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SIMULTANEOUS APPROXIMATION OF ZERO BY VALUES OF INTEGRAL POLYNOMIALS WITH RESPECT TO DIFFERENT VALUATIONS

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ABSTRACT. We prove an analogue of the convergence part of Khinchine's theorem for the simultaneous approximation of zero in $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ by the values of polynomials $P_n(y) \in \mathbb{Z}[y]$. This is a proof of a stronger version of V. Sprindžuk's conjecture (1980).

1. Introduction

The problem under consideration belongs to the metric theory of Diophantine approximation of dependent values. This theory was formed in papers of V. Sprindžuk [14], [15], W. M. Schmidt [13]. Nowadays it is intensively developed ([1], [3]–[6], [9], [10], [16]).

Let $P_n = P_n(y) = a_n y^n + \dots + a_1 y + a_0 \in \mathbb{Z}[y]$, $\deg P_n = n$ and $H = H(P_n) = \max\limits_{0 \le i \le n} |a_i|$. Let $p \ge 2$ be a prime number, $\mathbb{Q}_p$ be the field of p-adic numbers, $|\cdot|_p$ be the p-adic valuation. V. Sprindžuk (1965) proved Mahler's problem for P_n in the fields $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}_p$. An analogue of Mahler's problem in the field $\mathbb{R}^k \times \mathbb{C}^l \times \prod\limits_{p \in S} \mathbb{Q}_p$, where $k \ge 1$, $l \ge 1$ are integers and S is a finite set of prime numbers, $n \ge k + 2l$, was formulated by him in 1980 and proved by F. Želudevich [16].

In 1924 A. Khintchine [8] proved the metric theorem about an exact order of approximation of real number α by rationals p/q. It says that the inequality $|\alpha - p/q| < f(q)/q$, where $f\colon \mathbb{N} \to \mathbb{R}^+$, $f \in \mathcal{C}(\mathbb{R})$ and the function $xf(x)$ is nonincreasing, has infinitely many solutions in integers p, $q > 0$ for almost all real numbers α (in the sense of Lebesque measure), provided that

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the integral $\int_c^\infty f(x)\,\mathrm{d}x = \infty$ for some $c > 0$. On the other hand, if the integral converges, then the given inequality has no more than a finite number of solutions in integers p, $q > 0$ for almost all α.

After 1986 some generalizations of the convergence part of this theorem were obtained for polynomials P_n. V. Bernik (1989), D. Vasiliyev (1998) and E. Kovalevskaya [11], [12] proved results of this type for $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}_p$ respectively (in the sense of Lebesque measures in $\mathbb{R}$, $\mathbb{R}^2$ and the Haar measure in $\mathbb{Q}_p$).

Let $\psi\colon \mathbb{N} \to \mathbb{R}^+$ be a monotonically decreasing function and $\sum_{n=1}^{\infty} \psi(n) < \infty$. Here we prove an analogue of the convergence part of the Khintchine theorem for the simultaneous approximation of zero by values of polynomials P_n in the field $\mathcal{O} = \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$. Further, we define a measure μ in $\mathcal{O}$ as a product of the Lebesque measures μ_1, μ_2 in $\mathbb{R}$, $\mathbb{C}$ and the Haar measure μ_h in $\mathbb{Q}_p$, that is, $\mu = \mu_1 \mu_2 \mu_h$. We consider the system of inequalities

$$|P_n(x)| < H^{\lambda_1} \psi^{\nu_1}(H)\,, \quad |P_n(z)| < H^{\lambda_2} \psi^{\nu_2}(H)\,, \quad |P_n(\omega)|_p < H^{\lambda_3} \psi^{\nu_3}(H)\,, \tag{1}$$

where $(x, z, \omega) \in \mathcal{O}$, $\lambda_i \le 1$ $(i = 1, 2)$, $\lambda_3 \le 0$, $\lambda_1 + 2\lambda_2 + \lambda_3 = n - 3$, $\nu_i \ge 0$ $(i = 1, 2, 3)$, $\nu_1 + 2\nu_2 + \nu_3 = 1$, $\lambda_i - \nu_i < 1$ $(i = 1, 2)$, $\lambda_3 - \nu_3 < 0$. We prove the following theorem.

THEOREM. *The system of inequalities* (1) *is satisfied by at most finitely many polynomials $P_n \in \mathbb{Z}[y]$ for almost all $(x, z, \omega) \in \mathcal{O}$.*

In order to prove the Theorem we develop Sprindžuk's method of essential and inessential domains, use a proof scheme from [2] and one lemma of Bernik–Kalosha from [7]. Proving the Theorem we investigate 7 cases dependent on the values of the derivative $|P_n'(y)|$, that is, we consider the domains where the value of $|P_n'(y)|$ is large and the domains where the value of $|P_n'(y)|$ is small. Then, we combine these domains with respect to above-mentioned valuations.

2. Notations and results

According to the metric ideas [14] we may put $x \ll 1$, $z \ll 1$, $|\omega|_p \ll 1$, where $\ll$ is Vinogradov's symbol ($x \ll y$ means that $x = O(y)$). Let $\alpha_1^{(n)}, \dots, \alpha_n^{(n)}$ be the roots of the polynomials P_n in $\mathbb{C}$ and $\beta_1^{(n)}, \dots, \beta_n^{(n)}$ be the roots of the polynomials P_n in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the least field containing $\mathbb{Q}_p$ and all algebraic numbers. As in [14], the investigation of the system (1) can be reduced to the case of primitive irreducible polynomials when $H(P_n) = |a_n|$ and

$|a_n|_p > p^{-n}$. We denote the set of those polynomials as $\mathcal{P}_n$. As in [14], we assume that $|\alpha_i^{(n)}| \le 2$, $|\beta_i^{(n)}| < p^n$ $(1 \le i \le n)$.

Let $\mathcal{P}_n(H)$ be a set of polynomials $P_n \in \mathcal{P}_n$ with the condition $H(P_n) = H$ where $H \in \mathbb{N}$. We order the roots of the polynomial $P_n \in \mathcal{P}_n(H)$ so that

$$|\alpha_1^{(n)} - \alpha_2^{(n)}| \le \cdots \le |\alpha_1^{(n)} - \alpha_n^{(n)}|, \qquad |\beta_1^{(n)} - \beta_2^{(n)}|_p \le \cdots \le |\beta_1^{(n)} - \beta_n^{(n)}|_p .$$

Suppose that

$$S_1(\alpha_i^{(n)}) = \left\{x \in \mathbb{R} : |x - \alpha_i^{(n)}| = \min_{1\le j\le n} |x - \alpha_j^{(n)}|\right\},$$
$$S_2(\alpha_i^{(n)}) = \left\{z \in \mathbb{C} : |z - \alpha_i^{(n)}| = \min_{1\le j\le n} |z - \alpha_j^{(n)}|\right\},$$
$$S_p(\beta_i^{(n)}) = \left\{\omega \in \mathbb{Q}_p : |\omega - \beta_i^{(n)}|_p = \min_{1\le j\le n} |\omega - \beta_j^{(n)}|_p\right\}.$$

It is clear that, for example, $S_p(\beta_i^{(n)})$ is a set of those points ω for which $\beta_i^{(n)}$ is the nearest root. Hence, $\bigcup_{i=1}^{n} S_2(\alpha_i^{(n)}) = \mathbb{C}$ and $\bigcup_{i=1}^{n} S_p(\beta_i^{(n)}) = \mathbb{Q}_p$. It is possible that some of $S_2(\alpha_i^{(n)})$ and $S_p(\beta_i^{(n)})$ are empty. Then, the following estimates for $P_n \in \mathcal{P}_n(H)$, $x \in S_1(\alpha_i^{(n)})$, $z \in S_2(\alpha_i^{(n)})$, $\omega \in S_p(\beta_i^{(n)})$ are known ([3; pp. 36, 131]):

$$|u - \alpha_1^{(n)}| \le \frac{2^n |P_n(u)|}{|P_n'(\alpha_1^{(n)})|}, \qquad |\omega - \beta_1^{(n)}|_p \le \frac{|P_n(\omega)|_p}{|P_n'(\beta_1^{(n)})|_p}, \tag{2}$$

$$\begin{gathered} |u - \alpha_1^{(n)}| \le \min_{2\le j\le n} \left(2^{n-j} \frac{|P_n(u)|}{|P_n'(\alpha_1^{(n)})|} \prod_{k=2}^{j} |\alpha_1^{(n)} - \alpha_k^{(n)}|\right)^{1/j}, \\ |\omega - \beta_1^{(n)}|_p \le \min_{2\le j\le n} \left(\frac{|P_n(\omega)|_p}{|P_n'(\beta_1^{(n)})|_p} \prod_{k=2}^{j} |(n) - \beta_k^{(n)}|_p\right)^{1/j}, \end{gathered} \tag{3}$$

where $u = x$ or $u = z$.

Since $|\alpha_i^{(n)}| \le 2$, $|\beta_i^{(n)}|_p < p^n$ $(1 \le i \le n)$, $|\omega|_p \ll 1$, then under $j = n$ and $H \ge H_0$ we obtain from (2) that the set of points $(x, z, \omega) \in \mathcal{O}$ for which (1) is satisfied is inside the set $\mathbb{A} = I \times K \times D$ where $I = [-3, 3]$, $K = \{z : |z| \le 3\}$, $D = \{\omega : |\omega|_p \ll 1\}$. Fix $\varepsilon > 0$ where ε is sufficiently small number. Suppose that $\varepsilon_1 = \varepsilon d^{-1}$ where $d > 0$ is sufficiently large number and $T = \varepsilon_1^{-1}$. We define real numbers $\rho_{ij} = \rho_{ij}(P_n)$ $(i = 1, 2, 3)$ and integers k_j, l_j, m_j from the following relations

$$|\alpha_1 - \alpha_j| = H^{-\rho_{1j}}, \quad |\alpha_0 - \alpha_j| = H^{-\rho_{2j}}, \quad |\beta_1 - \beta_j|_p^{-\rho_{3j}} \qquad (2 \le j \le n),$$

$\alpha_0 \neq \alpha_j$, where $|\alpha_0|$ is one of the least modulo complex conjugate roots of P_n,

$$\begin{aligned} &(k_j - 1)/T \leq \rho_{1j} < k_j/T\,, \\ &(l_j - 1)/T \leq \rho_{2j} < l_j/T\,, \\ &(m_j - 1)/T \leq \rho_{3j} < m_j/T \qquad (2 \leq j \leq n)\,. \end{aligned}$$

For brevity we write α_j, β_j instead $\alpha_j^{(n)}$, $\beta_j^{(n)}$. It is not difficult to show that $0 \leq k_j, l_j, m_j \leq nT$. Also we define numbers q_i, r_i, s_i $(1 \leq i \leq n)$:

$$q_i = (k_{i+1} + \cdots + k_n)/T\,, \quad r_i = (l_{i+1} + \cdots + l_n)/T\,, \quad s_i = (m_{i+1} + \cdots + m_n)/T\,.$$

Now we associate each polynomial $P_n \in \mathcal{P}_n(H)$ with three integer vectors $\overline{q} = (k_2, \dots, k_n)$, $\overline{r} = (l_2, \dots, l_n)$, $\overline{s} = (m_2, \dots, m_n)$. As in [14; pp. 46, 99 100], we can show that the number of these vectors is finite and dependent on n, p, T only and is independent of H.

3. Proof of Theorem

Further we describe the main steps of the proof of the Theorem. First of all, one makes a classification of polynomials $P_n \in \mathcal{P}_n(H)$ so that the class $\mathcal{P}_n(H, \overline{q}, \overline{r}, \overline{s})$ contains all polynomials P_n having the same triple of vectors, $(\overline{q}, \overline{r}, \overline{s})$. The further investigations are dependent on the values $q_1 + k_2/T$, $r_1 + l_2/T$, $s_1 + m_2/T$ which characterize the behaviour of the first derivative $|P_n'(y)|$ at the roots α_1, α_0, β_1 and values $|\alpha_1 - \alpha_i|$, $|\alpha_0 - \alpha_i|$, $\alpha_0 \neq \alpha_i$, $|\beta_1 - \beta_i|_p$ $(i = 2, \dots, n)$. Note also that from the convergence of series $\sum\limits_{n=1}^{\infty} \psi(n)$ it follows that $\psi(H) < cH^{-1}$ for sufficiently large H, $H \geq H_0$, where $c > 0$ is a constant independent of H.

Denote by $\mathcal{O}_1$ the set of the points $(x, z, \omega) \in \mathcal{O}$ for which the system (1) holds. Then, the set $\mathcal{O}_1$ is measurable with respect to measure μ according to the theory of measure. Hence, proving the Theorem can be reduced to proving it for one of the elementary sets $\mathcal{O}_{1i} = I_i \times K_i \times D_i$ $(i = 1, 2, \dots)$ from the counting covering of the set $\mathcal{O}_1$.

Case 1.
Let

$$q_1 + 2r_1 + s_1 + (k_2 + 2l_2 + m_2)/T > n - 1 + 6n\varepsilon_1\,,$$
$$q_1 + k_2/T \leq -\lambda_1 + \nu_1 + 1\,,$$

and

$$r_1 + l_2/T \leq -\lambda_2 + \nu_2 + 1\,, \quad s_1 + m_2/T \leq -\lambda_3 + \nu_3\,. \tag{4}$$

Estimating from above the resultant $R(P_{ni}, P_{nj})$ as in [2; Proposition 1], [12; Proposition 1]), we obtain $\mathcal{O}_{1i} \cap \mathcal{O}_{1j} = \emptyset$ and $\mu(\mathcal{O}_{1i}) = 0$.

Case 2.
Let the condition (4) be true and

$$4 - \varepsilon/2 < q_1 + 2r_1 + m_1 + (k_2 + 2l_2 + m_2)/T \le n - 1 + 6n\varepsilon_1 . \tag{5}$$

As in [2; Proposition 2], [12; Proposition 2], we introduce the numbers $\theta = n + 1 - [q_1 + 2r_1 + m_1 + (k_2 + 2l_2 + m_2)/T]$, $\beta = \theta - 1 - 0.1\varepsilon$, $\sigma_1 = k_2/T + n\varepsilon_1$, $\sigma_2 = l_2/T + n\varepsilon_1$, $\sigma_3 = m_2/T + n\varepsilon_1$. Further, fix $H \ge H_0$ and divide $\mathcal{O}_{1i}$ into the elementary sets $\mathcal{M}_{1i} = I_i \times K_i \times D_i$ so that $\mu_1(I_i) = H^{-\sigma_1}$, $\mu_2(K_i) = H^{-2\sigma_2}$, $\mu_h(D_i) = H^{-\sigma_3}$, that is, $\mu(\mathcal{M}_{1i}) = H^{-\sigma_1 - 2\sigma_2 - \sigma_3}$.

Definition. We say that polynomial $P_n \in \mathcal{P}_n(H, \overline{q}, \overline{r}, \overline{s})$ *belongs to the set* $\mathcal{M}_{1i}$ if there exists such point $(x, z, \omega) \in \mathcal{M}_{1i}$ that $|P_n(x)| < H^{\lambda_1 - \nu_1}$, $|P_n(z)| < H^{\lambda_2 - \nu_2}$, $|P_n(\omega)| < H^{\lambda_3 - \nu_3}$.

Let $\{\theta\} \ge \varepsilon$ where $\{\theta\}$ is a fractional part of θ. Consider the sets $\mathcal{M}_{1i}$ with more than H^{β} polynomials from $\mathcal{P}_n(H, \overline{q}, \overline{r}, \overline{s})$ belonging to each of them. Fix one of such set $\mathcal{M}_{1i_0}$. Divide all polynomials belonging to it into classes in the following manner. Two polynomials

$$P_{n1}(y) = Hy^n + a_{n-1}^{(1)} y^{n-1} + \cdots + a_1^{(1)} y + a_0^{(1)} ,$$
$$P_{n2}(y) = Hy^n + a_{n-1}^{(2)} y^{n-1} + \cdots + a_1^{(2)} y + a_0^{(2)}$$

belong to one class if $a_{n-1}^{(1)} = a_{n-1}^{(2)}, \ldots, a_{n-d}^{(1)} = a_{n-d}^{(2)}$, where $d = [\theta] - 1$ and $[\theta]$ is the integer part of θ. Since the number of different classes is not greater than $(2H+1)^d$ and the number of considered polynomials is greater than H^{β}, then there exits a class which has at least $\ll H^{\beta - d} = H^{0,9\varepsilon}$ polynomials according to Dirichlet's principle. Denote the polynomials of this class by $P_{n1}, \ldots, P_{nt}$ and construct $(t-1)$ new polynomials $R_{nj}(y) = P_{n(j+1)} - P_{nj}$ $(1 \le j \le t-1)$. Thus, starting from polynomials of degree n we reduce the problem to polynomials of degree not greater than $(q_1 + 2r_1 + m_1 + (k_2 + 2l_2 + m_2)/T) - 1$, satisfying the system of inequalities

$$|R_{nj}(x)| < H(R)^{(1 - q_1 - k_2/T - \varepsilon_1)/(1 - \{\theta\} - 0,1\varepsilon)} ,$$
$$|R_{nj}(z)| < H(R)^{(1 - r_1 - l_2/T - \varepsilon_1)/(1 - \{\theta\} - 0,1\varepsilon)} ,$$
$$|R_{nj}(\omega)|_p < H(R)^{(1 - s_1 - m_2/T - \varepsilon_1)/(1 - \{\theta\} - 0,1\varepsilon)} .$$

These inequalities are obtained from the estimation of the Taylor series for polynomial $R_{nj}(y)$ in the neighbourhoods of roots $(\alpha_1^{(nj)}, \alpha_q^{(nj)}, \beta_1^{(nj)})$ defined by the relations (3).

Assumption, that there exist a class containing more than H^β polynomials $P_{nt} \in \mathcal{P}_n(H, \overline{q}, \overline{r}, \overline{s})$ with condition (5), leads to a contradiction with the result from [7]. Whenever a class is containing less than H^β polynomials P_{nt}, the Theorem is proved without difficulty with the help of the inequalities (2) and the Borel-Cantelly lemma. This method with details can be found in [2; Proposition 2], [12; Proposition 2].

If $\{\theta\} < \varepsilon$, we use the former argument with the other parameters: $\beta = \theta - 1 + \varepsilon$, $\sigma_1 = k_2/T + 0.8$, $\sigma_2 = l_2/T$, $\sigma_3 = m_2/T$.

Case 3.
Let the condition (4) be true and $q_1 + 2r_1 + s_1 + (k_2 + 2l_2 + m_2)/T \leq 4 - \varepsilon/2$. Then the system (1) is investigated by the method of essential and inessential domains as in [2; Propositions 3, 4], [12; Propositions 3, 4].

Case 4.
Let

$$\begin{aligned} q_1 + k_2/T &> -\lambda_1 + \nu_1 + 1\,, \\ r_1 + l_2/T &\leq -\lambda_2 + \nu_2 + 1\,, \\ s_1 + m_2/T &\leq -\lambda_3 + \nu_3\,, \end{aligned}$$

and

$$2r_1 + s_1 + (2l_2 + m_2)/T > 3 + \lambda_1 - \nu_1 - \varepsilon/2\,. \tag{6}$$

Now we apply the argument of the case 2 with the parameters: $\theta = -2\lambda_2 - \lambda_3 + 2\nu_2 + \nu_3 + 2 - 2r_1 - s_1 - (2l_2 + m_2)/T$, $\sigma_1 = -\lambda_1 + \nu_1 + 1 - q_1$, $\sigma_2 = l_2/T$, $\sigma_3 = m_2/T$, $\beta = \theta - 1 - 0.1\varepsilon$.

Case 5.
Let the condition (6) be true and $2r_1 + s_1 + (2l_2 + m_2)/T \leq 3 + \lambda_1 - \nu_1 - \varepsilon/2$. Then the system (1) is investigated by the method of essential and inessential domains as in [2; Propositions 6, 7].

Case 6.
Let $q_1 + k_2/T > -\lambda_1 + \nu_1 + 1$, $r_1 + l_2/T > -\lambda_2 + \nu_2 + 1$, $s_1 + m_2/T \leq -\lambda_3 + \nu_3$. Following the argument of [2; Propositions 8, 9], we show that there exist integers a, b: $2 \leq a$, $b \leq n - 1$ such that

$$\begin{aligned} k_a/T &> (-\lambda_1 + \nu_1 + 1 - q_a - 1/T)/a \geq k_{a+1}/T\,, \\ l_b/T &> (-\lambda_2 + \nu_2 + 1 - r_b - 1/T)/b \geq l_{b+1}/T\,. \end{aligned} \tag{7}$$

In other words, there exist large derivations $\left|P_n^{(a)}(x)\right|$, $\left|P_n^{(b)}(z)\right|$ $\left|P_n'(\omega)\right|_p$ when $(x, z, \omega) \in \mathcal{O}_{1i}$. Then we use the argument of the case 2.

Case 7.
Let $q_1 + k_2/T > -\lambda_1 + \nu_1 + 1$, $r_1 + l_2/T > -\lambda_2 + \nu_2 + 1$, $s_1 + m_2/T > -\lambda_3 + \nu_3$. Following the argument of the case 6, we get (7) and show that there exits integer

$c\colon 2 \le c \le n-1$ such that $m_c/T > (-\lambda_3+\nu_3-s_c-1/T)/c \ge m_{c+1}/T$. Hence, we find the large derivations $\left|P_n^{(a)}(x)\right|$, $\left|P_n^{(b)}(z)\right|$, $\left|P_n^{(c)}(\omega)\right|_p$ when $(x,z,\omega) \in \mathcal{O}_{1i}$. Further we use the argument of the case 2 again.

Thus, the Theorem is proved.

4. Remarks

1. If $n = 2$ and y^2 is replaced by a function $f(y)$, where $f(y)$ is a normal function (by Mahler), $f\colon \mathbb{R} \times \mathbb{C} \times \mathbb{Z}_p \to \mathbb{R} \times \mathbb{C} \times \mathbb{Z}_p$, then an analogue of the convergence part of the Khintchine theorem was proved by N. Silaeva (2003).

2. Regarding the divergent part of the Khintchine theorem for $P_n(y)$ or $f(y)$, when $y \in \mathbb{R}$ and $f \in \mathcal{C}^3(\mathbb{R})$ or $f\colon \mathbb{C} \to \mathbb{C}$ and $f(y)$ is an analytic function, or $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ and $y \in \mathbb{Z}_p$, respectively, we can mention that these results were obtained by V. Beresnevich (1999-2003).

3. The divergent part of our result is the next step of the investigation.

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