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A MEASURE DECOMPOSITION THEOREM

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There is shown in this note that various measure decomposition theorems may be proved by the same technique. Let $(X, \mathscr{S})$ be a measurable space (in the sense of [2]) and M the set of all measures on $\mathscr{S}$. If $\nu \in M$ *and* A *is locally* $\mathscr{S}$*-measurable, then* ν_A *denotes the measure defined via* $\nu_A(E) = \nu(A \cap E)$, $E \in \mathscr{S}$. $\mathscr{N}(\nu)$ *denotes the family of all* ν*-null sets.*

Theorem. *Let for every* $\tau \in M$ *a* σ*-ring* $\mathscr{S}(\tau) \subset \mathscr{S}$ *be given, in such a way that the following conditions* I—IV *are true*:

I. $\mathscr{N}(\tau) \subset \mathscr{S}(\tau)$

II. $E \in \mathscr{S}(\tau)$, $F \in \mathscr{S}$, $F \subset E$ *implies* $F \in \mathscr{S}(\tau)$

III. $A \in \mathscr{S}(\tau)$ *iff* $A \in \mathscr{S}(\tau_A)$

IV. *If* $A \in \mathscr{S}$ *and* $\tau(B) = \sup\{\tau(C)\colon C \subset B,\ C \in \mathscr{S}(\tau)\}$ *for every* $\mathscr{S}$*-measurable subset* $B \subset A$, *then* $A \in \mathscr{S}(\tau)$.

Then every $\nu \in M$ *may be written as a sum of measures* ν_1, ν_2 *where* $\mathscr{S}(\nu_1) = \mathscr{S}$ *and* $\mathscr{S}(\nu_2) = \mathscr{N}(\nu_2)$.

Proof. For every $\tau \in M$, denote $\mathscr{Z}(\tau)$ the σ-ring of all sets $A \in \mathscr{S}$ such that $B \subset A$, $B \in \mathscr{S}(\tau)$ implies $\tau(B) = 0$. Clearly,

(1) $$E \in \mathscr{Z}(\tau),\ F \subset E,\ F \in \mathscr{S} \text{ implies } F \in \mathscr{Z}(\tau)$$

(2) $$\mathscr{S}(\tau) \cap \mathscr{Z}(\tau) = \mathscr{N}(\tau) \text{ for every } \tau \in M. \qquad (2).$$

If $\nu \in M$ is given, define ν_1 and ν_2 by the formulas

$$\nu_1(E) = \sup\{\nu_A(E)\colon A \in \mathscr{S}(\nu)\},\ E \in \mathscr{S}$$
$$\nu_2(E) = \sup\{\nu_B((E)\colon B \in \mathscr{Z}(\nu)\},\ E \in \mathscr{S}.$$

Families $\{\nu_A\}_{A \in \mathscr{S}(\nu)}$ and $\{\nu_B\}_{B \in \mathscr{Z}(\nu)}$ are increasingly directed, hence ν_1 and ν_2 are measures (see [1], Theorem 10.1.). Let $E \in \mathscr{S}$ be given. If $\nu_1(E) = \infty$, then the equality $\nu(E) = \nu_1(E) + \nu_2(E)$ is obvious. Let $\nu_1(E)$ be finite. There exists a sequence $A_n \in \mathscr{S}(\nu)$, $A_n \subset E$ and $\nu_1(E) = \lim_{n \to \infty} \nu(A_n)$. Denoting $F = \bigcup_{n=1}^{\infty} A_n$, we have $F \in \mathscr{S}(\nu)$ and $\nu_1(E) = \nu(F)$. Moreover, $E \setminus F \in \mathscr{Z}(\nu)$. Let $B \in \mathscr{Z}(\nu)$ be arbitrary. $\nu(B \cap F) = 0$ by (2), hence $\nu(E \setminus F) \geqq \nu(B \cap E)$. Thus, $\nu_2(E) = \nu(E \setminus F)$. Consequently, $\nu(E) = \nu_1(E) + \nu_2(E)$.

Let us observe the following consequences of II and (1):

$$A \in \mathscr{S}(\nu) \text{ implies } (\nu_1)_A = \nu_A \tag{3}$$

$$A \in \mathscr{Z}(\nu) \text{ implies } (\nu_2)_A = \nu_A. \tag{4}$$

Let A be an arbitrary set of $\mathscr{S}$. $\nu_1(A) = \sup\{\nu(B)\colon B \subset A,\ B \in \mathscr{S}(\nu)\}$. $B \in \mathscr{S}(\nu)$ implies $B \in \mathscr{S}(\nu_B)$. By (3), $\nu(B) = \nu_1(B)$ and $\mathscr{S}((\nu_1)_B) = \mathscr{S}(\nu_B)$. Thus, $B \in \mathscr{S}((\nu_1)_B)$. Hence $B \in \mathscr{S}(\nu_1)$. Summarizing, $\nu_1(A) = \sup\{\nu_1(B)\colon B \subset A,\ B \in \mathscr{S}(\nu_1)\}$. Hence by IV, $\mathscr{S}(\nu_1) = \mathscr{S}$.

Suppose, $A \in \mathscr{S}(\nu_2)$ and $\nu_2(A) > 0$. Then there exists $B \in \mathscr{Z}(\nu)$ such that $B \subset A$ and $\nu(B) > 0$. Clearly, $B \in \mathscr{S}(\nu_2)$. By (4), $(\nu_2)_B = \nu_B$ and $\mathscr{S}((\nu_2)_B) = \mathscr{S}(\nu_B)$. Using III, we have $B \in \mathscr{S}(\nu)$. Thus, $\nu(B) = 0$ by (2), a contradiction. Thus, $\mathscr{S}(\nu_2) = \mathscr{N}(\nu_2)$. The theorem is proved.

We show four possible applications of this theorem. If μ, $\nu \in M$, then ν is said to be absolutely continuous with respect to μ (written $\nu \ll \mu$) if $\mu(E) = 0$ implies $\nu(E) = 0$, $E \in \mathscr{S}$. μ, ν are said to be singular (written $\mu \perp \nu$) if $\mu_A = \nu_B = 0$ for some disjoint locally $\mathscr{S}$-measurable sets A, B such that $A \cup B = X$. Denoting by $\mathscr{S}(\tau)$ the family of all sets $A \in \mathscr{S}$ such that $\tau_A \ll \mu$, one obtains that $\nu = \nu_1 + \nu_2$ where $\nu_1 \ll \mu$ and $\mathscr{S}(\nu_2) = \mathscr{N}(\nu_2)$. If ν fulfils the Countably Chain Condition, then the last equality implies the singularity of ν_2 and μ. Thus, we have obtained the Lebesgue decomposition.

A set $A \in \mathscr{S}$ is called τ-atom (briefly atom) if $\tau(A) > 0$ and $B \subset A$, $B \in \mathscr{S}$ implies $\tau(B) = 0$ or $\tau(B) = \tau(A)$. τ is called non atomic if it possesses no atom. τ is called purely atomic it every set of positive measure τ contains an atom. Defining $\mathscr{S}(\tau) = \{A \in \mathscr{S}\colon \tau_A \text{ is non atomic}\}$, one obtains from the Theorem that every $\nu \in M$ is a sum of a non atomic and purely atomic measure.

Denote by $\mathscr{S}(\tau)$ the family of all sets $A \in \mathscr{S}$ such that τ_A is semifinite. Then it follows from the Theorem that every $\nu \in M$ is a sum of a semifinite measure ν_1 and of a measure ν_2 which attains only values 0 or ∞.

Assume that $\mathscr{S}$ contains every countable subset of X. We say that $\tau \in M$ is determined by countable sets if for every $E \in \mathscr{S}$ there exists a countable set C such that $C \subset E$ and $\tau(C) = \tau(E)$. Denote by $\mathscr{S}(\tau)$ the family of all sets $A \in \mathscr{S}$ such that τ_A is zero on countable sets. Then it follows from the Theorem that every $\nu \in M$ is a sum of ν_1 nad ν_2 where ν_1 is zero on countable sets and ν_2 is determined by countable sets.

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ТЕОРЕМА ОБ РАЗЛОЖЕНИИ МЕРЫ

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Резюме

В работе доказывается абстрактная теорема об разложении меры, определенной на σ-кольце. Из нее следуют четыре конкретные теоремы, например теорема об разложении Лебега и другие.