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PROPERTY (A) OF ADVANCED FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, property (A) of the advanced functional differential equation

$$L_n u(t) + F\big(t, u[g(t)]\big) = 0 \qquad (*)$$

is derived from the asymptotic behaviour of a set of ordinary functional equations

$$\alpha_i u(t) + F\big(t, u(t)\big) = 0 .$$

On the basis of this comparison principle the sufficient conditions for property (A) of equation $(*)$ are deduced.

We consider the functional differential equation with advanced argument

$$L_n u(t) + F\big(t, u[g(t)]\big) = 0 , \tag{1}$$

where $n \geq 3$ and L_n denotes the disconjugate differential operator

$$L_n = \frac{1}{r_n(t)} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{r_{n-1}(t)} \frac{\mathrm{d}}{\mathrm{d}t} \cdots \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{r_1(t)} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\cdot}{r_0(t)} . \tag{2}$$

We always assume that

(i) r_i, $g\colon [t_0, \infty) \to \mathbb{R}$ are continuous, $r_i(t) > 0$, $0 \leq i \leq n$, and $g(t) \geq t$;

(ii) $F\colon [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and $\operatorname{sgn} F(t,x) = \operatorname{sgn} x$ for each $t \in [t_0, \infty)$.

We will assume that

$$\int^{\infty} r_i(s) \, \mathrm{d}s = \infty \qquad \text{for} \quad 1 \leq i \leq n-1 . \tag{3}$$

We say that operator L_n is in the *canonical form* if (3) holds. In the sequel we will suppose that operator L_n is in its canonical form. It is well known that any

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differential operator of the form (2) can always be represented in a canonical form in an essentially unique way (see Trench [11]).

We introduce the following notation:

$$D_0u(t) = D_0(u; r_0)(t) = \frac{u(t)}{r_0(t)},$$

$$D_iu(t) = D_i(u; r_0, \dots, r_i)(t) = \frac{1}{r_i(t)} \frac{\mathrm{d}}{\mathrm{d}t} D_{i-1}u(t), \qquad 1 \le i \le n.$$

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $u\colon [T_u, \infty) \to \mathbb{R}$ such that $D_iu(t)$, $0 \le i \le n$, exist and are continuous on $[T_u, \infty)$. A nontrivial solution of (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory*. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

LEMMA 1. *If $u(t)$ is a nonoscillatory solution of (1), then there exists a $t_1 \in [t_0, \infty)$ and an integer $\ell \in \{0, 1, \dots, n-1\}$ such that $\ell \not\equiv n \pmod 2$ and*

$$\begin{aligned} u(t)D_iu(t) &> 0, & 0 \le i \le \ell, \\ (-1)^{i-\ell}u(t)D_iu(t) &> 0, & \ell + 1 \le i \le n, \end{aligned} \tag{4}$$

for all $t \ge t_1$.

This lemma is a generalization of a lemma of Kiguradze ([5; Lemma 3]).

A function $u(t)$ satisfying (4) is said to be a function of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then, by Lemma 1,

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \dots \cup \mathcal{N}_{n-1} \qquad \text{if} \quad n \text{ is odd},$$

and

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \dots \cup \mathcal{N}_{n-1} \qquad \text{if} \quad n \text{ is even}.$$

DEFINITION 1. Equation (1) is said to *have property* (A) if for n even (1) is oscillatory (i.e. $\mathcal{N} = \emptyset$) and for n odd $\mathcal{N} = \mathcal{N}_0$.

The main purpose of this paper is to establish a comparison principle between advanced equation (1) and the corresponding ordinary equation and to obtain sufficient conditions for equation (1) to have property (A).

We remark that for delay equations ($g(t) \le t$) of the form (1), efforts in this direction have been undertaken by several authors, see e.g. Mahfoud [9], Erbe [1], [2], and Kusano and Naito [8] in which delay equations of the form (1) are compared with ordinary equations without delay, and on the basis of such a comparison theorem many sufficient conditions for property (A) of delay equation (1) are deduced.

Let us consider the set of the disconjugate differential operators

$$\alpha_i = \frac{1}{r_n(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_{n-1}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\cdots\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_i(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_{i-1}[g(t)]g'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\cdots$$
$$\cdots\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_1[g(t)]g'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\cdot}{r_0[g(t)]}$$

for $i = 1, 2, \dots, n-1$.

THEOREM 1. *Suppose that*

(5) $F(t,x)$ *is nondecreasing in* x,

(6) $g(t) \in C^1([t_0,\infty))$, $g'(t) > 0$, $g(t) \ge t$.

Further assume that for $i = 1, 3, \dots, n-1$ *if* n *is even and for* $i = 2, 4, \dots, n-1$ *if* n *is odd, the functional equation*

$$\alpha_i u(t) + F\big(t, u(t)\big) = 0 \tag{E_i}$$

has not any solution of degree i. *Then equation* (1) *has property* (A).

P r o o f. Let $u(t)$ be a function of degree ℓ, satisfying (1). We may suppose that $u(t)$ is eventually positive. For the sake of contradiction we assume that $\ell \in \{1, 2, \dots, n-1\}$. Let t_1 be a number associated with $u(t)$ by Lemma 1. Integrating (1) from t ($\geqslant t_1$) to ∞ we have

$$D_{n-1}u(t) \ge \int_t^\infty r_n(s)F\big(s, u[g(s)]\big)\,\mathrm{d}s\,.$$

Repeating this procedure, we arrive at

$$D_\ell u(t) \ge \int_t^\infty r_{\ell+1}(s_{\ell+1})\int_{s_{\ell+1}}^\infty \cdots \int_{s_{n-1}}^\infty r_n(s_n)F\big(s_n, u[g(s_n)]\big)\,\mathrm{d}s_n\dots\mathrm{d}s_{\ell+2}\,\mathrm{d}s_{\ell+1}\,. \tag{7}$$

We multiply (7) by $r_\ell(t)$ and integrate over $[t_1, t]$ to obtain

$$\begin{aligned} &D_{\ell-1}u\big[g(t)\big] \ge D_{\ell-1}u(t)\\ &\ge \int_{t_1}^t r_\ell(s_\ell)\int_{s_\ell}^\infty r_{\ell+1}(s_{\ell+1})\int_{s_{\ell+1}}^\infty \cdots \int_{s_{n-1}}^\infty r_n(s_n)F\big(s_n, u[g(s_n)]\big)\,\mathrm{d}s_n\dots\mathrm{d}s_\ell\,, \end{aligned} \tag{8}$$

where we also used the facts that $g(t) \ge t$ and $D_{\ell-1}u(t)$ is an increasing function as $\ell \ge 1$. If $\ell \ge 2$, then we multiply (8) by $r_{\ell-1}\big[g(t)\big]g'(t)$ and integrate the resulting inequality over $[t_1, t]$. Continuing in this manner we obtain

$$D_0 u[g(t)] \geq \int_{t_1}^{t} r_1[g(s_1)]g'(s_1) \int_{t_1}^{s_1} \dots \int_{t_1}^{s_{\ell-2}} r_{\ell-1}[g(s_{\ell-1})]g'(s_{\ell-1}) \int_{t_1}^{s_{\ell-1}} r_\ell(s_\ell) \int_{s_\ell}^{\infty} \dots$$
$$\dots \int_{s_{n-1}}^{\infty} r_n(s_n) F\big(s_n, u[g(s_n)]\big) \, \mathrm{d}s_n \dots \mathrm{d}s_1 \,, \qquad t \geq t_1 \,. \tag{9}$$

Denote the right hand side of (9) by $v(t)$ and define $z(t) = r_0[g(t)]v(t)$. By repeated differentiation of $z(t)$, one can verify that $z(t)$ is a function of degree ℓ and, on the other hand,

$$\alpha_\ell z(t) + F\big(t, u[g(t)]\big) = 0 \,. \tag{10}$$

Since $u[g(t)] \geq z(t)$, we obtain in view of (10) that $z(t)$ is a solution of the differential inequality

$$\big\{\alpha_\ell z(t) + F\big(t, z(t)\big)\big\} \operatorname{sgn} z(t) \leq 0 \,.$$

But then [8; Corollary 1] of Kusano and Naito ensures that equation (E_ℓ) has also a solution of degree ℓ, which contradicts the hypotheses.

If $\ell = 1$, then (8) implies

$$D_0 u[g(t)] \geq \int_{t_1}^{t} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \int_{s_2}^{\infty} \dots \int_{s_{n-1}}^{\infty} r_n(s_n) F\big(s_n, u[g(s_n)]\big) \, \mathrm{d}s_n \dots \mathrm{d}s_1 \,.$$

Again, denote the right hand side of the above inequality by $v(t)$ and define $z(t) = r_0[g(t)]v(t)$. Proceeding similarly as above we can verify that equation (E_1) has a solution of degree 1, which contradicts the hypotheses. The proof is complete. □

Now, we apply our comparison principle to the linear form of equation (1), namely, to the advanced equation

$$L_n u(t) + p(t) u[g(t)] = 0 \,, \tag{11}$$

where function $p(t)$ is continuous and positive on $[t_0, \infty)$.

Let $i_k \in \{1, 2, \dots, n-1\}$, $1 \leqslant k \leqslant n-1$, and $t, s \in [t_0, \infty)$, we define

$$I_0 = 1 \,,$$

$$I_k(t, s; r_{i_k}, \dots, r_{i_1}) = \int_{s}^{t} r_{i_k}(x) \, I_{k-1}(x, s; r_{i_{k-1}}, \dots, r_{i_1}) \, \mathrm{d}x \,.$$

For simplicity of notation, we put

$$J_i(t,s) = r_0\big[g(t)\big] I_i\big(t,s;r_1(g)g',\dots,r_i(g)g'\big)\,,$$
$$K_i(t,s) = r_n(t)\, I_i(t,s;r_{n-1},\dots,r_{n-i})\,.$$

It is easy to see that

$$J_i(t,s) = r_0\big[g(t)\big] I_i\big(g(t),g(s);r_1,\dots,r_i\big)\,.$$

We define

$$q_i(t) = r_{i+1}(t)\int\limits_t^\infty K_{n-i-2}(s,t)\,J_{i-1}(s,t)p(s)\;\mathrm{d}s\,, \qquad i=1,2,\dots,n-3\,,$$

$$q_{n-1}(t) = r_{n-2}\big[g(t)\big]g'(t)\int\limits_t^\infty J_{n-3}(s,t)K_0(s,t)p(s)\;\mathrm{d}s\,,$$

where we suppose that both integrals are convergent.

THEOREM 2. *Suppose that* (6) *holds. Assume that the second order equations*

$$\left(\frac{1}{r_i(t)}z'(t)\right)' + q_i(t)z(t) = 0 \tag{e_i}$$

are oscillatory for $i=2,4,\dots,n-1$ *if* n *is odd and for* $i=1,3,\dots,n-1$ *if* n *is even. Then equation* (11) *has property* (A).

P r o o f. Let $\ell \in \{1,2,\dots,n-1\}$ be fixed. By Theorem 1, equation (11) has not any solution of degree ℓ if the equation

$$\alpha_\ell u(t) + p(t)u(t) = 0$$

has not any solution of degree ℓ, which, by [7; Theorem B] and [10; Theorem 2], comes if equation (e_ℓ) is oscillatory. The proof is complete. □

Let us consider another form of equation (11), namely, the advanced equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}u'(t)\right)'\right)' + p(t)u\big[g(t)\big] = 0\,. \tag{12}$$

Let us denote

$$R_i(t) = \int\limits_{t_0}^t r_i(s)\;\mathrm{d}s\,, \qquad \text{for} \quad i=1,2\,.$$

COROLLARY 1. *Suppose that* (6) *holds. Equation* (12) *has property* (A) *if*

$$\liminf_{t\to\infty} R_2(t) \int_t^{\infty} \Big(R_1[g(s)] - R_1[g(t)]\Big) p(s)\, \mathrm{d}s > \frac{1}{4}\,. \tag{13}$$

P r o o f . By Theorem 2, equation (12) has property (A) if equation (e_2) is oscillatory. By the well-known criterion of H i l l e [3], condition

$$\liminf_{t\to\infty} \left(\int_{t_0}^{t} r_2(s)\, \mathrm{d}s\right)\left(\int_t^{\infty} q_2(s)\, \mathrm{d}s\right) > \frac{1}{4}\,,$$

which is equivalent to (13), ensures oscillation of all solutions of (e_2). □

E x a m p l e 1. Let us consider the advanced equation

$$y'''(t) + p(t)y\big[g(t)\big] = 0\,. \tag{14}$$

By Corollary 1, equation (14) has property (A) if (6) holds and moreover if

$$\liminf_{t\to\infty} t \int_t^{\infty} \big(g(s) - g(t)\big)p(s)\, \mathrm{d}s > \frac{1}{4}\,. \tag{15}$$

E x a m p l e 2. The ordinary equation

$$y'''(t) + \frac{2}{3\sqrt{3}t^3}y(t) = 0\,, \qquad t > 0\,, \tag{16}$$

has a nonoscillatory solution $y(t) = t^{1+1/\sqrt{3}}$, which is of degree 1, but the corresponding differential equation with advanced argument

$$y'''(t) + \frac{2}{3\sqrt{3}t^3}y(\lambda t) = 0\,, \qquad t > 0\,, \quad \lambda > 1\,, \tag{17}$$

in view of condition (15), has property (A) if $\lambda > \frac{3\sqrt{3}}{4}$.

As a matter of fact we are able to relax the condition of monotonicity imposed on the advanced argument in Theorems 1 and 2. Let us consider functional equation of the form (1) with larger advanced argument $Q(t)$, where $Q(t)\colon [t_0,\infty) \to \mathbb{R}$ is continuous.

THEOREM 3. *Suppose that* (5) *and* (6) *hold and* $Q(t) \geq g(t)$. *Let* r_0 *be nondecreasing. Further assume that for* $i = 1, 3, \dots, n-1$ *if* n *is even and for* $i = 2, 4, \dots, n-1$ *if* n *is odd functional equation* (E_i) *has not any solution of degree* i. *Then the equation*

$$L_n u(t) + F\big(t, u[Q(t)]\big) = 0 \tag{18}$$

has property (A).

P r o o f. By Theorem 1, equation (1) has property (A), and, by [8; Theorem 1], equation (18) has property (A). □

Using similar arguments we can prove the following result.

THEOREM 4. *Suppose that* (6) *holds and* $Q(t) \geq g(t)$. *Let* r_0 *be nondecreasing. Further assume that equations* (e_i) *are oscillatory for* $i = 1, 3, \dots, n-1$ *if* n *is even and for* $i = 2, 4, \dots, n-1$ *if* n *is odd. Then the equation*

$$L_n u(t) + p(t) u\big[Q(t)\big] = 0$$

has property (A).

E x a m p l e 3. Let us consider the advanced equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}u'(t)\right)'\right)' + p(t)u[2t + \cos t] = 0\,. \tag{19}$$

Letting $g(t) = 2t - 1$ and applying Corollary 1 and Theorem 4, one gets that equation (19) has property (A) if (6) holds and (13) is satisfied with $g(t) = 2t - 1$.

Let $0 \leq i \leq n-1$. We denote

$$M_i(t,s) = r_0(t)\, I_i(t, s; r_1, \dots, r_i)\,, \qquad M_i(t) = M_i(t, t_0)\,,$$
$$K_i(t) = K_i(t, t_0)\,.$$

We show that the conclusions of Theorems 1 – 4 can be strengthened as follows:

THEOREM 5. *Let* (5) *hold. Assume that equation* (1) *has property* (A). *Then every nonoscillatory solution* $u(t)$ *of* (1) *satisfies*

$$\lim_{t\to\infty} \frac{u(t)}{r_0(t)} = 0 \tag{20}$$

if and only if

$$\int^{\infty} K_{n-1}(t) r_n(t) \big| F\big(t, c r_0[g(t)] M_0[g(t)]\big)\big| \, \mathrm{d}t = \infty\,,$$

for some $c \in \mathbb{R} - \{0\}$.

The proof follows immediately from the following result, which can be found in [6].

LEMMA 2. *Let* (5) *hold. Let* i, $0 \le i \le n-1$, *be fixed. Equation* (1) *has a solution* $u(t)$ *satisfying*

$$\lim_{t\to\infty} D_i u(t) = a_i \in \mathbb{R} - \{0\},$$

for some a_i *if and only if*

$$\int^{\infty} K_{n-i-1}(t) r_n(t) \big| F\big(t, c r_0[g(t)] M_i[g(t)]\big)\big| \,\mathrm{d}t < \infty .$$

for some $c \in \mathbb{R} - \{0\}$.

Remark. The previous lemma provides effective sufficient and necessary condition for equation (1) not to have any solution $u(t)$ of degree 0 satisfying $\lim_{t\to\infty} D_0 u(t) = c \ne 0$. On the other hand, for any nonoscillatory solution $u(t)$ of equation (1), condition (20) implies that

$$\lim_{t\to\infty} D_i u(t) = 0, \qquad \text{for} \quad i = 0, 1, \dots, n-1. \tag{21}$$

Therefore, property (A) of equation (1) can be defined as above, or we can use the following definition.

DEFINITION 1'. Equation (1) is said to *have property* (A) if for n even (1) is oscillatory and for n odd every nonoscillatory solution $u(t)$ of (1) satisfies (21).

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