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OSCILLATORY BEHAVIOUR OF CERTAIN DIFFERENCE EQUATIONS

S. R. GRACE

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ABSTRACT. Some new criteria for the oscillation of second order difference equations of the form

$$\Delta^2 x_n + p_n \Delta x_{n-h} = q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n},$$

and

$$\Delta^2 x_n = p_n \Delta x_{n+h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n}$$

are established.

1. Introduction

In this paper, we are concerned with the oscillation of the solutions of certain second order difference equations of the form

$$\Delta^2 x_n + p_n \Delta x_{n-h} = q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n}, \tag{E$_1$}$$

and

$$\Delta^2 x_n = p_n \Delta x_{n+h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n}, \tag{E$_2$}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}_{n\ge 0}$ and $\{q_n\}_{n\ge 0}$ are sequences of nonnegative real numbers, $\{g_n\}_{n\ge 0}$ is a nondecreasing sequence of nonnegative integers with $g_n \to \infty$ as $n \to \infty$, h is a positive integer and c is a positive real number.

A nontrivial solution $\{x_k\}_{k\ge 0}$ of (E$_1$) (or (E$_2$)) is said to be *oscillatory* if for every positive integer N, there exists an $n \ge N$ such that $x_n x_{n+1} \le 0$ and *nonoscillatory* otherwise.

Equation (E$_i$), $i = 1, 2$, is said to be *almost oscillatory* if for every solution $\{x_n\}$ of (E$_i$), either $\{x_n\}$ is oscillatory or $\{\Delta x_n\}$ is oscillatory.

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There is an extensive literature on the topic of oscillation criteria for the generalized Emden-Fowler functional differential equation

$$x''(t) + q(t)\left|x(g(t))\right|^c \operatorname{sgn} x(g(t)) = 0\,, \qquad c > 0\,, \tag{F}$$

where $g, q\colon [t_0, \infty) \to \mathbb{R}$ are continuous and $g(t) \to \infty$ as $t \to \infty$. Few results are known regarding the oscillatory behaviour of the continuous analogues of equations (E_i), $i = 1, 2$, namely the functional differential equations

$$x''(t) + p(t)x'(t-h) = q(t)\left|x(g(t))\right|^c \operatorname{sgn} x(g(t))\,, \tag{F$_1$}$$

and

$$x''(t) = p(t)x'(t+h) + q(t)\left|x(g(t))\right|^c \operatorname{sgn} x(g(t))\,, \tag{F$_2$}$$

where c and h are positive constants, $p, q\colon [t_0, \infty) \to [0, \infty)$ are continuous and the function $g(t)$ is defined as in (F). For recent contributions we refer to the papers [1] - [4] and the references cited therein.

Oscillation criteria for the discrete analogue of (F), namely the difference equation

$$\Delta^2 x_n + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n} = 0\,, \qquad c > 0\,, \tag{E}$$

have been investigated by a number of authors in recent years (see for example [5] - [10] and the reference cited therein), but the literature is relatively limited. It seems that nothing is known about the oscillation of (E_i), $i = 1, 2$. Therefore, the purpose of this paper is to establish some new criteria for the oscillation of (E_i), $i = 1, 2$. We also mention that the results of this paper are not applicable to equations of type (E_i), $i = 1, 2$, with either $h = 0$ or $p_n = 0$.

The following properties of Δ are needed. For every N, $n \geq N$

(i) $\Delta u_i = u_{i+1} - u_i$,

(ii) $\sum\limits_{i=N}^{n} u_i \Delta v_i = u_{n+1}v_{n+1} - u_N v_N - \sum\limits_{i=N}^{n} v_{i+1}\Delta u_i$,

(iii) $\Delta(u_n v_n) = v_{n+1}\Delta u_n + u_n \Delta v_n = u_{n+1}\Delta v_n + v_n \Delta u_n$.

2. Almost oscillatory character of (E_1)

The following result concerns the almost oscillatory character of (E_1) when $c > 1$.

THEOREM 1. *Suppose that $\Delta p_n \geq 0$, $0 < p_n < 1$ and $g_n \geq n + 1$ for $n \geq n_0 \geq 0$. If*

$$\liminf_{n\to\infty} \left(\frac{1}{h} \sum_{k=n-h}^{n-1} p_n \right) > \frac{h^h}{(1+h)^{(1+h)}}\,, \tag{1}$$

and

$$\sum_{k\geq n_0}^{\infty} P_{k+1,g_k-1}q_k = \infty\,, \qquad n_0 \geq 0\,, \tag{2}$$

where

$$P_{k+1,g_k-1} = \sum_{j=k+1}^{g_k-1}\left(\prod_{i=n_0}^{j}\frac{1}{1-p_i}\right)^{1-c},$$

then (E_1) *is almost oscillatory.*

P r o o f . Assume for the purpose of contradiction that (E_1) has a nonoscillatory solution $\{x_n\}$, which we may (and do) assume to be eventually positive. There exists $n_1 \geq n_0 + h + 1$ such that $x_n > 0$ and $x_{g_n} > 0$ for $n \geq n_1$. Next, we consider the following two cases:

(a) $\Delta x_n < 0$ eventually, (b) $\Delta x_n > 0$ eventually.

(a) Assume $\Delta x_n < 0$ eventually. From (E_1), we see that

$$\Delta^2 x_n + p_n \Delta x_{n-h} = q_n x_{g_n}^c \geq 0 \quad \text{eventually.}$$

Set $y_n = \Delta x_n < 0$ eventually. Then

$$\Delta y_n + p_n y_{n-h} \geq 0 \quad \text{eventually.}$$

Now, by a result similar to [8; Lemma 1.1(a)], we see that the equation

$$\Delta y_n + p_n y_{n-h} = 0 \tag{3}$$

has an eventually negative solution. But, in view of [10; Theorem 3] and condition (1), equation (3) is oscillatory, which is a contradiction.

(b) Assume $\Delta x_n > 0$ for $n \geq N \geq n_2 + h$. From (E_1), we obtain

$$\Delta x_n - \Delta x_N + \sum_{k=N}^{n-1} p_k \Delta x_{k-h} = \sum_{k=N}^{n-1} q_k x_{g_k}^c,$$

and since

$$\begin{aligned}\sum_{k=N}^{n-1} p_k \Delta x_{k-h} &= p_n x_{n-h} - p_N x_{N-h} - \sum_{k=N}^{n-1} x_{k-h}\Delta p_k\\ &\leq p_n x_{n-h} \leq p_n x_n\,,\end{aligned}$$

we have

$$\Delta x_n + p_n x_n \geq \sum_{k=N}^{n-1} q_k x_{g_k}^c, \qquad n \geq N\,. \tag{4}$$

Define the sequence $\{r_n\}$, $n \ge 0$, by the recurrence relation

$$r_{n+1} = \frac{r_n}{1-p_n}, \quad n = 0, 1, 2, \dots, \qquad r_0 > 0. \tag{5}$$

Next, multiply (4) by r_{n+1}. We get

$$\Delta(r_n x_n) \ge r_{n+1} \sum_{k=N}^{n-1} q_k x_{g_k}^c \qquad \text{for} \quad n \ge N. \tag{6}$$

Choose $N_1 \ge N$ and define $m = \max\left\{N_1, \max_{N \le n \le N_1} g_n\right\}$. Dividing (6) by $(r_{n+1}x_{n+1})^c$ and summing from $N+1$ to m, we obtain

$$\begin{aligned}\sum_{n=N+1}^{m} \frac{\Delta(r_n x_n)}{(r_{n+1}x_{n+1})^c} &\ge \sum_{n=N+1}^{m} (r_{n+1})^{1-c} \sum_{k=N}^{n-1} q_k (x_{g_k}/x_{n+1})^c \\ &\ge \sum_{k=N}^{m} q_k \sum_{n=k+1}^{g_k-1} (r_{n+1})^{1-c} (x_{g_k}/x_{n+1})^c.\end{aligned}$$

Since $x_{g_k} \ge x_{n+1}$ for $k+1 < n < g_k - 1$, we have

$$\sum_{n=N+1}^{m} \Delta(r_n x_n)/(r_{n+1}x_{n+1})^c \ge \sum_{k=N}^{m} q_k \left(\sum_{n=k+1}^{g_k-1} \left(\prod_{j \ge n_0 \ge 0}^{n} r_{n_0}/(1-p_n) \right)^{1-c} \right). \tag{7}$$

Now, from the proof of Theorem 4.1 in [7], it follows that

$$\sum^{\infty} \Delta z_i / z_{i+1}^c < \infty,$$

which is a contradiction. This completes the proof. □

The following theorem is concerned with the almost oscillatory character of (E_1) when $c = 1$.

THEOREM 2. *Suppose that* $\Delta p_n \ge 0$, $g_n \ge n+1$ *and* $0 < p_n < 1$ *for* $n \ge n_0 \ge 0$. *If condition* (1) *holds and*

$$\limsup_{n \to \infty} \sum_{k=n}^{g_n - 1} B_{k, g_n - 1} q_k > 1, \tag{8}$$

where

$$B_{k,g_n-1} = \sum_{s=k}^{g_n-1} \left(\prod_{j=s+1}^{g_n-1} (1-p_j) \right),$$

then (E_1) *is almost oscillatory.*

Proof. Let $\{x_n\}$ be an eventually positive solution of (E_1), say $x_n > 0$ and $x_{g_n} > 0$ for $n \ge n_1 \ge n_0 \ge 0$. As in the proof of Theorem 1, we consider the following two cases:

(a) $\Delta x_n < 0$ eventually, (b) $\Delta x_n > 0$ eventually.

(a) Assume $\Delta x_n < 0$ eventually. The proof of this case is similar to that of Theorem 1(a) and hence is omitted.

(b) Assume $\Delta x_n > 0$ for $n \ge N \ge n_2 + h$. Proceeding as in the proof of Theorem 1(b) and defining the sequence $\{r_n\}$ as in (5) we obtain

$$\Delta(r_s x_s) \ge r_{s+1} \sum_{k=n}^{s-1} q_k x_{g_n} \qquad \text{for} \quad s \ge n \ge N. \tag{9}$$

Summing both sides of (9) from n to g_{n-1}, we have

$$r_{g_n} x_{g_n} \ge r_{g_n} x_{g_n} - r_n x_n \ge \sum_{s=n}^{g_n-1} r_{s+1} \sum_{k=n}^{s-1} q_k x_{g_k},$$

or

$$\begin{aligned} 1 &\ge \sum_{s=n}^{g_n-1} (r_{s+1}/r_{g_n}) \sum_{k=n}^{s-1} q_k (x_{g_k}/x_{g_n}) \\ &\ge \sum_{k=n}^{g_n-1} q_k (x_{g_k}/x_{g_n}) \left(\sum_{s=k}^{g_n-1} r_{s+1}/r_{g_n} \right). \end{aligned}$$

Since $x_{g_k} \ge x_{g_n}$ for $n \le k \le g_n - 1$, we obtain

$$1 \ge \sum_{k=n}^{g_n-1} q_k \left(\sum_{s=k}^{g_n-1} \prod_{j=s+1}^{g_n-1} (1-p_j) \right),$$

which contradicts (8). This completes the proof. □

The following criterion deals with the almost oscillation of all bounded solutions of (E_1) for any $c > 0$.

THEOREM 3. *Suppose that* $\Delta p_n \geq 0$, $g_n \geq n+1$ *and* $0 < p_n < 1$ *for* $n \geq n_0 \geq 0$. *If condition* (1) *holds and*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n} \left(\prod_{i=s+1}^{n} (1-p_i)^{-1} \right) \sum_{k=n_1}^{s-1} q_k = \infty , \qquad n_1 \geq n_0 , \tag{10}$$

then every bounded solution $\{x_n\}$ *of* (E_1) *is oscillatory or* $\{\Delta x_n\}$ *is oscillatory.*

Proof. Let $\{x_n\}$ be a bounded and eventually positive solution of (E_1), say $x_n > 0$ and $x_{g_n} > 0$ for $n \geq n_1 \geq n_0 \geq 0$. Proceeding as in the proof of Theorem 1, we see that the case (a) is impossible. Next, we consider:

(b) $\Delta x_n > 0$ for $n \geq n_2$. There exists a constant $c_1 > 0$ and $N \geq n_2 + h$ such that

$$x_{g_n} \geq c_1 \qquad \text{for} \quad n \geq N . \tag{11}$$

As in the proof of Theorem 1(b) we obtain (4) and then define the sequence $\{r_n\}$ as in (5) and obtain (6) which takes the form

$$\Delta(r_n x_n) \geq c_1^c \, r_{n+1} \sum_{k=N}^{n-1} q_k \qquad \text{for} \quad n \geq N . \tag{12}$$

Summing both sides of (12) from N to $m \geq N$, we have

$$r_{m+1}x_{m+1} \geq r_{m+1}x_{m+1} - r_N x_N \geq c_1^c \sum_{n=N}^{m} r_{n+1} \sum_{k=N}^{n-1} q_k ,$$

or

$$\begin{aligned} x_{m+1} &\geq c_1^c \sum_{n=N}^{m} (r_{n+1}/r_{m+1}) \sum_{k=N}^{n-1} q_k \\ &= c_1^c \sum_{n=N}^{m} \left(\prod_{i=n+1}^{m} (1-p_i)^{-1} \right) \sum_{k=N}^{n-1} q_k \to \infty \qquad \text{as} \quad m \to \infty , \end{aligned}$$

which contradicts the fact that $\{x_n\}$ is bounded. This completes the proof. □

3. Almost oscillatory character of (E_2)

In this section, we present two criteria for the almost oscillation of (E_2) when $0 < c \le 1$.

THEOREM 4. *Suppose that $c = 1$, $g_n \le n$ and $\Delta p_n \le 0$ for $n \ge n_0 \ge 0$. If*

$$\liminf_{n\to\infty}\left(\frac{1}{h-1}\sum_{k=n+1}^{n+h-1} p_k\right) > \frac{(h-1)^{(h-1)}}{h^h} \tag{13}$$

and

$$\limsup_{n\to\infty}\sum_{k=g_n}^{n-1} C_{g_n,k}\, q_k > 1\,, \tag{14}$$

where

$$C_{g_n,k} = \sum_{s=g_n}^{k}\left[\prod_{j=g_n+1}^{s}(1+p_j)^{-1}\right],$$

then (E_2) is almost oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of (E_2), say $x_n > 0$ and $x_{g_n} > 0$ for $n \ge n_1 \ge n_0 \ge 0$. Now, there are two cases to consider:

(a) $\Delta x_n > 0$ eventually, (b) $\Delta x_n < 0$ eventually.

(a) Suppose $\Delta x_n > 0$ eventually. From (E_2) we see that

$$\Delta y_n - p_n y_{n+h} = q_n x_{g_n} \ge 0 \quad \text{eventually},$$

where $y_n = \Delta x_n > 0$ eventually. Now, by [8; Lemma 1.1(b)], the equation

$$\Delta y_n - p_n y_{n+h} = 0 \tag{15}$$

has an eventually positive solution. But, in view of [10; Theorem 3'] and condition (13), equation (15) is oscillatory, which is a contradiction.

(b) Suppose $\Delta x_n < 0$ for $n \ge N \ge n_2 + 1$. Then from (E_2), we have

$$\Delta x_n - \Delta x_s = \sum_{k=s}^{n-1} p_k \Delta x_{k+h} + \sum_{k=s}^{n-1} q_k x_{g_k} \qquad \text{for} \quad n \ge s \ge N\,. \tag{16}$$

Since

$$\sum_{k=s}^{n-1} p_k \Delta x_{k+h} = p_n x_{n+h} - p_s x_{s+h} - \sum_{k=s}^{n-1} x_{k+h}\Delta p_k\,,$$

and $\Delta p_n \le 0$ and $\{x_n\}$ is nonincreasing, $n \ge N$, we have

$$\sum_{k=s}^{n-1} p_k \Delta x_{k+h} \ge -p_s x_s \qquad \text{for} \quad n \ge s \ge N\,.$$

Now, (16) takes the form

$$-(\Delta x_s - p_s x_s) \geq \sum_{k=s}^{n-1} q_k x_{g_k} \qquad \text{for} \quad n \geq s \geq N. \tag{17}$$

Define the sequence $\{r_n\}$ by

$$r_{n+1} = r_n/(1+p_n), \quad n = 0,1,2,\dots \qquad \text{and} \qquad r_{n_0} > 0 \quad \text{for } n_0 \geq 0, \tag{18}$$

and multiply (17) by r_{s+1}. Then we have

$$-\Delta(r_s x_s) \geq r_{s+1} \sum_{k=s}^{n-1} q_k x_{g_k} \qquad \text{for} \quad n \geq s \geq N. \tag{19}$$

Summing both sides of (19) from $g_n \geq N$ to $n-1 \geq g_n$, we have

$$r_{g_n} x_{g_n} \geq r_{g_n} x_{g_n} - r_n x_n \geq \sum_{s=g_n}^{n-1} r_{s+1} \sum_{k=s}^{n-1} q_k x_{g_k}.$$

Now,

$$\begin{aligned} 1 &\geq \sum_{s=g_n}^{n-1} \left(r_{s+1}/r_{g_n}\right) \sum_{k=s}^{n-1} q_k \left(x_{g_k}/x_{g_n}\right) \\ &\geq \sum_{k=g_n}^{n-1} q_k \left(x_{g_k}/x_{g_n}\right) \sum_{s=g_n}^{k} \left(r_{s+1}/r_{g_n}\right). \end{aligned}$$

Since $x_{g_k} \geq x_{g_n}$ for $g_n \leq k \leq n-1 \leq n$, we see that

$$\begin{aligned} 1 &\geq \sum_{k=g_n}^{n-1} q_k \sum_{s=g_n}^{k} \left(r_{s+1}/r_{g_n}\right) \\ &= \sum_{k=g_n}^{n-1} \left(\sum_{s=g_n}^{k} \prod_{j=g_n+1}^{s} (1+p_j)^{-1} \right) q_k. \end{aligned}$$

Taking lim sup of both sides of the above inequality as $n \to \infty$, we obtain a contradiction to (14). This completes the proof. □

THEOREM 5. *Suppose that* $0 < c < 1$, $\Delta p_n \leq 0$ *and* $g_n < n$ *for* $n \geq n_0 \geq 0$, *and let condition* (13) *hold. If*

$$\sum_{k=n_0}^{\infty} A_{g_k,k} q_k = \infty, \tag{20}$$

where

$$A_{g_k,k} = \sum_{s=g_k}^{k} (1+p_s)^{-1} \left(\prod_{j=1}^{s-1} (1+p_j)^{-1} \right)^{1-c},$$

then (E_2) *is almost oscillatory.*

P r o o f. Let $\{x_n\}$ be an eventually positive solution of (E_2), $x_n > 0$ and let $x_{g_n} > 0$ for $n \ge n_1 \ge n_0 \ge 0$. As in the proof of Theorem 4, we see that case (a) is impossible. Next we consider:

(b) Suppose $\Delta x_n < 0$ for $n \ge N \ge n_2 + 1$. Define the sequence $\{r_n\}$ as in (18) and proceed as in the proof of Theorem 4(b) to obtain (19) which takes the form

$$-\Delta(r_s x_s) \ge r_{s+1} \sum_{k=s}^{n-1} q_k x_{g_k}^c \qquad \text{for} \quad n \ge s \ge N. \tag{21}$$

Choose $N^* > N$ such that $g_s \ge N$ for $s \ge N^*$ and let $m > N^*$ be fixed. We see that

$$-\Delta(r_s x_s) \ge r_{s+1} \sum_{k=s}^{m} q_k x_{g_k}^c \qquad \text{for} \quad m \ge s \ge N. \tag{22}$$

Dividing (22) by $(r_s x_s)^c$ and summing from N to m, we obtain

$$\begin{aligned} \sum_{s=N}^{m} -\Delta(r_s x_s)/(r_s x_s)^c &\ge \sum_{s=N}^{m} (r_{s+1}/r_s^c) \sum_{k=s}^{m} q_k (x_{g_k}/x_s)^c \\ &= \sum_{s=N}^{m} r_s^{1-c}/(1+p_s) \sum_{k=s}^{m} q_k (x_{g_k}/x_s)^c \\ &\ge \sum_{k=N^*}^{m} q_k \sum_{s=g_k}^{k} r_s^{1-c}/(1+p_s)(x_{g_k}/x_s)^c, \qquad N^* \ge N. \end{aligned}$$

Since $x_{g_k} > x_s$ for $g_k \le s \le k$, $m \ge k \ge N^*$, we have

$$\sum_{s=N}^{m} -\Delta(r_s x_s)/(r_s x_s)^c \ge \sum_{k=N^*}^{m} q_k \sum_{s=g_k}^{k} r_s^{1-c}/(1+p_s).$$

It follows from the proof of Theorem 4.3 in [7], that

$$\sum_{s=N}^{m} -\Delta z_s / z_s^c \ \text{ is bounded below,} \qquad m \ge N,$$

which contradicts condition (20). This completes the proof. □

As an application of Theorems 2 and 4, we consider the special cases of (E_i), $i = 1, 2$, namely, the constant coefficients equations:

$$\Delta^2 x_n + p x_{n-h} = q x_{n+g}, \tag{L_1}$$

and

$$\Delta^2 x_n = p x_{n+h} + q x_{n-g}, \tag{L_2}$$

where p and q are positive real numbers and h and k are positive integers.

Now, we have the following two oscillation results for (L_i), $i = 1, 2$.

Corollary 1. *Let $g \geq 1$ and $0 < p < 1$. If*

$$p > \frac{h^h}{(1+h)^{1+h}} \tag{23}$$

and

$$(q/p)\left[g + \frac{1-p}{p}\Big((1-p)^g - 1\Big)\right] > 1, \tag{24}$$

then (L_1) is almost oscillatory.

Corollary 2. *If*

$$p > \frac{(h-1)^{h-1}}{h^h} \tag{25}$$

and

$$(q/p)(1+p)\left[g - \frac{1}{p}\Big(1 - (1+p)^{-g}\Big)\right] > 1, \tag{26}$$

then (L_2) is almost oscillatory.

As an illustration, we see that the difference equations

$$\Delta^2 x_n + \frac{1}{2}\Delta x_{n-3} = q_1 x_{n+3}, \tag{27}$$

and

$$\Delta^2 x_n = \Delta x_{n+4} + q_2 x_{n-4} \tag{28}$$

are almost oscillatory if $q_1 > 4/17$ and $q_2 > 8/49$ by Corollaries 1 and 2 respectively.

Remarks.

1. If we let $p_n = 0$ in the results presented in this paper, the remaining conditions in our results are not enough to describe the oscillatory character of the equation

$$\Delta^2 x_n = q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n}, \qquad c > 0, \tag{E*}$$

and hence our results are not applicable to (E*).

2. It would be interesting to study the oscillatory character of (E_1) and (E_2) instead of almost oscillation and to obtain results similar to these presented here for (E_1) with $c < 1$ and for (E_2) with $c > 1$.

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