Mathematica Slovaca

Jaroslav Smítal

Iterates of piecewise monotonic continuous functions

Mathematica Slovaca, Vol. 32 (1982), No. 2, 143--146

Persistent URL: `http://dml.cz/dmlcz/129283`

Math. Slovaca 32, 1982, No. 2, 143—146

ITERATES OF PIECEWISE MONOTONIC CONTINUOUS FUNCTIONS

JAROSLAV SMÍTAL

Let f be a continuous piecewise monotonic function from a compact real interval I into itself. Denote by $P(k)$ the number of monotonic pieces in the graph of the k-th iterate f^k of f. A. Sklar [1] recently has set the problem to give the best possible lower estimate of $P(k)$ in the case when f contains a 3-cycle (and hence when' it contains cycles of all orders, c.f. [2]). The corresponding best upper estimate is given by $P(k) \leqq P(1)^k$ (cf. [1]).

In this note we give a complete solution of the above quoted problem. We begin with the following two easily verified lemmas which must be known in literature, but we are not able to give any references.

Lemma 1. *Let f, g be continuous functions from a real interval I into itself, which are piecewise monotonic and nonconstant on every subinterval of I. If f has a local extremum at a point x_0, so has $f \circ g$.*

Proof. Let f have a local maximum at x_0. If g is decreasing on a left neighbourhood of $f(x_0)$, then $f \circ g$ (first apply f, then g) has at x_0 a local minimum, and if g is increasing on a left neighbourhood of $f(x_0)$, then $f \circ g$ has at x_0 a local maximum. Similarly in other cases.

Lemma 2. *Under the same assumptions as in Lemma 1 $g \circ f$ has a local extremum at $t_0 \in I$ if f has a local extremum at x_0, and if $g(t_0) = x_0$.*

Proof is similar to that in the preceding lemma.

Now we are able to prove the following

Proposition 1. *Let f be a continuous function from the set R of reals into itself, which is non-constant on each interval and which contains a 3-cycle $t_1 \mapsto t_2 \mapsto t_3 \mapsto t_1$, where $t_1 < t_2 < t_3$. Assume that f is piecewise monotonic and that f has $p+1$ monotonic pieces in the interval (t_1, t_3) and $q+1$ monotonic pieces in the interval (t_2, t_3). Then f^n has at least*

$$q \cdot F_{n-1} + p \cdot F_n + F_{n+1} \leqq P(n)$$

monotonic pieces in the interval (t_1, t_3) where $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$.

Proof. We say that an interval I is of the first type with respect to f^n if $f^n(I) \supset (t_2, t_3)$ and that I is of the second type with respect to f^n if $f^n(I) \supset (t_1, t_3)$. Clearly, if I is of the first type with respect to f^n, then I is of the second type with respect to f^{n+1} since $f^{n+1}(I) = f(f^n(I)) \supset f((t_2, t_3)) \supset (f(t_3), f(t_2)) = (t_1, t_3)$.

Choose some $t_0 \in (t_1, t_3)$ such that $f(t_0) = \max f(t)$ for $t \in (t_1, t_3)$; such t_0 exists, since $f(t_1), f(t_3) < f(t_2)$.

We say that f^n has the property $P(k, s)$ provided there is a sequence

(1) $$t_1 = a_0 < a_1 < \ldots < a_{k+s} = t_3$$

such that f_n has a local extremity at each a_i with $i = 1, \ldots, k+s-1$, and if the corresponding intervals (a_{i-1}, a_i), where $i = 1, \ldots, k+s$, can be rearranged as

(2) $$I_1, \ldots, I_k, J_1, \ldots, J_s$$

where the $I_i's$ are of the first type and $J_i's$ of the second type with respect to f^n.

Now we show that if f^n has the property $P(k, s)$, then f^{n+1} has the property $P(s, k+s)$. Since for each interval J_i from (2) we have $f^n(J_i) \supset (t_1, t_3)$, there is an interior point c_i of J_i such that $f^n(c_i) = t_0$. Now divide J_i into two intervals J_i^1, J_i^2 with c_i as a common end-point such that $(t_1, t_0) \subset f^n(J_i^1)$ and $(t_0, t_3) \subset f^n(J_i^2)$. In this case, however, $f^{n+1}(J_i^1) \supset f((t_1, t_0)) \supset (t_2, t_3)$ and $f^{n+1}(J_i^2) \supset f((t_0, t_3)) \supset (t_1, t_3)$, hence J_i^1 is of the first type and J_i^2 of the second type with respect to f^{n+1}. Moreover, f^{n+1} has a local extremum at each c_i for $i = 1, \ldots, s$ (see Lemma 2), as well as at each a_i for $i = 1, \ldots, k+s-1$ (Lemma 1). Hence by the above quoted construction we obtain from (1), (2) a new system

$$\{a_0, \ldots, a_{k+s}\} \cup \{c_1, \ldots, c_s\}$$

of dividing points, and a new system

$$J_1^1, \ldots, J_s^1, J_1^2, \ldots, J_s^2, I_1, \ldots, I_k$$

of corresponding intervals where each of the intervals J_i^1 is of the first type and each of the intervals J_i^2, I_i of the second type with respect to f^{n+1}. Thus f^{n+1} has the property $P(s, k+s)$.

Clearly f has the property $P(1, 1) = P(F_1, F_2)$ with t_0 as the dividing point, thus by induction we involve that f^n has the property $P(F_n, F_{n+1})$.

Now let

(3) $$I_1, \ldots, I_{F_n}, J_1, \ldots, J_{F_{n+1}}$$

be intervals of the first and second type, respectively, corresponding to f^n. It is easy to see that f^{n+1} takes on at least q local extremities in the interior of each I_i (i.e. f^{n+1} has at least $p+1$ monotonic pieces on I_i), and at least p local extremities in the interior of each J_i. Moreover, f^{n+1} takes on, by Lemma 2, a local extremity at each of the $(F_n + F_{n+1} + 1)$ end-points of the intervals (3), with the possible exceptions of t_1 and t_3, hence at $(F_n + F_{n+1} - 1)$ points. Thus f^{n+1} has at least

$$q \cdot F_n + p \cdot F_{n+1} + (F_n + F_{n+1} - 1)$$

local extremities on the interval (t_1, t_3), and so

$$P(n+1) \geqq q \cdot F_n + p \cdot F_{n+1} + F_{n+2}$$

and the proposition is proved.

Proposition 2. *Proposition* 1 *holds also for any function* $f\colon R \to R$ *with a cycle* $t_1 \mapsto t_2 \mapsto t_3 \mapsto t_1$, *where* $t_3 < t_2 < t_1$ *if the intervals* (t_1, t_3) *and* (t_2, t_3) *in Proposition* 1 *are replaced by* (t_3, t_1) *and* (t_3, t_2), *respectively.*

Proof. Put $f^*(t) = -f(-t)$ for all t. Then clearly f^* is piecewise monotonic and has a 3-cycle $t_1^* \mapsto t_2^* \mapsto t_3^* \mapsto t_1^*$, where $t_i^* = -t_i$ for $i = 1, 2, 3$ and $t_1^* < t_2^* < t_3^*$. Moreover, $(f^*)^k(t) = -(f^k)(-t) = (f^k)^*(t)$, hence $f^*)^k$ has the same number of monotonic pieces as f^k.

Remark 1. The cases $t_1 < t_2 < t_3$ and $t_3 < t_2 < t_1$, where $t_1 \mapsto t_2 \mapsto t_3 \mapsto t_1$, are, under suitable notation, clearly all possible cases of the 3-cycles.

Now we are able to prove the main result.

Theorem. *Let f be a continuous piecewise monotonic function from a compact interval $I \subset R$ to I, which is nonconstant on every subinterval of I, and which contains a* 3*-cycle. Then*

$$P(n) \geqq P(1) - 2 + F_{n+2}.$$

Proof. Choose a 3-cycle of f and let J be the smallest interval containing it. Assume that f has $p+1$ monotonic pieces on J, and denote $P(1) = m$. Then by the above propositions and Remark 1, f^n has at least $(p \cdot F_n + F_{n+1} - 1)$ local extrema in the interior of J, and by Lemma 1, at least $(m - p - 1)$ local extrema outside the interior of J, hence

$$\begin{aligned} P(n) &\geqq (p \cdot F_n + F_{n+1} - 1) + (m - p - 1) + 1 \\ &\geqq P(1) - 2 + F_{n+2}, \end{aligned}$$

q.e.d.

Remark 2. The estimation given in the theorem is the best possible in the sense that there is a function f with $P(1) = 2$ such that $P(k) = F_{k+2}$, for each k; such a function can be defined, e.g., as follows: $f\colon [0, 2] \to [0, 2]$, $f(0) = 1$, $f(1) = 2$, $f(2) = 0$ and f is linear in the intervals $[0, 1]$ and $[1, 2]$.

Moreover, one can easily construct a function f with arbitrarily large $P(1)$ such that $P(k) = P(1) - 2 + F_{k+2}$ for each k.

REFERENCES

[1] SKLAR, A.: Problem P 168. Aequationes Math., 15, 1977, 297.
[2] ŠARKOVSKIJ, A. N.: Co-existence of the cycles of a continuous mapping of the line into itself. Russian. Ukrain. Mat. Žurnal, 16, 1964, 61—71.

Received March 21, 1980.

Katedra matematickej štatistiky
Matematicko-fyzikálnej fakulty UK
Mlynská dolina
842 15 Bratislava

ИТЕРАЦИИ КУСОЧНО МОНОТОННЫХ НЕПРЕРЫВНЫХ ФУНКЦИЙ

Ярослав Смитал

Резюме

Пусть $f: I \to I$ — непрерывная кусочно монотонная функция и I — компактной интервал. Пусть $P(k)$ — число монотонных частей k-той итерации f^k функции f. В статье предлагается нижняя оценка $P(k)$ в том случае, если функция f содержит 3-цикл.