuple $\bar{x}_n$. If B is L-determining, then there is a finite subset $C \subseteq B$ such that the set $C^* = \{\vartheta(\bar{x}_n);\ \vartheta \in L(\bar{x}_n, C)\}$ is equal to B^*, hence it determines the n-tuple $\bar{x}_n$ "as well as" the set B^* does. There are similar reasons for the term "(m, L)-determining set".

3.3. Lemma. *Let $L \subseteq$* Eq (A) *and m be an infinite cardinal. Then*

1) *every set $B \subseteq A^n$,* card $(B) < m$ *is (m, L)-determining;*

2) *if B_1, $B_2 \subseteq A^n$ are (m, L)-determining, then the set $B_1 \cup B_2$ is also (m, L)-determining.*

The proof is obvious; notice that the second statement can be generalized to the union of systems of cardinality less than m.

3.4. Lemma. *Let $L \subseteq$* Eq (A) *and n, k be natural numbers. Then the set*

$$A_{n,k} = \{(x_1, \ldots, x_n) \in A^n\,;\ \text{card}\,(\{x_1, \ldots, x_n\}) < k\}$$

is an L-determining subset of A^n.

Proof. Let $\bar{x}_n \in A^n$. The set $B = A_{n,k} \cap \{x_1, \ldots, x_n\}^n$ is finite. We shall show that $L(x_n, A_{n,k}) = L(\bar{x}_n, B)$; it suffices to show $\subseteq$. Let $\bar{z}_n \in A_{n,k}$, $\vartheta \in L$ and $\bar{x}_n \vartheta \bar{z}_n$. For every $i \in \{1, \ldots, n\}$ let $\mathrm{r}(i)$ be the least integer satisfying $z_{\mathrm{r}(i)} = z_i$. For all $i = 1, \ldots, n$ we have $x_{\mathrm{r}(i)} \vartheta z_{\mathrm{r}(i)} = z_i \vartheta x_i$, hence $(x_{\mathrm{r}(1)}, \ldots, x_{\mathrm{r}(n)} \vartheta \bar{x}_n$. Further, card $(\{\mathrm{r}(1), \ldots, \mathrm{r}(n)\} < k$ and thus $(x_{\mathrm{r}(1)}, \ldots, x_{\mathrm{r}(n)}) \in B$. Therefore $\vartheta \in L(\bar{x}_n, B)$, q. e. d.

In fact we have proved that the set $A_{n,k}$ is (n^n+1, L)-determining; Lemma 3.4 also holds for $k>n$ (then $A_{n,k} = A^n$) and $k=1$ (then $A_{n,k}$ $\emptyset$).

3.5. Extension lemma. *Let L be an m-arithmetical complete sublattice of* Eq (A), *g be an n-ary L-compatible partial function and* Dom (g) *be an (m, L)-determining set. Then to every $\bar{x}_n \in A^n$ there is such a $y \in A$ that the partial function $g \cup \{(\bar{x}_n, y)\}$ is L-compatible.*

Proof. If $\bar{x} \in \mathrm{Dom}(g)$, it suffices to take $y = g(\bar{x}_n)$, assume $\bar{x}_n \notin \mathrm{Dom}(g)$. Let $B = \{t_i ; i \in I\} \subseteq \mathrm{Dom}(g)$, card $(I)<m$ and $L(\mathrm{Dom}(g), \bar{x}_n) = L(B, \bar{x}_n)$. For every $t_i \in B$ denote by ϑ_i the least element of L satisfying $t_i \vartheta \bar{x}_n$. Then we have $t_i \vartheta_i \vartheta_j t_j$ for all $i, j \in I$ and since g is L-compatible we also have $g(t_i) \vartheta_i \vartheta_j g(t_j)$ for all $i, j \in I$. Hence by the Infinite Chinese remainder theorem 2.3 there is a $y \in A$ such that $y \vartheta_i g(t_i)$ for all $i \in I$. We shall prove that $g \cup \{(\bar{x}_n, y)\}$ is L-compatible. It suffices to show $g(\bar{z}_n) \vartheta_y$ for all $\vartheta \in L$, $\bar{z}_n \in \mathrm{Dom}(g)$, $\bar{z}_n \vartheta \bar{x}_n$. If $\bar{z}_n \vartheta \bar{x}_n$, then $\vartheta \in L(\mathrm{Dom}(g), x_n) = L(B, \bar{x}_n)$. Therefore there is an $i \in I$ such that $t_i \vartheta \bar{x}_n$; then $\vartheta_i \leqq \vartheta$. By the choice of y we have $y \vartheta_i g(t_i)$ and hence $y \vartheta g(t_i)$. On the other hand $\bar{z}_n \vartheta \bar{x}_n \vartheta t_i$, hence $g(\bar{z}_n) \vartheta g(t_i)$. Together we have $g(\bar{z}_n) \vartheta y$, q. e. d.

3.6. Completion theorem. *Let* card $(A) \leqq m$, *let L be an m-arithmetical complete sublattice of* Eq (A), *let g be an n-ary L-compatible partial function and let its domain* Dom (g) *be an (m, L)-determining set. Then there is an n-ary L-compatible function f which is a completion og g.*

Proof. Let τ be the least ordinal of the cardinality m and let $A^n - \mathrm{Dom}(g) = \{t_\alpha ; \alpha < \tau\}$. By the transfinite induction we shall construct an ascending chain $\{g_\alpha ; \alpha \leqq \tau\}$ of L-compatible extensions of g such that $\mathrm{Dom}(g_\alpha) = \mathrm{Dom}(g) \cup \{t_\alpha ; \beta < \alpha\}$ for all α ; then it is sufficient to take $f = g_\tau$.

1) For $\alpha = 0$ we define $g_0 = g$.

2) If α is a limit ordinal (especially, if $\alpha = \tau$), let $g_\alpha = \bigcup \{g_\beta ; \beta < \alpha\}$.

3) Let an L-compatible extension g_α of g be constructed, and we have to construct

$$g_{\alpha+1}, \mathrm{Dom}(g_{\alpha+1}) = \mathrm{Dom}(g_\alpha) \cup \{t_\alpha\}.$$

Since card $(\mathrm{Dom}(g_\alpha) - \mathrm{Dom}(g)) < m$, the set $\mathrm{Dom}(g_\alpha)$ is (m, L)-determining; the other assumptions of Lemma 3.5 are also fulfilled. Hence there is an $a_\alpha \in A$ such that $g_{\alpha+1} = g_\alpha \cup \{(t_\alpha, a_\alpha)\}$ is L-compatible, q. e. d.

3.7. Lemma. *The ternary partial function g with the domain* $\mathrm{Dom}(g) = A_{3,3} = \{(x, y, z) \in A^3$; card $(\{x, y, z\}) < 3\}$ *such that*

(3.7.1) $$g(x, x, z) = g(z, x, x) = g(z, x, z) = z$$

for all $x, z \in A$, is Eq (A)*-compatible.*

The proof of Lemma 3.7 is obvious. Now we can prove the following generalization of Theorem 1.7.

3.8. Theorem. *Let* card $(A) \leqq m$ *and let* L *be an* m*-arithmetical complete sublattice of* Eq (A). *Then there is a ternary* L*-compatible function* f *which satisfies* (1.5.1) *for all* $x, z \in A$.

Proof. Every completion f of the partial function g of Lemma 3.7 fulfils (1.5.1). Hence it suffices to show that g has an L-compatible completion. Since g is Eq (A)-compatible, it is L-compatible. The set Dom $(g) = A_{3,3}$ is L-determining, and thus (m, L)-determining. Therefore by the Completion theorem 3.6 there is an L-compatible completion of g, q. e. d.

3.9. Lemma. *Let* card $(A) \leqq m$, L *be an* m*-arithmetical complete sublattice of* Eq (A) *and* $\eta \in$ Eq $(A) - L$. *Then there is a unary* L*-compatible function* g *which is not* η*-compatible.*

Proof. By [4] there are a, b, c, $d \in A$ such that the unary partial function $g = \{(a, c), (b, d)\}$ is L-compatible and not η-compatible. Then no completion of g is η-compatible. On the other hand, g is L-compatible and Dom (g) is (m, L)-determining, hence by Theorem 3.6 g has an L-compatible completion, q. e. d.

Lemma 3.9 and Theorem 3.8 imply the following representation theorem.

3.10. Theorem. *Let* card $(A) \leqq m$ *and let* L *be an* m*-arithmetical complete sublattice of* Eq (A). *Then there is an algebra with the congruence lattice* L*, among the fundamental operations of which there is a ternary function* f *satisfying* (1.5.1) *for all* $x, z \in A$.

Lemma 1.5 and the direct implication in Theorem 1.6 can be obtained from 3.10 by putting $m = \aleph_0 +$ card (A). Then the lattice L is m-arithmetical because it is $\aleph_0$-arithmetical and finite. The implications (i)$\rightarrow$(ii) and (i)$\rightarrow$(iii) in Theorem 1.8 can be obtained from 3.8 and 3.10 by putting $m = \aleph_0$.

4. Representation of equivalence lattices of type 0.

We shall begin the section with a characterization of equivalence lattices of type 0. All the conditions will be formulated not only for sublattices of Eq (A) but even for directed subsets of Eq (A); a partial ordered set L is said to be directed if every two its elements have an upper bound and a lower bound in L. The proof of Lemma 4.1 is straightforward and will be omitted.

4.1. Lemma. *For every directed subset* L *of* Eq (A) *the following conditions are equivalent:*

(i) *every interval of* Part (L) *is a chain;*
(ii) *for every* B, $C \in \text{Part}(L)$ *there holds* $B \cap C = \emptyset \vee B \subseteq C \vee C \subseteq B$
(iii) $\xi \vee \eta = \xi \cup \eta$ *for all* $\xi, \eta \in L$;
(iv) $\xi\eta = \xi \cup \eta$ *for all* $\xi, \eta \in L$;
(v) *for all* $\xi, \eta \in L$ *and all* $x, y \in A$, $x\xi\eta y \to x\xi y \vee x\eta y$.

4.2. Lemma. *Let* L *be a sublattice of* Eq (A) *and let every interval of* Part (L) *be a chain. Let* g be an *n-ary L-compatible partial function satisfying*

$$g(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\} \tag{4.2.1}$$

for all $\bar{x}_n \in \text{Dom}(g)$. *Then for every* $\bar{a}_n = (a_1, \ldots, a_n) \in A^n$ *there is a* $b \in \{a_1, \ldots, a_n\}$ *such that the partial function* $g \cup \{(\bar{a}_n, b)\}$ *is L-compatible.*

Proof. We may obviously assume $a_n \notin \text{Dom}(g)$. Consider the set $M = \{\xi(g(\bar{x}_n));\ \bar{x}_n \in \text{Dom}(g) \wedge \xi \in L\bar{x}_n\xi\bar{a}_n\}$. For every $\xi, \eta \in L$, $\bar{x}_n, \bar{y}_n \in \text{Dom}(g)$, $x_n\xi\bar{a}_n$, $\bar{y}_n\eta\bar{a}_n$ we have $\bar{x}_n\xi\eta\bar{y}_n$, $\xi\eta \in L$, hence $g(\bar{x}_n)\xi\eta g(\bar{y}_n)$, $\xi(g(\bar{x}_n) \cap \eta(g(\bar{y}_n)) \neq \emptyset$. Then by (ii) of Lemma 4.1 the sets $\xi(g(\bar{x}_n))$, $\eta(g(\bar{y}_n))$ are comparable. Therefore the set M is a chain. Now consider the sets $M_i = \{\xi(x_i);\ x_n \in \text{Dom}(g) \wedge g(\bar{x}_n) = x_i \wedge \xi \in L \wedge \bar{x}_n\xi\bar{a}_n\}$. Since obviously $M = M_1 \cup \ldots \cup M_n$ and M is a chain, we can choose a number $i \in \{1, \ldots, n\}$ such that $\bigcap M = \bigcap M_i$. Take $b = a_i$. We have to prove that $g \cup \{(\bar{a}_n, b)\}$ is L-compatible. It suffices to prove $g(\bar{x}_n)\xi a_i$ for every $\bar{x}_n \in \text{Dom}(g)$ and $\xi \in L$ satisfying $\bar{x}_n\xi\bar{a}_n$. For every such $\bar{x}_n$ and ξ there are $\bar{y}_n \in \text{Dom}(g)$ and $\eta \in L$ such that $\bar{y}_n\eta\bar{a}_n$, $g(\bar{y}_n) = y_i$ and $\eta(y_i) \subseteq \xi(g(\bar{x}_n))$. Then $a_i \in \eta(y_i) \subseteq \xi(g(\bar{x}_n))$, i.e. $g(\bar{x}_n)\xi a_i$, q. e. d.

4.3. Theorem. *Let* L *be a sublattice of* Eq (A) *of type* 0 *and let* g *be an n-ary L-compatible partial function satisfying* (4.2.1) *for all* $\bar{x}_n \in \text{Dom}(g)$. *Then there is an L-compatible completion* f *of* g *such that*

$$f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\} \tag{4.3.1}$$

for all $\bar{x}_n \in A^n$.

Proof. Consider the set M of all L-compatible extensions f of the partial function g which satisfy (4.3.1) for all $\bar{x}_n \in \text{Dom}(f)$. Zorn's lemma implies that the set M has a maximal element. By Lemma 4.2 it must be a function, q. e. d.

4.4. Theorem. *For every sublattice* L *of* Eq (A) *the following conditions are equivalent:*

(i) L *is of type* 0;
(ii) *there is an L-compatible function* f *which satisfies* (1.5.1) *for all* $x, z \in A$ *and*

$$f(x, y, z) \in \{x, y, z\} \tag{4.4.1}$$

for all $x, y, z \in A$.

Proof. Do not let (i) hold and let (ii) hold. There are $a, b, c \in A$ and $\xi, \eta \in L$ satisfying $a\xi b$, $\neg b\xi\, c$, $b\eta c$, $\neg a\eta b$. Denote $d = f(a, b, c)$. Since $(a, a, c)\ \xi\ (a, b, c)$ $\eta\ (a, c, c)$, we have $c\xi d\eta a$. However, no element $d \in \{a, b, c\}$ satisfies this condition, which is a contradiction.

Conversely, let (i) hold and let g be the partial function of Lemma 3.7. Every completion f of g fulfils the condition (1.5.1), g is L-compatible and fulfils (4.2.1). Hence by Theorem 4.3 g has an L-compatible completion f, which satisfies (1.5.1) and (4.4.1) q. e. d.

4.5. Lemma. *Let L be a complete sublattice of* Eq (A) *of type* 0 *and let $\eta \in \mathrm{Eq}(A) - L$. Then there is a ternary L-compatible function f which is not η-compatible and satisfies* (4.4.1) *for all $x, y, z \in A$.*

Proof. Let a, b, c, d be chosen similarly as in the proof of Lemma 3.9, i.e. in such a way that the partial function $g' = \{(a, c), (b, d)\}$ is L-compatible and not η-compatible. Let $g = \{(a, c, d, c), (b, c, d, d)\}$ (i.e. $g(a, c, d) = c$, $g(b, c, d) = d$). Then g is L-compatible and not η-compatible. Moreover, g fulfils (4.2.1) and hence it has an L-compatible completion f satisfying (4.4.1) which is not η-compatible, q.e.d.

4.6. Theorem. *For every complete sublattice of* Eq (A) *the following conditions are equivalent:*

(i) *L is of type* 0;

(ii) *L is the congruence lattice of an algebra, among the fundamental operation of which there is a ternary function f satisfying* (1.5.1) *and* (4.4.1) *for all $x, y, z \in A$;*

(iii) *L is the congruence lattice of such an algebra $\mathcal{A} = (A ; (f_i ; i \in I))$ that every (nonempty) subset of A forms a subalgebra of $\mathcal{A}$ and among the fundamental operations of $\mathcal{A}$ there is a ternary function f satisfying* (1.5.1) *and* (4.4.1) *for all $x, y, z \in A$.*

Proof. (i)$\rightarrow$(iii) follows from Theorem 4.4 and Lemma 4.5, (iii)$\rightarrow$(ii) is obvious and (ii)$\rightarrow$(i) follows from Theorem 4.4.

Clearly, Theorem 3.10 does not follow from Theorem 4.6. The example below shows that the Representation theorem 4.6 is not a corollary of Theorem 3.10.

4.7. Example. Let Z be the set of integers and let A be the set of irrational reals. For every integer k let us denote by ϑ_k the equivalence relation on A satisfying

$$x\vartheta_k y \leftrightarrow [x \cdot 2^k] = [y \cdot 2^k]$$

for all $x, y \in A$. Further, denote $\xi = \bigvee\{\vartheta_k ; k \in Z\}$. Then $L = \{\vartheta_k ; k \in Z\} \cup \{\xi, \mathrm{id}_A, A \times A\}$ is a complete sublattice of Eq (A); since L is a chain, it is of type 0. Hence by Theorem 4.6 there is an L-compatible ternary function f

satisfying (1.5.1) for all $x, z \in A$. Theorem 3.10 could not be applied because card $(A) = c$ and the lattice L is not c-arithmetical. Indeed, the set Part (L) contains the countable chain

$$\{\{x \in A\,;\,[2^k \cdot x] = 0\}\,;\,k \in Z\},$$

the intersection of which is empty.

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КОНКРЕТНОЕ ПРЕДСТАВЛЕНИЕ НЕКОТОРЫХ РЕШЕТОК ОТНОЧЕНИЙ ЭКВИВАЛЕНТНОСТИ

Иван Корец

Реуюме

Пусть L-подрешетка решетки Eq (A) всех отношений эквивалентности на множестве A. Обозначим Part (L) множество всех классов всех отношений эквивалентности из L. Решетка L называется арифметической, если она дистрибутивна и все ее элементы попарно перестановочны. Решетка L называется m-арифметической (m — бесконечная мощность), если L является арифметической и если всякая цепь из Part (L) с мощностью меньше m имеет непустое пересечение. Решетка L называется решеткой типа 0, если теоретико-множественное объединение любых двух ее элементов принадлежит L. Доказывается следующее обобщение некоторых результатов из [4] и [5]: Если A есть множество с мощностью не больше m и L — полная m-арифметическая подрешетка решетки Eq (A), то существует трехместная функция f на множестве A, совместимая с L, для которой выполняется (1.5.1). Дальше доказывается: Если L — полная подрешетка решетки Eq (A), то L является решеткой типа 0 тогда и только тогда, когда существует трехместная функция f на множестве A, для которой выполняются условия (1.5.1) и (4.4.1).