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B*-IDEALS AND Q*-IDEALS IN NON-COMMUTATIVE SEMIRINGS

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Introduction. A semiring $(\mathfrak{S}, +, \cdot)$ is a set $\mathfrak{S}$ with two binary operations $+$ and $\cdot$, such that $(\mathfrak{S}, +)$ and $(\mathfrak{S}, \cdot)$ are semigroups and $\cdot$ distributes over $+$: $a(b+c) = ab + ac$, $(b+c)a = ba + ca$. A subset $\boldsymbol{a}$ of a semiring $\mathfrak{S}$ will be called an ideal if $a, b \in \boldsymbol{a}$ and $s \in \mathfrak{S}$ implies $a + b \in \boldsymbol{a}$, $sa \in \boldsymbol{a}$ and $as \in \boldsymbol{a}$. The nuclear ideal, i.e., the intersection of all non empty ideals, will be noted by $\mathfrak{J}$. We shall consider B-ideals, B*-ideals, Q-ideals and Q*-ideals; all these define congruences over $\mathfrak{S}$. The first two types of ideals extend the concept of congruence due to Bourne [3]; those congruences which depend either on Q — or on Q*-ideals extend the concept of congruence due to Allen [1]. We examine the relations between the different mentioned ideals and Henriksen's k-ideals, and this leads to a generalization of LaTorre's results [5]. In defining k_a-ideals by a unilateral condition, we extend the notion of the k-ideal. Epimorphisms will be characterized according to the nature of their kernel. One can then state (theorem 6) a result on semi-isomorphisms in the sense of Bourne [4] and another (corollary 3) on isomorphisms, in which one uses in a convenient form Allen's notion [1] of maximal epimorphism. Theorem 4′ and 5′ exploit an idea of Margarita Ramalho and extend a theorem by LaTorre [5] and another by Allen [1]. Finally, assuming the existence of a semiring of quotients of $\mathfrak{S}$, we shall prove several statements concerning ideals.

2. B*-ideals

An idèal $\boldsymbol{c} \neq \emptyset$ defines a reflexive and symetric relation β_c over $\mathfrak{S}$ (Bourne's relation denoted simply by β when there is no danger of ambiguity) in the following way: $x\beta y$ if and only if there exists $c, c_0 \in \boldsymbol{c}$ such that $x + c = y + c_0$. This relation which is such that $x_1\beta y_1$, $x_2\beta y_2$ imply $(x_1x_2)\beta(y_1y_2)$ is not transitive in general.

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An ideal $\boldsymbol{a} \neq \emptyset$ is said to be *normal,* if and only if, given any $a \in \boldsymbol{a}$ and any $x \in \mathfrak{S}$, there exists $a_0 \in \boldsymbol{a}$ such that $a + x = x + a_0$. In this case, β is already transitive. Furthermore, β is a congruence relation. One extends thus the Bourne congruence given in the case of a commutative addition.

The elements of a normal ideal $\boldsymbol{a}$ all belong to the same β-congruence class $C_{\boldsymbol{a}}$. $C_{\boldsymbol{a}}$ is an ideal and the zero element for the quotient semiring $\mathfrak{S}/\beta$. From this fact one concludes that $C_{\boldsymbol{a}}$ defines over $\mathfrak{S}$ a congruence relation $\beta_{C_{\boldsymbol{a}}} = \beta_{\boldsymbol{a}}$. It may be either $\boldsymbol{a} = C_{\boldsymbol{a}}$ or $\boldsymbol{a} \subset C_{\boldsymbol{a}}$. In the latter case it may happen that $C_{\boldsymbol{a}}$ is not normal. An example is the following:

+	a	b	c
a	a	a	c
b	b	b	c
c	c	c	c

,

·	a	b	c
a	c	c	c
b	c	c	c
c	c	c	c

.

In the above the ideal $\boldsymbol{a} = \{c\}$ is normal, but $C_{\boldsymbol{a}} = \{a, b, c\}$ is not.

Consequently, a non normal ideal can define a congruence. We have another example taking a non empty ideal $\boldsymbol{a} \neq \mathfrak{S}$ of a G-semiring (characterized by the following rule for addition: $x + y = y$, $\forall x, y \in \mathfrak{S}$).

An ideal $\boldsymbol{b} \neq \emptyset$ is said to be a *B-ideal,* if and only if the corresponding relation β is a congruence and the elements of $\boldsymbol{b}$ are congruent. We shall call $\mathfrak{S}/\beta$ a *Bourne quotient semiring* and denote by $C_{\boldsymbol{b}}$ the class which contains $\boldsymbol{b}$. $C_{\boldsymbol{b}} \supseteq \boldsymbol{b}$ is an ideal and a right zero for the addition in $\mathfrak{S}/\beta$. Consequently, $\boldsymbol{b}$ and $C_{\boldsymbol{b}}$ define the same congruence and $C_{\boldsymbol{b}}$ is a B-ideal.

Keeping in mind that $C_{\boldsymbol{b}}$ is also a multiplicative zero of $\mathfrak{S}/\beta$, we have:

Theorem 1. *A B-ideal $\boldsymbol{b}$ defines a congruence β and the congruence class $C_{\boldsymbol{b}} \supseteq \boldsymbol{b}$ is a B-ideal that defines the same congruence and that is a right additive and multiplicative zero of $\mathfrak{S}/\beta$.*

In LaTorre [5], one finds the concept of the k-ideal of M. Henriksen. $\boldsymbol{k}$ is a k-ideal, if and only if $a + x \in \boldsymbol{k}$, $y + a' \in \boldsymbol{k}$, with $a, a' \in \boldsymbol{k}$ and $x, y \in \mathfrak{S}$ imply x, $y \in \boldsymbol{k}$. Since $\mathfrak{S}$ is a k-ideal and the intersection of k-ideals is a k-ideal, there exists the k-ideal generated by a subset of $\mathfrak{S}$.

We recognize the importance of the k-ideals by the following example. Let $\mathfrak{S}$ be a lattice semiring [2], i.e., a semiring which is a lattice for which $x + y = x \vee y$, $xy \leqq x \wedge y$. The ideals of the lattice are ideals of the semiring; however the converse assertion is not true. One calls these ideals *lattice ideals.* Lattice ideals are identical with k-ideals, so that the k-ideals generated by a set of elements may be obtained in the following way: first construct the ideal $\boldsymbol{s}$ of the semiring generated by the said elements, then take the elements $x \in \mathfrak{S}$ such that $x \leqq x$ for some $x \in \boldsymbol{s}$.

Theorem 2. *If $\boldsymbol{b}$ is a B-ideal which is also a k-ideal, then $\boldsymbol{b} = C_b$.*

In fact, let $c \in C_b$; then $b\beta c$, with $b \in \boldsymbol{b}$, implies $b + b_0 = c + b_{00}$ $(b_0, b_{00} \in \boldsymbol{b})$. Since $\boldsymbol{b}$ is a k-ideal, $c + b_{00} \in \boldsymbol{b}$ and $b_{00} \in \boldsymbol{b}$ imply $c \in \boldsymbol{b}$.

Taken a B-ideal $\boldsymbol{b}$, whenever C_b is an additive zero of $\mathfrak{S}/\beta$, $\boldsymbol{b}$ will be said to be a B_0-ideal [6]. If $\boldsymbol{a}$ is a normal ideal, then $\boldsymbol{a}$ and C_a are B_0-ideals. In a G-semiring a non empty ideal is a B_0-ideal. A B-ideal $\boldsymbol{b}$ that contains a B_0-ideal is also a B_0-ideal.

Theorem 3. *If $\boldsymbol{b}$ is a B_0-ideal, the class C_b is a B_0-ideal and also the k-ideal generated by $\boldsymbol{b}$.*

In view of theorem 1, C_b is a B-ideal and $C_b \supseteq \boldsymbol{b}$ implies that C_b is a B_0-ideal. It is a k-ideal as well, because, for instance, $x + c \in C_b$, with $c \in C_b$, yields $C_{x+c} = C_b$, therefore $C_x = C_b$. Furthermore, C_b is the k-ideal generated by $\boldsymbol{b}$, since if $\boldsymbol{b} \subseteq \boldsymbol{k}$, where $\boldsymbol{k}$ is a k-ideal, from $c\beta b$ one gets $c + b_0 = b + b_{00}$, $(c \in C_b;\ b,\ b_0,\ b_{00} \in \boldsymbol{b})$, hence $c + b_0 \in \boldsymbol{k}$ and $c \in \boldsymbol{k}$.

Corollary 1. *If $\boldsymbol{b}$ is a B_0-ideal, $\boldsymbol{b}$ is a k-ideal if and only if $\boldsymbol{b} = C_b$.*

Corollary 2. *In a lattice semiring, $\boldsymbol{b}$ is a lattice ideal if and only if $\boldsymbol{b} = C_b$.*

Extension of results. Let σ be a congruence over $\mathfrak{S}$. In general an ideal $\boldsymbol{a}$ is partitioned by a set of classes $\{C_a\}$, $(a \in \boldsymbol{a},\ C_a \in \mathfrak{S}/\sigma)$, and $\cup C_a$ is an ideal of $\mathfrak{S}$. If every class C_a is a right zero for the addition in $\mathfrak{S}/\sigma$, The relation β defined by a will imply σ, for, if $x\beta y$, from $x + a = y + a_0$, $(a,\ a_0 \in \boldsymbol{a})$, one gets $C_x = C_y$, i.e. $x\sigma y$. Conversely, if $\beta \leqq \sigma$, then, since $(x + a)\beta x$, $(a \in \boldsymbol{a})$, one has $C_x + C_a = C_x$.

This being so, let us consider, following [7], an ideal $\boldsymbol{b}$ that defines a congruence relation β, but whose elements are not necessarily congruent with other. We shall call it *B*-ideal.* If $\boldsymbol{b}'$ is an ideal such that $\boldsymbol{b} \subseteq \boldsymbol{b}' \subseteq C_b$, $(b \in \boldsymbol{b})$, $\boldsymbol{b}'$ will define the same congruence, since each class of $\mathfrak{S}/\beta$ containing an element of $\boldsymbol{b}'$ is a right zero for the addition in $\mathfrak{S}/\beta$ and so $\beta_{\boldsymbol{b}'} \leqq \beta$; on the other hand $\beta \leqq \beta_{\boldsymbol{b}'}$. We can state:

Theorem 1′. *A B*-ideal $\boldsymbol{b}$ defines a congruence that is defined as well by every ideal $\boldsymbol{b}'$ such that $\boldsymbol{b} \subseteq \boldsymbol{b}' \subseteq \cup C_b$, $(b \in \boldsymbol{b},\ C_b \in \mathfrak{S}/\beta)$. $\cup C_b$ is a B*-ideal and its image $\{C_b, \ldots\}$ is an ideal of $\mathfrak{S}/\beta$, each element C_b being an additive right zero for $\mathfrak{S}/\beta$.*

We shall say that an ideal $\boldsymbol{k}$ is a *k_d-ideal* if and only if $x + a \in \boldsymbol{k}_d$, with $a \in \boldsymbol{k}_d$, implies $x \in \boldsymbol{k}_d$. In any natural epimorphism $\mathfrak{S} \to \mathfrak{S}/\beta$, defined by a B*-ideal $\boldsymbol{b}$, the image of this ideal is the k_d-ideal $\{C_b \mid b \in \boldsymbol{b}\}$, whose complete inverse image is $\cup C_b$. However, in any epimorphism $\mathfrak{S} \to \mathfrak{S}'$ a subset $\mathfrak{S}' \subseteq \mathfrak{S}'$ is a k_d-ideal if and only if its complete inverse image is one as well. Therefore $\cup C_b$ is a k_d-ideal of $\mathfrak{S}$, precisely the k_d-ideal generated by $\boldsymbol{b}$. Thus:

Theorem 2′. *If $\boldsymbol{b}$ is a B*-ideal, $\boldsymbol{b}$ is a k_d-ideal if and only if $\boldsymbol{b} = \cup C_b$.*

3. Q*-ideals

The B-ideal $\boldsymbol{b}$ will be called a Q-ideal if and only if it satisfies the two following conditions: i) given $\mathfrak{S}/\beta = \{C_b, C_a, \ldots, C_q, \ldots\}$, there exists a set $Q = \{b_0, a, \ldots, q, \ldots\}$, $(b_0 \in C_b)$, of class representatives such that $q + b = C_q$, $\forall q \in Q$; ii) $\boldsymbol{b} + q \subseteq q + \boldsymbol{b}$, $\forall q \in Q$. This concept can be found in [1] for the case where addition is commutative.

We wish to remark that: 1) the set Q is not, in general, uniquely determined; 2) $q \in q + \boldsymbol{b}$; 3) if $C_x = C_q = q + \boldsymbol{b}$, then $x + \boldsymbol{b} \subseteq q + \boldsymbol{b}$, so that in the family of sets $\{x + \boldsymbol{b}\}$, $(x \in \mathfrak{S})$, the sets $\{q + \boldsymbol{b}\}$, $(q \in Q)$, are maximal; 4) in the natural epimorphism $\mathfrak{S} \to \mathfrak{S}/\beta$ one has $x \to C_x = q + \boldsymbol{b}$ and this is a class independent of Q; 5) every Q-ideal is a B_0-ideal, since, if $C_x = C_1$, $C_b + C_x = C_{b_0} + C_q = C_{b_0+q} = C_{q+b_{00}} = C_q = C_x$, $(b_0, b_{00} \in \boldsymbol{b})$; 6) every Q-ideal is a k-ideal. Summing up:

Theorem 4. *If $\boldsymbol{b}$ is a Q-ideal, the quotient $\mathfrak{S}/\beta$ will be independent of the sets Q which may be considered. Furthermore, $\boldsymbol{b}$ is a B_0-ideal and equal to C_b, therefore a k-ideal.*

And also:

Theorem 5. *The Q-ideal $\boldsymbol{b}$ will be normal if and only if, given $b_0, b_1 \in \boldsymbol{b}$, there exists b_0' such that $b_0 + b_1 = b_1 + b_0'$, $(b_0' \in \boldsymbol{b})$.*

It is enough to verify the sufficiency. Put $x = q + b_1$, with $x \in \mathfrak{S}$, $b_1 \in \boldsymbol{b}$; given $b \in \boldsymbol{b}$ one has $b + x = b + q + b_1$. The property ii) yields $b + q = q + b_0$, $(b_0 \in \boldsymbol{b})$, therefore $b + x = q + b_0 + b_1$; since, by hypothesis, $b_0 + b_1 = b_1 + b_0'$, one has $b + x = q + b_1 + b_0' = x + b_0'$, and $\boldsymbol{b}$ is normal.

Extension of results. Take an ideal $\boldsymbol{a}$ and assume the existence of a set Q^* such that $\{q + \boldsymbol{a}\}$, $(q \in Q^*)$ is a partition of $\mathfrak{S}$. When the equivalence relation α so defined is a congruence, $\boldsymbol{a}$ is said to be a *Q*-ideal.* The following statement holds:

Theorem 3′. *If $\boldsymbol{a}$ is a Q*-ideal, one has: i) $x \in q + \boldsymbol{a}$ if and only if $q + \boldsymbol{a} = q + \boldsymbol{a}$; ii) the congruence classes $q + \boldsymbol{a}$, $(q \in Q^*)$, containing elements of $\boldsymbol{a}$ are right additive zeros in $\mathfrak{S}/\alpha$; iii) an ideal $\boldsymbol{a}'$ such that $\boldsymbol{a} \subseteq \boldsymbol{a}' \subseteq \cup(q_a + \boldsymbol{a})$, where the $q_a \in Q^*$ are such that $q_a + \boldsymbol{a}$ contains elements of $\boldsymbol{a}$, is a Q*-ideal, as well; iv) $\cup(q_a + a)$ is a k_d-ideal generated by $\boldsymbol{a}$.*

With regard to *i*): if $x + \boldsymbol{a} \subseteq q + \boldsymbol{a}$, assume that $x = q_1 + a \in q_1 + \boldsymbol{a}(q_1 \in Q^*$, $a \in \boldsymbol{a})$; then $x + \boldsymbol{a} \subseteq q_1 + \boldsymbol{a}$ and $q + \boldsymbol{a} = q_1 + \boldsymbol{a}$, $q = q_1$. With regard to *ii*): assume that $a \in q_1 + \boldsymbol{a}$; since $q \in q + \boldsymbol{a}$, $(q + \boldsymbol{a}) + (q_1 + \boldsymbol{a})$ is a class containing $q + a$ and this class can only be $q + a$. As regards *iii*), one notices that $q + \boldsymbol{a} = q + \boldsymbol{a}'$, $\forall q \in Q$ is satisfied, since, if $x \in \boldsymbol{a}'$, then $x \in q_a + \boldsymbol{a}$, for some q_a, and one has $q + x \in q + q_a + \boldsymbol{a} \subseteq (q + \boldsymbol{a}) + (q_a + \boldsymbol{a}) = q + \boldsymbol{a}$. As to *iv*), $\cup(q_a + \boldsymbol{a})$ is certainly a k_d-ideal, and if one suppose $\boldsymbol{a} \subseteq \boldsymbol{k}_d$, then, if $a \in q_a + \boldsymbol{a}$, $(a \in \boldsymbol{a})$, one has $a = q_a + a_1$, $(a_1 \in \boldsymbol{a})$, since a, $a_1 \in \boldsymbol{k}_d$, $q_a \in \boldsymbol{k}_d$ and $q_a + \boldsymbol{a} \subseteq \boldsymbol{k}_d$, $\cup(q_a + \boldsymbol{a}) \subseteq \boldsymbol{k}_d$.

4. Morphisms

Let $\mathfrak{S} \rightarrow \mathfrak{S}'$ be an epimorphism of semirings and assume that $\mathfrak{I}'$ is the nuclear ideal of $\mathfrak{S}'$. The complete inverse image $\mathfrak{N}$ of $\mathfrak{I}'$ is the kernel of the epimorphism. Following [4], we shall call an epimorphism a semi-isomorphism if $\mathfrak{N} = \mathfrak{I}$. Now we have: the epimorphism cannot be a semi-isomorphism, unless $\mathfrak{I}$ and $\mathfrak{I}'$ are both empty or both non empty. In the former case, one always has a semi-isomorphism, while in the latter it will be a semi-isomorphism if and only if $\mathfrak{I}$ is an ideal saturated with regard to the congruence defined by the epimorphism.

An epimorphism $\mathfrak{S} \rightarrow \mathfrak{S}'$ is said to be a *B-epimorphism (B_0-epimorphism),* if the kernel is a B-ideal (B_0-ideal). This certainly happens in the natural epimorphism $\mathfrak{S} \rightarrow \mathfrak{S}/\beta$, if $\boldsymbol{b}$ is a B-ideal (B_0-ideal). One can, as a general rule, write either $\mathfrak{S}/\boldsymbol{b}$ or $\mathfrak{S}/\beta$ on the quotient semiring.

Theorem 6. *Let $\mathfrak{S} \rightarrow \mathfrak{S}'$ be a B-epimorphism and suppose $\mathfrak{I}' = C_{\mathfrak{I}'}$ a B-ideal; then there exists a semi-isomorphism $\mathfrak{S}/\mathfrak{N} \rightarrow \mathfrak{S}/\mathfrak{I}'$,* ($\mathfrak{N} = \operatorname{Ker} \varphi$).

The morphism obtained by the composition $\mathfrak{S} \rightarrow \mathfrak{S}' \rightarrow \mathfrak{S}'/\mathfrak{I}'$ is a B-epimorphism whose kernel is $\mathfrak{N}$; then it suffices to show if $\mathfrak{S} \rightarrow \mathfrak{S}'$ is a B-epimorphism and $\mathfrak{S}'$ has a right additive and multiplicative zero, there exists a semi-isomorphism $\mathfrak{S}/\mathfrak{N} \rightarrow \mathfrak{S}'$ ($\mathfrak{N} = \operatorname{Ker} \varphi_0$). If σ is the congruence relation defined by φ_0 and β the congruence defined by $\mathfrak{N}$, it suffices to show that $\beta \leqq \sigma$. Since the ideal $\mathfrak{N}$ is contained in a unique class defined by σ, and this is $\mathfrak{N}$ itself, which is a right additive zero for $\mathfrak{S}/\sigma$, we have shown in § 2 (extension of results) that as required $\beta \leqq \sigma$.

We shall give now an isomorphism theorem concerning B*-ideals which extends a known result [5]. The theorem is based on two lemmas.

Lemma 1'. *Let $\mathfrak{S} \rightarrow \mathfrak{S}'$ be an epimorphism and σ the congruence it defines. Then: i) if $\boldsymbol{a}$ is an ideal such that $\sigma \leqq \beta_{\boldsymbol{a}}$, then $x\beta_{\boldsymbol{a}}y$ if and only if $\varphi(x)\beta_{\varphi(\boldsymbol{a})}\varphi(y)$; ii) also with $\sigma \leqq \beta_{\boldsymbol{a}}$, $\boldsymbol{a}$ is a B*-ideal if and only if $\varphi(\boldsymbol{a})$ is one as well; iii) in the last case, we have $\mathfrak{S}/\beta_{\boldsymbol{a}} \simeq \mathfrak{S}'/\beta_{\varphi(\boldsymbol{a})}$.*

With regard to *i*): the necessity is obvious. We show the sufficiency. Assuming that $\varphi(x)\beta_{\varphi(\boldsymbol{a})}\varphi(y)$, one will have $\varphi(x) + \varphi(a) = \varphi(y) + \varphi(a_0)$, $(a, a_0 \in \boldsymbol{a})$. Then from $\varphi(x+a) = \varphi(y+a_0)$, one gets $(x+a)\sigma(y+a_0)$, therefore $(x+a)\beta_{\boldsymbol{a}}(y+a_0)$, hence $x\beta_{\boldsymbol{a}}y$. With regard to *ii*) and *iii*), it suffices to verify that there is a $1-1$ correspondence between the congruence classes "modulo-$\beta_{\boldsymbol{a}}$" and the congruence classes "modulo $\beta_{\varphi(\boldsymbol{a})}$" and that this correspondence preserves addition and multiplication.

Lemma 2'. *Let us suppose $\boldsymbol{a}$ a B*-ideal and $\boldsymbol{b} \supseteq \boldsymbol{a}$ a k_d-ideal; then in the natural epimorphism $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}/\beta_{\boldsymbol{a}}$ the image $\bar{\boldsymbol{b}}$ of $\boldsymbol{b}$ is a k_d-ideal and one has $\bar{\boldsymbol{b}} = \boldsymbol{b}/\beta_{\boldsymbol{a}}$.*

We begin by verifying that the ideal $\boldsymbol{b}$ is saturated with regard to the congruence $\beta_{\boldsymbol{a}}$. Assuming $x\beta_{\boldsymbol{a}}b$, $(b \in \boldsymbol{b})$, from $x+a=b+a_0$, $(a, a_0 \in \boldsymbol{a})$, one gets $x+a \in \boldsymbol{b}$, therefore $x \in \boldsymbol{b}$. From what we saw in §2, $\boldsymbol{b}$ is a k_d-ideal since $\boldsymbol{b}$ is one as well; moreover, one will have $\bar{\boldsymbol{b}}=\boldsymbol{b}/\beta_{\boldsymbol{a}}$. Now the theorem:

Theorem 4'. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$, with $\boldsymbol{b} \supseteq \boldsymbol{a}$, be B^*-ideals. If $\boldsymbol{b}$ is a k_d-ideal the following isomorphism will take place:* $\mathfrak{S}/\beta_{\boldsymbol{b}} \simeq (\mathfrak{S}/\beta_{\boldsymbol{a}})/(\boldsymbol{b}/\beta_{\boldsymbol{a}})$.

Let us consider now the case of the natural epimorphism $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}/\boldsymbol{b}$, where $\boldsymbol{b}$ is supposed to be a Q-ideal. Given $\bar{x} \in \mathfrak{S}/\boldsymbol{b}$, there is $q \in Q$ such that $\varphi(q)=\bar{x}$ and $\varphi^{-1}(x)=q+\boldsymbol{b}$, with $\boldsymbol{b}=\operatorname{Ker}\varphi$. We shall then call a B_0-epimorphism $\mathfrak{S} \to \mathfrak{S}'$ maximal if $\mathfrak{J}'$ is a Q-ideal and if, given $x' \in \mathfrak{S}'$, there exists $v \in \mathfrak{S}$ such that $\varphi(v)=x'$ and $\varphi^{-1}(x'+\mathfrak{J}') = v+\mathfrak{N}$, with $\mathfrak{N}=\operatorname{Ker}\varphi$.

Lemma 1. *When $\mathfrak{S}'$ contains a zero $0'$ the kernel of a maximal epimorphism $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}'$ is a Q-ideal.*

By hypothesis the kernel is a B_0-ideal; moreover, given $x' \in \mathfrak{S}'$, there is $q \in \mathfrak{S}$ such that $\varphi^{-1}(x')=q+\mathfrak{N}$. The classes $\varphi^{-1}(x')$ are disjoint, and the set of elements q, previously chosen, is such that $C_q=q+\mathfrak{N}$. On the other hand $\varphi(\mathfrak{N}+q)=x'$ and so $\mathfrak{N}+q \subseteq q+\mathfrak{N}$.

Theorem 7. *A Q-ideal $\boldsymbol{b}$ determines a maximal epimorphism $\mathfrak{S} \to \mathfrak{S}/\boldsymbol{b}=\mathfrak{S}'$ where $\mathfrak{S}'$ has a zero element. Conversely, a maximal epimorphism $\mathfrak{S} \xrightarrow{q} \mathfrak{S}'$, in case there exists $0' \in \mathfrak{S}'$, will have a kernel $\boldsymbol{b}$ which is a Q-ideal and the following isomorphism holds:* $\mathfrak{S}/\operatorname{Ker}\varphi \cong \mathfrak{S}'$.

In fact, the Bourne congruence defined by $\boldsymbol{b}$ coincides with the one defined by φ.

Corollary 3. *If $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}'$ is a maximal epimorphism, we have an isomorphism* $\mathfrak{S}/\mathfrak{N} \simeq \mathfrak{S}'/\mathfrak{J}'$, *with* $\mathfrak{N}=\operatorname{Ker}\varphi$.

The extension of this corollary is interesting when one introduces Q*-ideals. We begin with a lemma.

Lemma 3'. *Let $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}'$ be an epimorphism, $\boldsymbol{b}'$ a Q^*-ideal of $\mathfrak{S}'$ and $\{\ldots, q', \ldots\}$ the set Q^* which corresponds to $\boldsymbol{b}$; then, according to the hypothesis $\varphi^{-1}(q'+\boldsymbol{b}') = q+\varphi^{-1}(\boldsymbol{b}')$, for certain elements $q \in \mathfrak{S}$, the ideal $\varphi^{-1}(\boldsymbol{b}')=\boldsymbol{b}$ is a Q^*-ideal and one has* $\mathfrak{S}/\boldsymbol{b} \simeq \mathfrak{S}'/\boldsymbol{b}'$.

$\mathfrak{S}=\cup(q+\varphi^{-1}(\boldsymbol{b}'))$ is a disjoint union and so the set of elements q is a Q*-set. The equivalence relation defined by this partition is a congruence in view of the following. Let $q_1, q_2 \in Q^* \subseteq \mathfrak{S}$ and suppose q_1', q_2', q_3' conveniently chosen: one has

$$\varphi[(q_1+\varphi^{-1}(\boldsymbol{b}'))+(q_2+\varphi^{-1}(\boldsymbol{b}'))]\subseteq(q_1'+\boldsymbol{b}')+(q_2'+\boldsymbol{b}')\subseteq q_3'+\boldsymbol{b}',$$

$$(q_1+\varphi^{-1}(\boldsymbol{b}'))+(q_2+\varphi^{-1}(\boldsymbol{b}'))\subseteq\varphi^{-1}(q_3'+\boldsymbol{b}')=q_3+\varphi^{-1}(\boldsymbol{b}');$$

and similarly for products. Hence the isomorphism.

Theorem 5'. *In a maximal epimorphism* $\mathfrak{S}\to\mathfrak{S}'$, $\operatorname{Ker}\varphi$ *is a* Q^**-ideal and the following isomorphism takes place:* $\mathfrak{S}/\operatorname{Ker}\varphi\simeq\mathfrak{S}'/\mathfrak{J}'$.

5. Transfer problems

Following D. A. Smith [8], we say: *i*) a subset D of the multiplicative semigroup of $\mathfrak{S}$ will be called a *right divisor set* if it consists of cancellable elements; it is closed for multiplication; and it has the property of the *right common multiple*, i.e., given $a\in\mathfrak{S}$, $\delta\in D$, there are $\gamma\in D$, $x\in\mathfrak{S}$ such that a $\gamma=\delta x$; *ii*) a semiring $\mathfrak{Q}$ is said to be a semiring of right quotients of the semiring $\mathfrak{S}$ if $\mathfrak{Q}$ contains the identity and a subset isomorphic to $\mathfrak{S}$, so that the inclusion $\mathfrak{S}\subseteq\mathfrak{Q}$ has a meaning; moreover, $\mathfrak{S}$ contains cancellable elements and there is a subset D_0 of the set of such elements which is closed for multiplication and consists of elements invertible in $\mathfrak{Q}$; at last, every element of $\mathfrak{Q}$ can be written in the form $x\eta^{-1}$, $(x\in\mathfrak{S},\ \eta\in D_0)$.

Given $\mathfrak{S}$ and D, one can define in the cartesian product $\mathfrak{S}\times D$ an equivalence relation by putting $(a,b)\approx(c,d)$, with $a,c\in\mathfrak{S}$ and $b,d\in D$ if and only if $bx=dy$ implies $ax=cy$. Denoting by a/b the equivalence class which contains (a,b), the set of all classes will constitute a semiring if one defines addition and multiplication as follows:

$$\frac{a}{b}+\frac{c}{d}=\frac{az+cv}{d},\quad \text{if}\quad dv=bz,\quad (v\in D,\ z\in\mathfrak{S}),$$
$$\frac{a}{b}\cdot\frac{c}{d}=\frac{az}{dv},\quad \text{if}\quad cv=bz,\quad (v\in D,\ z\in\mathfrak{S}). \tag{1}$$

Then the following proposition holds: $\mathfrak{S}$ will have a semiring of right quotients $\mathfrak{S}_D$, if and only if its multiplicative semigroup contains a right divisor set D. The semiring of right quotients is the one defined by the rules (1).

In what follows we shall suppose that both operations $+$ and $\cdot$ are commutative in $\mathfrak{S}$, and we shall employ Greek letters to denote elements of D, although denominators such as b or d belong to D, as well.

We shall be concerned with several statements.

1) The operations $+$ and $\cdot$, in $\mathfrak{S}_D$, are also commutative. Let us put

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{az}{dv},\quad \frac{c}{d}\cdot\frac{a}{b}=\frac{cy}{b\mu},\quad \text{with}\quad cv=bz,\quad a\mu=dy.$$

We have to show $(az, dv) \approx (cy, b\mu)$ but, as was formulated by D. A. Smith, it is only necessary to show that $dvx = b\mu\xi$, $(x \in \mathfrak{S}, \xi \in D)$ implies $azx = cy\xi$. Let us assume that $dvx = b\mu\xi$, then we have successively

$$dvxcy = b\mu\xi cy, \quad a\mu vxc = b\mu\xi cy, \quad avxc = b\xi cy,$$

$$abzx = b\xi cy, \quad azx = cy\xi.$$

Further, let us put

$$\frac{a}{b} + \frac{c}{d} = \frac{az + c}{dv}, \quad \frac{c}{d} + \frac{a}{b} = \frac{cy + a\varrho}{b\varrho}, \quad \text{with} \quad dv = bz,$$

$$b\varrho = dy.$$

We wish to show that $(az + cv, dv) \approx (cy + a\varrho, b\varrho)$. From $dvx = b\varrho\eta$, $(x \in \mathfrak{S})$ we obtain successively

$$bzx = b\varrho\eta, \quad zx = \varrho\eta, \quad azx = a\varrho\eta,$$

and quite similarly, from the same equality we obtain, successively

$$dvx = dy\eta, \quad vx = y\eta, \quad cvx = cy\eta.$$

Consequently, $azx + cvx = cy\eta + a\varrho\eta$ and this completes the proof.

2) Now, let $\boldsymbol{a}$ be an ideal of $\mathfrak{S}$ and let $\boldsymbol{a}'$ be the ideal of $\mathfrak{S}_D$ generated by $\boldsymbol{a}$. Each element a' of $\boldsymbol{a}'$ can be written in the form

$$a' = \Sigma m_i a_i + \Sigma b_j t'_j, \quad (a_i, b_j \in \boldsymbol{a}\ ;\ t'_j \in \mathfrak{S}_D),$$

where the sums are finite and each m_i is a positive integer. Since a finite number of elements of $\mathfrak{S}_D$ can always be represented in such a way that they have the same denominator in D, the element a' of $\boldsymbol{a}'$ can be written

$$a' = a_0\sigma^{-1}, \quad \text{with} \quad a_0 \in \boldsymbol{a}, \ \sigma \in D.$$

For example, taken

$$a' = m_1a_1 + m_2a_2 + b_1t'_1 + b_2t'_2, \quad \left(t'_1 = \frac{t_1}{\xi}, \ t'_2 = \frac{t_2}{\eta}\right),$$

we have

$$b_1t_{1'} + b_2t_{2'} = \frac{b_1\xi}{\xi}\cdot\frac{t_1}{\xi} + \frac{b_2\eta}{\eta}\cdot\frac{t_2}{\eta} = \frac{b_1x}{v} + \frac{b_2y}{\mu},$$

$$\text{with} \quad t_1v = \xi x, \quad t_2\mu = \eta y,$$

and

$$m_1a_1 + m_2a_2 = \frac{m_1a_1v\mu}{v\mu} + \frac{m_2a_2v\mu}{v\mu},$$

therefore

$$a' = \frac{a_0}{\sigma}, \quad \text{with} \quad \sigma = \mu\nu,$$

$$a_0 = m_1 a_1 \nu\mu + m_2 a_2 \nu\mu + b_1 x\mu + b_2 y\nu.$$

3) Let $\boldsymbol{k}$ be a k-ideal of $\mathfrak{S}$ and denote by $\boldsymbol{k}'$ the ideal of $\mathfrak{S}_D$ generated by $\boldsymbol{k}$. If we take $a' + x' \in \boldsymbol{k}'$ and if we put

$$a' = \frac{a}{\xi}, \quad x' = \frac{x}{\eta}, \quad (a \in \boldsymbol{k}),$$

we obtain

$$\frac{a\eta + x\xi}{\xi\eta} = \frac{k}{\xi}, \quad (k \in \boldsymbol{k}),$$

and further

$$\frac{a\eta\xi + x\xi\xi}{\xi\eta\xi} = \frac{k\xi\xi}{\xi\eta\varrho},$$

which implies $a\eta\xi + x\xi\zeta = k\xi\zeta$, therefore

$$x\xi\zeta \in \boldsymbol{k}, \quad \text{and} \quad \frac{x\xi\zeta}{\xi\eta\zeta} = \frac{x}{\xi} \in \boldsymbol{k}'.$$

4) Let $\boldsymbol{a}$ and $\boldsymbol{a}'$ be as above and consider the classes $C_{\boldsymbol{a}}$, $C_{\boldsymbol{a}'}$ and the ideal $(C_{\boldsymbol{a}})'$ of $\mathfrak{S}_D$ generated by $C_{\boldsymbol{a}}$. Given $c\xi^{-1} \in (C_{\boldsymbol{a}})'$, with $c \in C_{\boldsymbol{a}}$, one has $c + a_0 = a + a_{00}$, for some a_0, a, $a_{00} \in \boldsymbol{a}$, therefore

$$\frac{c}{\xi} + \frac{a_0}{\xi} = \frac{a}{\xi} + \frac{a_{00}}{\xi},$$

and this implies $(C_{\boldsymbol{a}})' \subseteq C_{\boldsymbol{a}'}$. Assuming now

$$x'\beta_{\boldsymbol{a}'}a' \left(x' = \frac{x}{\xi}, \; a' = \frac{a}{\xi} \in \boldsymbol{a}'\right),$$

one gets

$$\frac{x}{\xi} + \frac{a_1}{\xi} = \frac{a}{\xi} + \frac{a_2}{\xi}$$

which implies $x + a_1 = a + a_2$, therefore $x \in C_{\boldsymbol{a}}$ and $x' \in (C_{\boldsymbol{a}})'$.

Summarizing

Theorem 8. *If the operations $+$ and $\cdot$ are commutative in $\mathfrak{S}$ and if there is a semiring of quotients $\mathfrak{S}_D$ of $\mathfrak{S}$, one has:* 1) *the operations $+$ and $\cdot$ are also commutative in $\mathfrak{S}_D$;* 2) *an ideal $\boldsymbol{a}$ (necessarily normal) of $\mathfrak{S}$ generated in $\mathfrak{S}_D$ an ideal $\boldsymbol{a}' = \{a' \in \mathfrak{S}_D \mid a' = a_0\xi^{-1}, a_0 \in \boldsymbol{a}, \xi \in D\}$;* 3) *the ideal $\boldsymbol{k}'$ generated in $\mathfrak{S}_D$ by* *a k-ideal of* $\mathfrak{S}$ *is also a k-ideal;* 4) *the class* $C_{a'}$ *is the ideal* $(C_a)'$ *generated in* $\mathfrak{S}_D$ *by the class* C_a.

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