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ON COMPLETE LATTICE ORDERED GROUPS WITH TWO GENERATORS I

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A. W. Hales [8] proved that there does not exist any free complete Boolean algebra with an infinite set of free generators. The main step in the proof is the following assertion:

(H) For each cardinal α there exists a complete Boolean algebra B_α such that B_α is generated by a countable subset and $\operatorname{card} B_\alpha \geqq \alpha$.

By using the assertion (H) it was proved in [12] that there does not exist any free complete lattice ordered group with an infinite set of free generators. An analogous result for complete vector lattices was established in [13].

The notion of a free complete lattice ordered group can be modified in such a way that instead of the class $\mathscr{C}_1$ of all complete lattice ordered groups we consider a nonempty subclass $\mathscr{C}$ of $\mathscr{C}_1$.

The definition of a free complete vector lattice used in [13] is not a precise analogy of the definition of a free complete lattice ordered group as used in [12] (though by translating both these definitions into the language of complete Boolean algebras we obtain equivalent notions). Thus in the case of complete lattice ordered groups and in the case of complete vector lattices we can consider two types of free structures; in the sequel we shall distinguish a-free generators and b-free generators (for definitions, cf. § 1 below).

In this paper there will be investigated complete lattice ordered groups with two generators. The existence of an a-free (or a b-free, respectively) complete lattice ordered group with two a-free (or b-free) generators in some classes of complete lattice ordered groups will be examined (without using the above mentioned result of Hales).

§ 1. Conditions (a) and (b)

For the basic notions and denotations concerning lattice ordered groups cf. Birkhoff [2], Fuchs [7] and Conrad [3]. The group operation will be denoted additively.

In this section the notions of a-free generators and b-free generators of a complete lattice ordered group will be introduced. We begin by recalling the definition of a free lattice ordered group. (Free lattice ordered groups and free abelian lattice ordered groups have been investigated in several papers; cf. e.g., Conrad [5] and Weinberg [20].)

Let G be a lattice ordered group, $\emptyset \neq M \subseteq G$. If $H = G$ for each l-subgroup H of G with $M \subseteq H$, then G is said to be generated by the set M. The set M is called a set of free generators of the lattice ordered group G if G is generated by M and if the following condition is fulfilled:

(a) For each lattice ordered group K and for each mapping φ of the set M into K there exists a homomorphism φ_1 of G into K such that $\varphi(m) = \varphi_1(m)$ for each $m \in M$.

Then G is said to be a free lattice ordered group with α free generators, where $\alpha = \operatorname{card} M$.

It is easy to verify that the condition (a) is equivalent with the following condition:

(b) For each lattice ordered group K and for each mapping φ of the set M into K having the property that K is generated by $\varphi(M)$ there exists a homomorphism φ_1 of G onto K such that $\varphi(m) = \varphi_1(m)$ for each $m \in M$.

Now let us consider analogous notions for complete lattice ordered groups. A lattice ordered group G is called complete if each nonempty upper-bounded subset of G possesses the least upper bound in G; if this is the case, then also each nonempty lower-bounded subset of G possesses the greatest lower bound in G.

Let H be an l-subgroup of a lattice ordered group G. Suppose that whenever $X \subseteq H$ and $\sup X = x_0$ holds in G, then x_0 belongs to H. Under this assumption H is said to be a closed l-subgroup of G. If this is the case, then the corresponding dual condition is valid as well.

Let G be a complete lattice ordered group and let $\emptyset \neq M \subseteq G$. Assume that for each closed l-subgroup H of G fulfilling the relation $M \subseteq H$ we have $H = G$. Then M is called a set of generators of the complete lattice ordered group G; we also say that the set M generates the complete lattice ordered group G.

A homomorphism f of a lattice ordered group G_1 into a lattice ordered group G_2 is said to be complete, if it fulfils the following condition: whenever $\emptyset \neq X \subseteq G_1$ and $\sup X$ exists in G_1, then $\sup f(x)$ exists in G_2 and

$$\sup f(X) = f(\sup X).$$

If f is a complete homomorphism, then also the corresponding dual condition is satisfied.

Let $\mathscr{C}$ be a class of complete lattice ordered groups, $G \in \mathscr{C}$, $\emptyset \neq M \subseteq G$. Consider the following conditions for G:

(a_1) For each $K \in \mathscr{C}$ and for each mapping φ of the set M into K there exists a complete homomorphism φ_1 of G into K such that $\varphi(m)=\varphi_1(m)$ for each $m\in M$.

(b_1) For each $K\in\mathscr{C}$ and for each mapping φ of the set M into K having the property that the set $\varphi(M)$ generates the complete lattice ordered group K there exists a complete homomorphism φ_1 of G onto K such that $\varphi_1(m)=\varphi(m)$ for each $m\in M$.

The conditions (a_1) and (b_1) are analogous to the conditions (a) and (b) with the distinction that instead of the category of all lattice ordered groups with the usual homomorphisms we now have the category whose objects are elements of $\mathscr{C}$ and morphisms are complete homomorphisms.

If the set M generates the complete lattice ordered group G and if (a_1) is valid, then M is said to be a set of a-free generators of the complete lattice ordered group G in the class $\mathscr{C}$. If this is the case and $\operatorname{card} M=\alpha$, then G is called an a-free complete lattice ordered group with α a-free generators in the class $\mathscr{C}$. Analogously we define a b-free lattice ordered group with α b-free generators in $\mathscr{C}$ (with (a_1) replaced by (b_1)).

Let us recall some further notions concerning lattice ordered groups.

For each subset X of a lattice ordered group G the polar X^δ in G is defined by

$$X^\delta=\{g\in G\colon |g|\wedge|x|=0 \text{ for each } x\in X\}$$

(cf. Šik [16]). If $X=\{x\}$ is a one-element set, then we denote $[x]=X^{\delta\delta}$; the set $[x]$ is said to be the principal polar generated by the element x.

For the definitions and denotations concerning direct products of lattice ordered groups cf., e.g., [14], § 2. If A is a direct factor of a lattice ordered group G and $g\in G$, then the component of g in A will be denoted by $g(A)$. If G is a complete lattice ordered group, then each polar in G is a direct factor of G; namely, for each $X\subseteq G$ we have

$$G=X^\delta\times X^{\delta\delta}.$$

If G is complete, $g\in G$ and $x\in G$, then we write $g[x]$ rather than $g([x])$.

A subset $\{x_i\}_{i\in I}$ of a lattice ordered group is called disjoint if $x_i\geqq 0$ for each $i\in I$ and $x_i\wedge x_j=0$ for each pair of distinct elements $i, j\in I$. The lattice ordered group G is said to be orthogonally complete provided each nonempty disjoint subset of G possesses the least upper bound in G.

An element $0\leqq s$ of a lattice ordered group G is said to be singular if $x\wedge(s-x)=0$ for each $x\in G$ with $0\leqq x\leqq s$. A lattice ordered group is called singular if for each $0<g\in G$ there exists a singular element $s\in G$ with $0<s\leqq g$.

We introduce the following denotations for classes of complete lattice ordered groups:

$\mathscr{C}_1$ — the class of all complete lattice ordered groups;

$\mathscr{C}_s$ — the class of all singular complete lattice ordered groups;

$\mathscr{C}_v$ — the class of all complete lattice ordered groups having no singular element distinct from 0;

$\mathscr{C}_0$ — the class of all complete lattice ordered groups that are orthogonally complete;

$\mathscr{C}_d$ — the class of all complete lattice ordered groups that are completely distributive.

The following theorem is well-known (cf. [3]):

(T) *For each complete lattice ordered group H there are uniquely determined l-subgroups A, B of G such that*

(a) $H = A \times B$,

(b) $A \in \mathscr{C}_s$, $\quad B \in \mathscr{C}_v$.

Orthogonally complete lattice ordered groups and completely distributive lattice ordered groups have been studied in several papers (cf., e.g., Bernau [1], Conrad [6], Rotkovič [15], or Conrad [4], Weinberg [19], respectively).

In Part I some relations between the a-freeness and the b freeness will be established and it will be shown that if $\mathscr{C} \in \{\mathscr{C}_1, \mathscr{C}_s, \mathscr{C}_d\}$, then there does not exist any b-free complete lattice ordered group with two b-free generators in the class $\mathscr{C}$.

In Part II it will be proved that the following scheme is valid.

	a	b
$\mathscr{C}_1$	?	$-$
$\mathscr{C}_s$	$+$	$-$
$\mathscr{C}_v$	$-$	$-$
$\mathscr{C}_0$	?	?
$\mathscr{C}_d$	$+$	$-$
$\mathscr{C}_s \cap \mathscr{C}_d$	$+$	$-$
$\mathscr{C}_s \cap \mathscr{C}_0$	$+$	$+$
$\mathscr{C}_d \cap \mathscr{C}_0$	$+$	$+$
$\mathscr{C}_s \cap \mathscr{C}_d \cap \mathscr{C}_0$	$+$	$+$

In this scheme we denote by $+$ and by $-$, respectively, the fact that an a-free lattice complete lattice ordered group with two a-free generators in the corresponding class does exist or does not exist; the same denotation will be used for the existence of b-free complete lattice ordered groups with two b-free generators.

In what follows the symbols N and N_0 denote the set of all positive integers or the set of all integers, respectively, with the natural linear order. Sometimes we consider N_0 as a linearly ordered group (with respect to the addition).

§ 2. Relations between the conditions (a_1) and (b_1)

In this paragraph the relations between the conditions (a_1) and (b_1) will be examined; we shall also consider conditions analogous to (a_1) and (b_1) for complete Boolean algebras.

2.1. Lemma. *Let $\mathscr{C}$ be a class of complete lattice ordered groups such that, whenever $H \in \mathscr{C}$ and H_1 is a closed l-subgroup of H, then H_1 belongs to $\mathscr{C}$. Let $G \in \mathscr{C}$, $\emptyset \neq M \subseteq G$ and suppose that the condition (b_1) holds. Then the condition (a_1) is valid as well.*

Proof. Let K and φ have the same meaning as in the condition (a_1) (cf. § 1). We denote by H the intersection of all closed l-subgroups H_i of K with $\varphi(M) \subseteq H_i$. Then H is a closed l-subgroup of K and the set $\varphi(M)$ generates the complete lattice ordered group H. According to the assumption the condition (b_1) is valid, hence there exists a complete homomorphism φ_1 of G onto H such that $\varphi_1(m) = \varphi(m)$ for each $m \in M$. Thus φ_1 is a homomorphism of G into K; it suffices to verify that φ_1 is a complete homomorphism of G into K.

Let $\emptyset \neq X \subseteq G$ and suppose that $\sup X = x_0$ holds in G. Since φ_1 is a complete homomorphism of G onto H, the relation

$$\varphi_1(x_0) = \sup \varphi_1(X)$$

is valid in H. Thus $\varphi_1(x_0)$ is an upper bound of the set $\varphi_1(X)$ in K. Because K is a complete lattice ordered group, there is $y \in K$ such that

$$y = \sup \varphi_1(X)$$

is valid in K. From the fact that H is a closed l-subgroup of K and from $\varphi_1(X) \subseteq H$ we obtain $y \in H$ and thus $\varphi_1(x_0) = y$. Therefore φ_1 is a complete homomorphism of G into K.

2.2. Corollary. *Let $\mathscr{C}$ be a class of complete lattice ordered groups fulfilling the same assumption as in 2.1. Suppose that M is a set of b-free generators of a complete lattice ordered group G in the class $\mathscr{C}$. Then M is a set of a-free generators of the complete lattice ordered group G in the class $\mathscr{C}$.*

2.3. Corollary. *Let $\mathscr{C}$ be as in 2.1 and let α be a cardinal. Suppose that there does not exist any a-free complete lattice ordered group with α a-free generators in the class $\mathscr{C}$. Then there does not exist any b-free complete lattice ordered group with α b-free generators in the class $\mathscr{C}$.*

The following result has been proved in [12] (by using another terminology):

2.4. Theorem. *Let α be an infinite cardinal. There does not exist any a-free complete lattice ordered group with α a-free generators in the class $\mathscr{C}_1$.*

From 2.4 and 2.3 we obtain:

2.5. Corollary. *Let α be an infinite cardinal. There does not exist any b-free lattice ordered group with α b-free generators in the class $\mathscr{C}_1$.*

We shall show below that the notion of the α-freeness and that of the β-freeness in a class $\mathscr{C}$ need not coincide.

Now let us consider an analogous notions for complete Boolean algebras. The notion of a closed subalgebra and the notion of a complete homomorphism of a complete Boolean algebra are defined analogously as in the case of complete lattice ordered groups.

Let G be a complete Boolean algebra, $\emptyset \neq M \subseteq G$. If for each closed subalgebra H of G with $M \subseteq H$ we have $G = H$, then M is said to be a set of generators of the complete Boolean algebra G.

Let $\mathscr{B}_1$ be the class of all complete Boolean algebras and let $\emptyset \neq M \subseteq G \in \mathscr{B}_1$. If we replace in the condition (a_1) the symbol $\mathscr{C}$ by $\mathscr{B}_1$, then we obtain the condition (a_2) for the Boolean algebra G. Analogously we get from (b_1) the condition (b_2) for the Boolean algebra G. By using the conditions (a_2) and (b_2) we can define the notion of an a-free (or b-free, respectively) complete Boolean algebra with α a-free (or b-free) generators in the class $\mathscr{B}_1$.

The following consideration shows that the notions of a-freeness and b-freeness in the class $\mathscr{B}_1$ are equivalent.

We have to verify that the conditions (a_2) and (b_2) are equivalent. Let $G \in \mathscr{B}_1$, $\emptyset \neq M \subseteq G$. If (b_2) holds, then by an analogous procedure to that in the proof of 2.1 we obtain that the condition (a_2) is valid as well. Conversely, suppose that (a_2) holds. Let K and φ have the same meaning as in (b_2). According to (a_2) there exists a complete homomorphism φ_1 of G into K such that $\varphi_1(m) = \varphi(m)$ for each $m \in M$. Put $\varphi_1(G) = H$. Our assertion will be proved if we verify that $H = K$ is valid. Since $\varphi_1(M)$ generates the complete Boolean algebra K, it suffices to prove the following assertion:

$(*)$ *Let G and K be complete Boolean algebras and let φ_1 be a complete homomorphism of G into K, $\varphi_1(G) = H$. Then H is a closed subalgebra of K.*

Proof. For any $X \subseteq K$ and $Y \subseteq G$ we denote by $\sup_K X$ and $\sup_G Y$ the corresponding suprema in K or in G, respectively. Let $\emptyset \neq X \subseteq H$. Put $Y = \varphi_1^{-1}(X)$. Then we have

$$\sup_K X = \sup_K \varphi_1(Y) = \sup_G Y \in H$$

and an analogous relation holds for the greatest lower bounds. Thus H is a closed subalgebra of G.

The essential diference between complete lattice ordered groups and complete Boolean algebras with respect to the conditions (a_1), (b_1) or (a_2), (b_2), respectively, consists in the fact that the assertion analogous to $(*)$ is not valid for complete lattice ordered groups. The reason for this lies in the following: each complete Boolean algebra can be considered as a structure with infinitary operations, while a complete lattice ordered group distinct from $\{0\}$ is only a structure with partial infinitary operations.

The following example shows that the assertion analogous to $(*)$ does not hold in general for complete lattice ordered groups.

Example 1. Let F be the set of all integer valued functions defined on the set N. In the set F we consider the usual operation $+$; the partial order in F is defined componentwise. Then F is a complete lattice ordered group. Let F_b be the set of all bounded functions belonging to F and let φ_1 be the identical mapping defined on F_b. Thus F_b is a complete lattice ordered group and φ_1 is a complete homomorphism of F_b into F. We have $\varphi_1(F_b)=F_b$ and F_b fails to be a closed l-subgroup of F.

§ 3. Remark on complete vector lattices

This paragraph contains some supplements and comments to the author's paper [13]. At first we recall some definitions concerning vector lattices.

A lattice ordered group H is said to be a vector lattice if H is a linear space over the field of all reals such that $\lambda h>0$ holds for each positive real λ and for each $0<h\in H$. If, moreover, H is a complete lattice ordered group, then H is called a complete vector lattice.

The following results are well known (cf. [3]):

If H is a vector lattice, then H does not contain any singular element distinct from 0. Let $H\in\mathscr{C}_v$ (cf. the denotations introduced in § 1). Then we can define a multiplication of elements of H with reals such that H turns out to be a complete vector lattice.

Let X be a vector lattice and let $\emptyset\neq Y\subseteq X$. Assume that

(i) Y is a closed l-subgroup of the lattice ordered group X;

(ii) $\lambda y\in Y$ for each $y\in Y$ and for each real λ.

Then Y is called a closed vector sublattice of X.

Let X be a complete vector lattice and let $\emptyset\neq M\subseteq X$. If $Y=X$ holds for each closed vector sublattice Y of X with $M\subseteq Y$, then M is said to be a set of generators of the complete vector lattice X.

Let X_1 and X_2 be vector lattices and let $f\colon X_1\to X_2$ be a mapping such that

(i) f is a complete homomorphism of the lattice ordered group X_1 into the lattice ordered group X_2;

(ii) $f(\lambda x)=\lambda f(x)$ for each real λ and each $x\in X$. Then f is said to be a complete homomorphism of the vector lattice X_1 into the vector lattice X_2.

Under these denotations we can replace in the conditions (a_1) and (b_1) the notions concerning complete lattice ordered groups by analogous notions concerning complete vector lattices. The corresponding conditions for a complete vector lattice G will be denoted as conditions (a_3) and (b_3). By means of (a_3) and (b_3) we can now define the notion of an a-free (or a b-free, respectively) complete vector lattice in a nonempty class $\mathscr{C}$ of complete vector lattices.

By a method analogous to that used in § 2 we obtain.

3.1. Lemma. *Let $\mathscr{C}$ be a class of complete vector lattices such that, whenever $H \in \mathscr{C}$ and H_1 is a closed vector sublattice of H, then $H_1 \in \mathscr{C}$. Let α be a cardinal. Suppose that there does not exist any a-free complete vector lattice with α a-free generators in $\mathscr{C}$. Then there does not exist any b-free complete vector lattice with α b-free generators in $\mathscr{C}$.*

The following result has been proved in [13] (Thm. 1) (under another terminology):

3.2. Theorem. *Let $\mathscr{V}_1$ be the class of all complete vector lattices and let α be an infinite cardinal. Then there does not exist any b-free complete vector lattice with α b-free generators in the class $\mathscr{V}_1$.*

By using the method of the proof of this theorem in [13] we infer that the following assertion is valid:

3.3. Theorem. *Let $\mathscr{V}_1$ be the class of all complete vector lattices and let α be an infinite cardinal. Then there does not exist any a-free complete vector lattice with α a-free generators in the class $\mathscr{V}_1$.*

Thm. 3.2 is an immediate consequence of Thm. 3.3 and Lemma 3.1. Let us remark that in the proof of Thm. 1 in [13] the assertion (H) on complete Boolean algebras (cf. the introduction) was used.

A vector lattice X is said to be orthogonally complete if the corresponding lattice ordered group X is orthogonally complete. We need the following result on orthogonal extensions (cf. [9]):

3.4. Lemma. *Let X be a complete vector lattice. There exists a complete vector lattice $o(X)$ such that:*

(i) *X is a convex vector sublattice of $o(X)$;*

(ii) *for each element $0 < y \in o(X)$ there exists a disjoint subset $\{x_i\}_{i \in I} \subseteq X$ having the property that $y = \bigvee_{i \in I} x_i$ holds in $o(X)$;*

(iii) *$o(X)$ is orthogonally complete.*

The vector lattice $o(X)$ is determined uniquely up to isomorphisms.

From 3.4 we obtain immediately:

3.5. Corollary. *Assume that a set M generates a complete vector lattice X. Then M generates the complete vector lattice $o(X)$.*

From 3.5 it follows that the conclusions of the proof of Thm. 1 in [13] remain valid if in this proof we replace X_0 and X by $o(X_0)$ and by $o(X)$. Thus we obtain the following assertion:

3.6. Theorem. *Let $\mathscr{V}_0$ be the class of all complete vector lattices that are orthogonally complete. Let α be an infinite cardinal. Then there does not exist any a-free complete vector lattice with α a-free generators in the class $\mathscr{V}_0$.*

§ 4. Complete lattice ordered groups with one generator

The natural question arises whether in each class $\mathscr{C} \in \{\mathscr{C}_1, \mathscr{C}_s, \mathscr{C}_v, \mathscr{C}_0, \mathscr{C}_d\}$ there exists an a-free (or b-free, respectively) complete lattice ordered group with one a-free (or b-free) generator.

Let G be a complete lattice ordered group and let $\emptyset \neq x \in G$. Put

$$G_1 = \{n_1 x^+ + n_2 x^-\},$$

where n_1 and n_2 run over the set of integers. Then G_1 is the least l-subgroup of G containing the element x. It is easy to verify that G_1 is a closed l-subgroup of G. Hence the one-element set $\{x\}$ generates the complete lattice ordered group G_1. If x is comparable with 0, then either $x^+ = 0$ or $x^- = 0$, hence in this case G_1 is isomorphic with N_0. If x is incomparable with 0, then G_1 is isomorphic with $N_0 \times N_0$ (the corresponding isomorphism φ being defined by $\varphi(n_1 x^+ + n_2 x^-) = (n_1, -n_2)$). From this we obtain the following assertion:

4.1. Lemma. *Let $\mathscr{C}$ be a class of complete lattice ordered groups, $H \in \mathscr{C}$. Then the following conditions are equivalent:*

(a) *H is an a-free complete lattice ordered group with one a-free generator in the class $\mathscr{C}$.*

(b) *H is a b-free complete lattice ordered group with one b-free generator in the class $\mathscr{C}$.*

(c) *H is isomorphic with $\mathrm{N}_0 \times \mathrm{N}_0$.*

Since $N_0 \times N_0 \in \mathscr{C}_s \cap \mathscr{C}_0 \cap \mathscr{C}_d$, from 4.1 it follows:

4.2. Corollary. *Let $\mathscr{C} \in \{\mathscr{C}_1, \mathscr{C}_s, \mathscr{C}_0, \mathscr{C}_d\}$, $c \in \{a, b\}$. Then there exists a c-free complete lattice ordered group with one c-free generator in the class $\mathscr{C}$.*

The class $\mathscr{C}_v$ is closed with respect to isomorphisms and $N_0 \times N_0 \notin \mathscr{C}_v$; hence from 4.1 we obtain:

4.3. Corollary. *Let $c \in \{a, b\}$. There does not exist any c-free complete lattice ordered group with one c-free generator in the class $\mathscr{C}_v$.*

Let G be a lattice ordered group and let R be a congruence relation on G. For $x \in G$ we denote by $R(x)$ the set of all elements $y \in G$ with $x \equiv y \pmod{R}$. If the natural homomorphism $x \to x(R)$ turns out to be a complete homomorphism of the lattice ordered group G onto G/R, then R will be said to be a complete congruence relation. For each set $\{R_i\}_{i \in I}$ of complete congruence relations on G

their meet $\bigwedge_{i \in I} R_i$ is also a complete congruence relation on G. If G is a complete lattice ordered group and if R is a complete congruence relation on G, then G/R is a complete lattice ordered group.

4.4. Lemma. *Let $\mathscr{C}$ be a class of complete lattice ordered groups that is closed with respect to complete homomorphisms. Let α, β be cardinals, $0<\alpha<\beta$ and let $c \in \{a, b\}$. Assume that there exists a c-free complete lattice ordered group with β c-free generators in $\mathscr{C}$. Then there exists a c-free complete lattice ordered group with α c-free generators in $\mathscr{C}$.*

Proof. According to the assumption there exist $G \in \mathscr{C}$ and $M \subseteq G$ such that M is a set of c-free generators of the complete lattice ordered group G in $\mathscr{C}$ and $\operatorname{card} M=\beta$. We can choose a subset $M_1 \subseteq M$ such that

$$\operatorname{card} M_1=\alpha \quad \text{if} \quad \alpha \quad \text{is infinite,}$$
$$\operatorname{card} M_1=\alpha-1, \quad \text{if} \quad \alpha \quad \text{is finite.}$$

Let m_0 be a fixed element of the set $M \backslash M_1$. Further let $\{R_i\}_{i \in I}$ be the set of all complete congruence relations on G fulfilling

$$m \equiv m_0 \,(\operatorname{mod} R_i) \quad \text{for each} \quad m \in M \backslash M_1 .$$

We put

$$R=\bigwedge_{i \in I} R_i, \quad G_1=G/R,$$
$$M'=\{m_0(R)\} \cup \{m(R)\}_{m \in M_1}.$$

Then M' is a set of c-free generators of the complete lattice ordered group G_1 in $\mathscr{C}$ and $\operatorname{card} M'=\alpha$.

The class $\mathscr{C}_v$ is closed with respect to complete homomorphisms. Hence from 4.3 and 4.4 it follows:

4.5. Corollary. *Let $c \in \{a, b\}$ and let $\alpha \geqq 1$ be a cardinal. There does not exist any c-free complete lattice ordered group with α c-free generators in $\mathscr{C}_v$.*

If G is a lattice ordered group generated by a finite set, then clearly $\operatorname{card} G \leqq \aleph_0$. If a one-element set generates a complete lattice ordered group H, then, as we have seen above, we have either $\operatorname{card} H=1$ or $\operatorname{card} H=\aleph_0$. Suppose that a set M with $\operatorname{card} M=2$ generates a complete lattice ordered group G; the following example shows that the power of the set G can be greather than $\aleph_0$.

Example 2. Let R_0 be the additive group of all reals with the natural linear order. Let x be a fixed irrational number and let G be the intersection of all closed l-subgroups H_i of R_0 such that $\{x, 1\} \subseteq H_i$. Hence G is a closed l-subgroup of R_0, thus G is a complete lattice ordered group. Thus either G is cyclic or $G=R_0$. In the first case there are $0<y\in G$, n_1, $n_2\in N$ with $n_1y=1$, $n_2y=x$, which is impossible. Thus $G=R_0$. The set $\{x, 1\}$ generates the complete lattice ordered group G.

This example also shows that if G is a complete lattice ordered group and $\emptyset\neq M\subseteq G$, then the following two conditions (i) and (ii) need not be equivalent:

(i) M generates the complete lattice ordered group G.

(ii) M generates the lattice ordered group G.

In fact, let us put $M=\{x, 1\}$ (under the denotations as above). We have verified that (i) holds. Put $G_1=\{n_1x+n_2\}$, where n_1 and n_2 run over the set N_0. Then G_1 is an l-subgroup of G, $G_1\neq G$ and $M\subseteq G_1$. Hence (ii) fails to be valid.

§ 5. Some auxiliary results

First let us investigate in more detail the lattice ordered group F from Example 1 in § 2.

Let f and g be elements of F such that

$$f(x)=1,\quad g(x)=x\quad \text{for each}\quad x\in N.$$

For each $n\in N$ we denote by e_n the element of F defined by

$$e_n(n)=1,\quad e_n(m)=0\quad \text{for each}\quad m\in N\backslash\{n\}.$$

5.1. Lemma. *Let G_1 be an l-subgroup of F, f, $g\in G_1$. Then $e_n\in G_1$ for each $n\in N$.*

Proof. We proceed by induction with respect to n. We have

$$e_1=(2f-g)^+,$$

hence $e_1\in G_1$. Let $n\in N$ and assume that the elements e_1, e_2, ..., e_n belong to G_1. Denote

$$\begin{aligned} f_n &= f-(e_1+e_2+\ldots+e_n),\\ g'_n &= g-(e_1+2e_2+\ldots+ne_n),\\ g_n &= g'_n-nf_n\,. \end{aligned}$$

Then

$$e_{n+1}=(2f_n-g_n)^+,$$

hence $e_{n+1}\in G_1$.

If n, $m\in N$, $n\neq m$, then $0<e_n$, $e_n\wedge e_m=0$. Hence from 5.1 it follows:

5.2. Corollary. *Let G_1 be an l-subgroup of F, f, $g\in G_1$. Then there exists an infinite disjoint subset of G_1.*

5.3. Lemma. *The set $\{f, g\}$ generates the complete lattice ordered group F.*

Proof. Let H be a closed l-subgroup of F and let f, $g \in H$. Let $h \in F$. Then the relation

$$h = \vee h(n)e_n \quad (n \in N)$$

holds in F and hence it follows from 5.1 that h belongs to H. From this we infer that $H = F$ is valid.

5.4. Lemma. *Let G be a complete lattice ordered group. Suppose that each disjoint subset of G is finite. Then G is a direct product of a finite number of linearly ordered groups.*

This is a consequence of [7], Chap. I, Thm. 14.

5.5. Lemma. *Let G be as in 5.4 and let ϱ be a congruence relation on G. Then each disjoint subset of the lattice ordered group G/ϱ is finite.*

The proof follows from 5.4 by routine induction steps; the details will therefore be ommited.

5.6. Lemma. *Let $\mathscr{C}$ be a class of complete lattice ordered groups, $F \in \mathscr{C}$. Let G be an a-free complete lattice group with two a-free generators in $\mathscr{C}$. Then G contains an infinite disjoint subset.*

Proof. Suppose that $f_1, f_2 \in G$, $f_1 \neq f_2$ and that f_1, f_2 is a set of a-free generators of G in the class $\mathscr{C}$. Since $F \in \mathscr{C}$, there exists a complete homomorphism φ_1 of G into F such that

$$\varphi_1(f_1) = f, \quad \varphi_2(f_2) = g .$$

Denote $H = \varphi_1(G)$. Then according to 5.2 there exists an infinite disjoint subset of H. Moreover, there exists a congruence relation ϱ on G such that H is isomorphic with G/ϱ. From this and from 5.5 it follows that there exists an infinite disjoint subset in G.

From 5.6 and 2.2 we obtain:

5.7. Corollary. *Let $\mathscr{C}$ be a class of complete lattice ordered groups such that, whenever $H \in \mathscr{C}$ and H_1 is a closed l-subgroup of H, then $H_1 \in \mathscr{C}$. Let $F \in \mathscr{C}$. Let G be a b-free complete lattice ordered group with two b-free generators in the class $\mathscr{C}$. Then G contains an infinite disjoint subset.*

Now suppose that a set $M \neq \emptyset$ generates a complete lattice ordered group G. Let us introduce the following denotations. Put $A_0 = M$. Let α_1 be an ordinal having the property that the set of all ordinals less than α_1 has the cardinality greater than the power of G. Let $\alpha \leqq \alpha_1$ be an ordinal and suppose that for each ordinal β with $\beta < \alpha$ we have defined the set A_β in such a way that whenever $\beta_1 \leqq \beta_2 < \alpha$, then

$$A_{\beta_1} \subseteq A_{\beta_2} \subseteq G .$$

Put $Z_\alpha = \bigcup_{\beta < \alpha} A_\beta$ and define A_α as the set of all elements $z \in G$ fulfilling some of the following conditions:

(i) there exist elements $z_1, z_2 \in Z_\alpha$ with $z = z_1 - z_2$;

(ii) there exists a subset X of Z_α such that either $z = \sup X$ or $z = \inf X$ holds in G.

Then we have $A_\beta \subseteq A_\alpha$ for each $\beta < \alpha$. Thus there exists an ordinal $\alpha_0 < \alpha_1$ such that

$$A_{\alpha_0+1} = A_{\alpha_0}.$$

Hence A_{α_0} is a closed l-subgroup of G and $M \subseteq A_{\alpha_0}$. Since the set M generates the complete lattice ordered group G, we obtain $A_{\alpha_0} = G$.

5.8. Lemma. *Suppose that a set $M \neq \emptyset$ generates a complete lattice ordered group G and that G is an l-subgroup of a lattice ordered group G_1. Let φ be a complete homomorphism of G into G_1 such that $\varphi(m) = m$ for each $m \in M$. Then φ is an identical mapping on G.*

Proof. Let A_α $(0 \leqq \alpha \leqq \alpha_0)$ have the same meaning as above. We have to verify that for each ordinal α with $0 \leqq \alpha \leqq \alpha_0$ the partial mapping

$$\varphi_{A_\alpha}: A_\alpha \to G_1$$

is an identical mapping on the set A_α. We proceed by transfinite induction with respect to α. According to the assumption we have $\varphi(x) = x$ for each $x \in A_0$. Let $0 < \alpha \leqq \alpha_0$ and suppose that for each $\beta < \alpha$ we have $\varphi(x) = x$ whenever $x \in A_\beta$. Thus $\varphi(x) = x$ for each $x \in Z_\alpha$. Since φ is a complete homomorphism, we infer that $\varphi(x) = x$ for each $x \in A_\alpha$. Now from $G = A_{\alpha_0}$ it follows that φ is an identity on G.

The following lemma 5.9 and Corollary 5.10 show that an a-free complete lattice ordered group with α a-free generators in a class $\mathscr{C}$ is defined uniquely up to isomorphisms.

5.9. Lemma. *Let $\mathscr{C}$ be a class of complete lattice ordered groups, $G, G' \in \mathscr{C}$, $\emptyset \neq M \subseteq G$, $\emptyset \neq M' \subseteq G'$. Let φ be a complete homomorphism of G into G'. Assume that the following conditions are fulfilled:*

(a) M is a set of a-free generators of the complete lattice ordered group G;

(b) M' is a set of a-free generators of the complete lattice ordered group G';

(c) the partial mapping $\varphi_M: M \to G'$ is a one-to-one mapping of M onto M'. Then φ is an isomorphism of G onto G'.

Proof. From (b) and (c) we obtain that there exists a homomorphism ψ of G' into G such that for each $m \in M$ and each $m' \in M'$, from $\varphi(m) = m'$ it follows $\psi(m') = m$. Put

$$\chi(x) = \psi(\varphi(x)), \quad \chi_1(y) = \varphi(\psi(y))$$

for each $x \in G$ and each $y \in G'$. Then χ is a complete homomorphism of G into G such that $\chi(m) = m$ for each $m \in M$. Similarly, χ_1 is a complete homomorphism of

G' into G' such that $\chi_1(m')=m'$ for each $m'\in M'$. From this and from 5.8 we obtain that χ is an identity on G and that χ_1 is an identity on G'. From this it follows that φ is an isomorphism of G onto G'.

5.10. Corollary. *Let $\mathscr{C}$ be a class of complete lattice ordered groups, G, $G'\in\mathscr{C}$, $\emptyset\neq M\subseteq G$, $\emptyset\neq M'\subseteq G'$. Suppose that the conditions* (a), (b) *from 5.9 are fulfilled and that* $\operatorname{card} M=\operatorname{card} M'$. *Then there exists an isomorphism φ of G onto G' such that the partial mapping φ_M is a bijection of M onto M'.*

Proof. According to the assumption there exists a one-to-one mapping φ of M onto M'. From (a) it follows that φ can be extended to a complete homomorphism of G into G'. Hence according to 5.9, φ is an isomorphism of G onto G'.

Let us denote by (a′) the condition that we obtain from the condition (a) of lemma 5.9 if we replace the expression 'a-free generators' by the expression 'b-free generators'. Further let (b′) have an analogous meaning with respect to the condition (b) of 5.9.

5.11. Lemma. *Let $\mathscr{C}$ be a class of complete lattice ordered groups such that, whenever $H\in\mathscr{C}$ and H_1 is a closed l-subgroup of H, then $H_1\in\mathscr{C}$. Assume that the conditions (a'), (b') are fulfilled and that the condition (c) from 5.9 holds. Then φ is an isomorphism of G onto G'.*

Proof. The assertion follows from 2.2 and 5.9.

5.12. Corollary. *Let $\mathscr{C}$ be as in 5.11. Let G, $G'\in\mathscr{C}$, $\emptyset\neq M\subseteq G$, $\emptyset\neq M'\subseteq G'$,* $\operatorname{card} M=\operatorname{card} M'$. *Assume that the conditions (a') and (b') are fulfilled. Then there exists an isomorphism φ of G onto G' such that the corresponding partial mapping φ_M is a monomorphism of M onto M'.*

The proof is analogous to that of 5.10 (with the distinction that instead of 5.9 we now use 5.11).

A lattice ordered group G is called σ-complete if each subset X of G with $\emptyset\neq X$, $\operatorname{card} X\leqq\aleph_0$ possesses the least upper bound in G.

5.13. Lemma. *Let G be a σ-complete lattice ordered group, f, $g\in G$, $f\geqq 0$.* Then

$$f=\bigvee_{n=1}^{\infty}(f\wedge n|g|).$$

This assertion is proved in [18] for σ-complete vector lattices, but the proof remains valid also for σ-complete lattice ordered groups.

5.14. Lemma. *Let G and G' be complete lattice ordered groups and let φ be a complete homomorphism of G into G'. Let f, $g\in G$, $f'=\varphi(f)$, $g'=\varphi(g)$. Then $\varphi(f[g])=f'[g']$.*

Proof. The element f can be expressed as $f=f_1-f_2$ with $f_1\geqq 0$, $f_2\geqq 0$. Then we have

$$\varphi(f[g]) = \varphi(f_1[g] - \varphi(f_2[g]).$$

From 5.13 we obtain

$$\varphi(f_1[g]) = \varphi\left(\bigvee_{n=1}^{\infty}(f_1 \wedge n|g|)\right) = \varphi(f_1)[g']$$

and an analogous relation is valid for $\varphi(f_2[g])$. Hence

$$\varphi(f[g]) = \varphi(f_1)[g'] - \varphi(f_2)[g'] = f'[g'].$$

§ 6. The case of two b-free generators

In this paragraph it will be shown that if $\mathscr{C} \in \{\mathscr{C}_1, \mathscr{C}_s, \mathscr{C}_d\}$, then there does not exist any b-free complete lattice ordered group with two b-free generators in $\mathscr{C}$.

We need the following result on the orthogonal extension of a complete lattice ordered group (analogous to Lemma 3.4); for the proof cf. [9], [11].

6.1. Lemma. *Let G be a complete lattice ordered group. There exists a complete lattice ordered group $o(G)$ having the following properties:*

(i) *G is a convex l-subgroup of $o(G)$;*

(ii) *for each element $y \in o(G)$ with $0 < y$ there exists a disjoint subset $\{x_i\}_{i \in I}$ of G such that $y = \bigvee_{i \in I} x_i$ holds in $o(G)$;*

(iii) *G is orthogonally complete.*

The lattice ordered group $o(G)$ is determined uniquely up to isomorphisms.

Let us remark that from (ii) we obtain the assertion: for a complete lattice ordered group G we have $G = o(G)$ if and only if G is orthogonally complete.

The lattice ordered group G is said to be the orthogonal hull of G.

In view of (ii), we obtain:

6.2. Lemma. *Let a set $M \neq \emptyset$ generate a complete lattice ordered group G. Then M generates the complete lattice ordered group $o(G)$.*

Let us consider the following conditions for a class $\mathscr{C}$ of complete lattice ordered groups:

(a) If $H \in \mathscr{C}$ and H_1 is a closed l-subgroup of $\mathscr{C}$, then $H_1 \in \mathscr{C}$.

(b) If $H \in \mathscr{C}$ and H_1 is a convex l-subgroup of $\mathscr{C}$, then $H_1 \in \mathscr{C}$.

(c) $F \in \mathscr{C}$ (F being as in Ex. 1 of § 2).

(d) If $H \in \mathscr{C}$, then $o(H) \in \mathscr{C}$.

(e) The class $\mathscr{C}$ is closed with respect to complete homomorphisms.

6.3. Lemma. *Let $\mathscr{C}$ be a class of complete lattice ordered groups fulfilling the conditions* (b) *and* (c). *Assume that a two-element set* $\{g_1, g_2\}$ *is a set of a-free generators of a complete lattice ordered group G in the class* $\mathscr{C}$. *Then G is not orthogonally complete.*

Proof. Assume that G is orthogonally complete. Put

$$g_3 = |g_1| \vee |g_2|,$$

$$K = \bigcup_{n \in N} [-ng_3, ng_3].$$

Then k is a convex l-subgroup of G, hence $K \in \mathscr{C}$. Obviously $g_1, g_2 \in K$. Hence there exists a complete homomorphism φ of G into K such that $\varphi(g_i) = g_i$ $(i = 1, 2)$. Then φ is a complete homomorphism of G into G. Thus according to 5.8, φ is the identical mapping on G, whence $G = K$.

By 5.7 there exists in G an infinite disjoint subset $\{a_i\}$ $(i \in N)$, where $a_i > 0$ for each $i \in N$. Denote

$$b_i = g_3[a_i]$$

for each $i \in N$. If $b_i = 0$ for some $i \in N$, then $[a_i]^\delta$ is a closed l-subgroup of G containing both g_1 and g_2, and $[a_i]^\delta \neq G$ (since $a_i \notin [a_i]^\delta$); this is a contradiction. Thus $b_i > 0$ for each $i \in I$. The set $\{ib_i\}$ $(i \in N)$ is disjoint, hence there exists $b \in G$ with

$$b = \bigvee_{i \in N} ib_i .$$

For each $n \in N$ we have

$$b[a_{n+1}] = (n+1)b_{n+1} = (n+1)g_3[a_{n+1}] > ng_3[a_{n+1}],$$

therefore $b \nleq ng_3$. From this it follows $b \notin K = G$, which is a contradiction.

6.4. Corollary. *Let $\mathscr{C}$ be a class of complete lattice ordered groups fulfilling the conditions* (a), (b) *and* (c). *If a two-element set is a set of b-free generators of a complete lattice ordered group G in $\mathscr{C}$, then G cannot be orthogonally complete.*

This follows from 6.3 and 2.2.

6.5. Theorem. *Let $\mathscr{C}$ be a class of complete lattice ordered groups fulfilling the conditions* (a), (b), (c) *and* (d). *Then there does not exist any b-free lattice ordered group with two b-free generators in the class $\mathscr{C}$.*

Proof. Assume (by way of contradiction) that a two element set $\{g_1, g_2\}$ is a set of b-free generators of a complete lattice ordered group G in the class $\mathscr{C}$. Hence according to 6.4 G is not orthogonally complete, thus $G \neq o(G)$. By 6.2, the set $\{g_1, g_2\}$ generates the complete lattice ordered group $o(G)$. According to (d) we have $o(G) \in \mathscr{C}$. Hence there exists a complete homomorphism φ of G onto $o(G)$ such that $\varphi(g_i) = g_i$ for $i = 1, 2$. In view of 5.8 we have arrived at a contradiction.

6.6. Corollary. *There does not exist any b-free complete lattice ordered group with two b-free generators in the class* $\mathscr{C}_1$.

Since the class $\mathscr{C}_s$ and also the class $\mathscr{C}_d$ fulfils the condition (a)—(d), we obtain:

6.7. Corollary. *Let* $\mathscr{C} \in \{\mathscr{C}_s, \mathscr{C}_d\}$. *Then there does not exist any b-free complete lattice ordered group with two b-free generators in the class* $\mathscr{C}$.

From 6.4, 4.4 and 2.2 it follows:

6.7. Corollary. *Let* $\mathscr{C}$ *be a class of complete lattice ordered groups fulfilling the conditions* (a)—(e). *Let* $\alpha > 1$ *be a cardinal. Then there does not exist any b-free complete lattice ordered group with* α *b-free generators in* $\mathscr{C}$.

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О ПОЛНЫХ СТРУКТУРНО УПОРЯДОЧЕННЫХ ГРУППАХ С ДВУМЯ ОБРАЗУЮЩИМИ I

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Резюме

Пусть $\mathscr{C}$ — класс полных структурно упорядоченных групп. В этой статье введены понятия двух типов свободной полной структурно упорядоченной группы в классе $\mathscr{C}$. Исследовано существование свободной полной структурно упорядоченной группы с двумя свободными образующими в некоторых классах полных структурно упорядоченных групп.