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A CLASS OF SUBADDITIVELY CONTINUOUS REAL FUNCTIONS

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1. Introduction and notations

In the paper [1] the schema of Daniell integral is generalized in such a way that one get a wide class of functionals not necessarily linear. The generalization is based mainly on replacing additivity by substantially weaker condition, called uniform subadditive continuity.

Also we shall consider a non uniform subadditive continuity.

In [1] the functionals are defined on a general vector lattice $\mathscr{F}$ of real valued functions on a set T. If T is a singleton, then $\mathscr{F}$ is in fact the lattice of real numbers, thus the investigated functionals are real functions of the real variable.

The characterization when such functionals are subintegrals in sense of [1] is easy and is given there. In this paper we investigate a wider class of subadditively continuous real functions of the real variable.

The investigation of such functions f is needed when we consider transformations of the type $f(I(\cdot))$, where I is a Daniell subintegral.

If f is a real function of the real variable, we write: f: $R \to R$, where R as usually means the set of all real numbers, $R^* = R \cup \{\infty\} \cup \{-\infty\}$.

Remind the following notations:

If $x_0 \in R^*$, $C(f, x_0)$ means the cluster set of the function f at the point x_0, that is the set of all limit numbers of f at x_0.

It means that $y \in C(f, x_0)$ if and only if there exists a sequence $\{x_n\}$, $x_n \neq x_0$, $x_n \to x_0$ such that $f(x_n) \to y$.

Further let $C'(f, x_0) = C(f, x_0) \cup \{f(x_0)\}$ for $x_0 \in R$. For $x_0 = +\infty$, or $x_0 = -\infty$ put $C'(f, x_0) = C(f, x_0)$.

$C(f, x_0)^+ (C(f, x_0)^-)$ denotes the set of all right (left) limit numbers of f at x_0. Finally $C'(f, x_0)^+ = C(f, x_0)^+ \cup \{f(x_0)\}$ and $C'(f, x_0)^- = C(f, x_0)^- \cup \{f(x_0)\}$.

Definition 1. *We say that the function* f: $R \to R$ *is subadditively continuous at the point* x *on the right (on the left), shortly* (s.c.)$^+$ *at* x ((s.c.)$^-$ *at* x), *if for every* $\varepsilon > 0$

there exists $\delta>0$ *such that* $|f(y)|<\delta$ *implies* $|f(x+y)-f(x)|<\varepsilon$ $(|f(x-y)-f(x)|<\varepsilon)$.

We say that f is subadditively continuous at x, shortly (s.c.) *at x, if it is both* $(\text{s.c.})^+$ *and* $(\text{s.c.})^-$ *at x.*

We say that f is (s.c.), $(\text{s.c.})^+$ *or* $(\text{s.c.})^-$ *on R, if it has the corresponding property at each point $x \in R$.*

Similarly we can define uniform subadditive continuity on the right (on the left) and uniform subadditive continuity. It is easily seen that these three notions are equivalent, therefore it is enough to have the next.

Definition 2. *We say that the function $f\colon R \to R$ is uniformly subadditive continuous* (u.s.c.), *if for every $\varepsilon>0$ there exists $\delta>0$ such that $|f(y)|<\delta$ implies $|f(x+y)-f(x)|<\varepsilon$, for every $x \in R$.*

2.

From the definitions in paragraph 1. it is clear that the additivity of f implies (u.s.c.) of f and (u.s.c.) of f implies (s.c.) of f. It is easy to see that converse implications in general do not hold. (For example $f(x)=x^2$ is an (s.c.) function, which is not (u.s.c.) and $f(x)=c$ (a constant) is a (u.s.c.) function, but not additive.)

If f is symmetric or antisymmetric, then f is (s.c.)$\Leftrightarrow f$ is $(\text{s.c.})^+ \Leftrightarrow f$ is $(\text{s.c.})^-$. This does not hold in general. For example: every function f such that $f(0)=0$ is $(\text{s.c.})^+$ in 0 (it is enough to put $\delta=\varepsilon$), while it need not be $(\text{s.c.})^-$ in 0 (for instance $f(x)=-1$ for $x<0$ and $f(x)=x$ for $x \geqslant 0$).

We will see that there is some correspondence between continuity and the property (s.c.) of f. Nevertheless there exist functions which are uniformly continuous, but which are not (s.c.) at any point (for example: if f is uniformly continuous, zero on an interval not containing 0 and strictly increasing elsewhere). On the other hand there exist functions continuous and (u.s.c.), but not uniformly continuous (for example $f(x)=x^2+\delta$, where $\delta>0$). The Dirichlet function is an example of a function which is discontinuous at every point, but is (u.s.c.).

It turns out that for the property (s.c.) of f the value $f(0)$ is important, as well as the null-points of f.

It holds namely

Theorem 1. *Let $f(0) \neq 0$. Then f is* (s.c.) *if and only if there exists an $\varepsilon>0$ such that $|f(x)| \geqslant \varepsilon$, for every $x \in R$.*

Proof. (s.c.) of f follows trivially.

Suppose f is (s.c.), $f(0) \neq 0$ and the assertion of the theorem is not valid. Then there exists a sequence $\{x_n\}$ such that $x_n \to x_0 \in R^*$ and $\lim\limits_{n\to\infty} f(x_n)=0$. (It can be a stationary sequence.)

By (s.c.) of f for arbitrary $x \in R$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(y)| < \delta$ implies $|f(x \pm y) - f(x)| < \varepsilon$.

Thus there exists a natural N such that $|f(x \pm x_n) - f(x)| < \varepsilon$ for $n \geq N$ and therefore $\lim_{n \to \infty} f(x \pm x_n) = f(x)$. Then $f(0) = \lim_{n \to \infty} f(0 \pm x_n) = \lim_{n \to \infty} f(\pm x_n) = 0$, contrary to the assumption.

Remark. It is obvious that in Theorem 1 (s.c.) can be replaced by (u.s.c.), $(\text{s.c.})^+$ or $(\text{s.c.})^-$.

Corollary. *Let f be a continuous function such that $f(x) \neq 0$ for all $x \in R$. Then f is* (s.c.) *if and only if $0 \notin C(f, +\infty) \cup C(f, -\infty)$.*

Further we have

Proposition 1. *Let f be $(s.c.)^+$ at some $x_0 \in R$ with $f(x_0) \neq 0$. Then $C(f, +\infty) \neq \{0\} \neq C(f, -\infty)$.*

Proof. Suppose for instance that $C(f, +\infty) = \{0\}$. Then $f(x_n) \to 0$ for any sequence $\{x_n\}$, $x_n \to \infty$. But from $(\text{s.c.})^+$ of f at x_0 it follows that $f(x_0 + x_n) \to f(x_0) \neq 0$, while $x_0 + x_n \to \infty$, a contradiction.

The proof for $C(f, -\infty) = \{0\}$ is analogous.

Proposition 2. *Let f be* $(\text{s.c.})^-$ *at some $x_0 \in R$ with $f(x_0) \neq 0$. Then $0 \in C(f, +\infty)$ implies $C(f, -\infty) \neq \{0\}$ and $0 \in C(f, -\infty)$ implies $C(f, +\infty) \neq \{0\}$.*

Proof. Let $0 \in C(f, +\infty)$. Then there exists a sequence $\{x_n\}$ such that $x_n \to \infty$ and $f(x_n) \to 0$. *By* $(\text{s.c.})^-$ *of f at x_0 $f(x_0 - x_n) \to f(x_0) \neq 0$, while $x_0 - x_n \to -\infty$*, hence $C(f, -\infty) \neq \{0\}$.

The second implication is analogous.

Theorem 1 asserts that if $f(0) \neq 0$, then f is (s.c.) if and only if $0 \in \bigcup_{x \in R^*} C'(f, x)$. For this reason in what follows we restrict our interest only to functions f with $f(0) = 0$.

Let now $f(0) = 0$, but $0 \notin \bigcup_{x \in R^* - \{0\}} C'(f, x)$.

In this case the property (s.c.) of f at arbitrary point is connected with the continuity of the function f:

Theorem 2. *Let $0 \notin \bigcup_{x \in R^* - \{0\}} C'(f, x)$ and $x \in R$ be a point of continuity of f. Then x is also the point of the* (s.c.) *of f.*

Proof. Let f be continuous at $x_0 \in R$. Then for every $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $|y| < \delta_1$ implies $|f(x_0 \pm y) - f(x_0)| < \varepsilon$. Further for δ_1 there exists δ_2 such that $|f(y)| < \delta_2 \Rightarrow |y| < \delta_1$. Hence for each $\varepsilon > 0$ there exists $\delta = \delta_2$ such that $|f(y)| < \delta \Rightarrow |f(x_0 \pm y) - f(x_0)| < \varepsilon$, i.e. f is (s.c.) at x_0.

The next theorem shows that if $f(0)=0$ and $0 \notin \bigcup_{x \in R^*-\{0\}} C'(f, x) \cup C(f, 0)$, then f is (u.s.c.).

Theorem 3. *Let* $0 \notin \bigcup_{x \in R^*-\{0\}} C'(f, x) \cup C(f, 0)$. *Then f is* (u.s.c.).

Proof. It is enough to show that in this case there exists an $\varepsilon > 0$ such that $|f(x)| > \varepsilon$ for $x \neq 0$. Suppose the contrary. Then there exists a sequence $\{x_n\}$, $x_n \in R$, $x_n \neq 0$ such that $|f(x_n)| \leq 1/n$ for each natural n. Further there exists a subsequence $\{x_{n_k}\}$ and an $x \in R^*$ such that $x_{n_k} \to x$. But then

$$0 = \lim_{n \to \infty} f(x_n) = \lim_{k \to \infty} f(x_{n_k}) \in \bigcup_{x \in R^*-\{0\}} C'(f, x) \cup C(f, 0),$$

a contradiction.

Now we consider the case that not only $f(0)=0$ but also $C(f, 0)=\{0\}$, while $0 \notin \bigcup_{x \in R^*-\{0\}} C'(f, x)$ as in preceding cases. This means in fact: $y \to 0 \Leftrightarrow f(y) \to 0$.

Theorem 4. *Let f be a function with the property $y \to 0 \Leftrightarrow f(y) \to 0$. Then f is* (u.s.c.) *if and only if f is uniformly continuous and f is* (s.c.) *(at x) if and only if f is continuous (at x).*

Proof. Let f be (u.s.c.). Then for $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $|f(y)| < \delta_1 \Rightarrow |f(x+y) - f(x)| < \varepsilon$, for every $x \in R$. Further, for δ_1 there exists $\delta_2 > 0$ such that $|y| < \delta_2 \Rightarrow |f(y)| < \delta_1$. Hence for $\varepsilon > 0$ there exists $\delta = \delta_2 > 0$ such that $|y| < \delta$ implies $|f(x+y) - f(x)| < \varepsilon$, for every $x \in R$, i.e. f is uniformly continuous.

Now let f be uniformly continuous. Then for $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $|y| < \delta_1 \Rightarrow |f(x+y) - f(x)| < \varepsilon$ for every $x \in R$. Further holds that for δ_1 there exists $\delta_2 > 0$ such that $|f(y)| < \delta_2 \Rightarrow |y| < \delta_1$. Hence for $\varepsilon > 0$ there exists $\delta = \delta_2$ such that $|f(y)| < \delta$ implies $|f(x+y) - f(x)| < \varepsilon$ for every $x \in R$, i.e. f is (u.s.c.).

The proof in the case of (s.c.) and continuity (at some x) is similar.

Corollary 1. *Let f be continuous or uniformly continuous, let $f^{-1}(0) = 0$, and let* $\lim_{x \to \infty} f(x) \neq 0 \neq \lim_{x \to -\infty} f(x)$. *Then f is* (s.c.) *or* (u.s.c.) *respectively.*

Corollary 2. *Let f be continuous or uniformly continuous one-to-one function with $f(0)=0$. Then f is* (s.c.) *or* (u.s.c.) *respectively.*

Similarly as the Theorem 4 one can prove the following assertions:

Proposition 3. *Let f be a function with the property: $y \to 0^+ \Leftrightarrow f(y) \to 0$. Then f is* $(\text{s.c.})^+$ *(at x) if and only if f continuous on the right (at x).*

Proposition 4. *Let f be a function with the property: $y \to 0^- \Leftrightarrow f(y) \to 0$. Then f is* $(\text{s.c.})^-$ *(at x) if and only if f is continuous on the left (at x).*

Corollary. *Let f be* (s.c.)$^+$, *continuous on the right at* 0 *with* $f(0)=0$. *Then f is continuous on the right on R.* Similar assertion is true for (s.c.)$^-$ and continuity on the left.

Finally we consider such (s.c.) functions that not only $f(0)=0$ but also $0 \in \bigcup_{x \in R^*-\{0\}} C'(f, x)$.

First we examine the case $0 \in \bigcup_{x \in R^*-R} C(f, x)$. We have

Proposition 5. *a) Let f be* (s.c.)$^+$ *and let* $0 \in C(f, \infty)$. *Then* $\{f(x): x \in R\} \subset C(f, \infty)$.

b) Let f be (s.c.)$^-$ *and let* $0 \in C(f, \infty)$. *Then* $\{f(x): x \in R\} \subset C(f, -\infty)$.

c) Let f be (s.c.)$^+$ *and let* $0 \in C(f, -\infty)$. *Then* $\{f(x): x \in R\} \subset C(f, -\infty)$.

d) Let f be (s.c.)$^-$ *and let* $0 \in C(f, -\infty)$. *Then* $\{f(x): x \in R\} \subset C(f, \infty)$.

Proof. We prove a), for example.

Since $0 \in C(f, \infty)$, there exists a sequence $\{x_n\}$ such that $x_n \to \infty$ and $f(x_n) \to 0$. Let $x \in R$ be arbitrary. Then from (s.c.)$^+$ at x follows that $f(x+x_n) \to f(x)$, while $x+x_n \to \infty$. Hence $f(x) \in C(f, \infty)$.

Corollary. *a) Let f be* (s.c.) *and let* $0 \in C(f, \infty) \cup C(f, -\infty)$. *Then* $\{f(x): x \in R\} \subset C(f, \infty) \cap C(f, -\infty)$.

b) Let f be (s.c.)$^+$ *or* (s.c.)$^-$ *and let* $0 \in C(f, \infty) \cap C(f, -\infty)$. *Then* $\{f(x): x \in R\} \subset C(f, \infty) \cap C(f, -\infty)$.

c) Let f be (s.c.)$^+$ *or* (s.c.)$^-$, *let the limits* $\lim_{x \to \infty} f(x)$ *and* $\lim_{x \to -\infty} f(x)$ *exist and let* $0 \in C(f, \infty) \cup C(f, -\infty)$. *Then* $f \equiv 0$.

Proof. a) and b) are trivial consequences of Proposition 5.

c) Suppose for example that f is (s.c.)$^-$ and $0 \in C(f, -\infty)$. Then $\lim_{x \to -\infty} f(x)=0$ and $\{f(x): x \in R\} \subset C(f, \infty)$ by d) of Proposition 5. By assumption $C(f, \infty)$ is a one-point set. Let $C(f, \infty)=\{a\}$, where $a \neq 0$. Then $\{f(x): x \in R\}=\{a\}$ and therefore $\lim_{x \to \infty} f(x)=a$, a contradiction.

The proof of other cases is similar.

Let now $f(0)=0$, and suppose f is (s.c.) at $x' \in R$. Let $x_0 \in R$, $x_0 \neq 0$ be such a point that $0 \in C(f, x_0)$.

Then $f(x') \in C(f, x'+x_0)$ by (s.c.)$^+$ and $f(x') \in C(f, x'-x_0)$ by (s.c.)$^-$ of f at x', hence $f(x') \in C(f, x'+x_0) \cap C(f, x'-x_0)$.

It holds

Theorem 5. *Let f be an* (s.c.) *function such that* $0 \in C'(f, x_0)$, *where* $x_0 \neq 0$. *If* $x' \in R$ *is a point of continuity of f then* $f(x')=f(x'+x_0)=f(x'-x_0)$.

On the other hand, if $x' \in R$ is such that $f(x'+x_0) \neq f(x'-x_0)$ then x' is a point of discontinuity of f.

Proof. Let $0 \in C'(f, x_0)$, where $x_0 \neq 0$. Then there exists a sequence $\{x_n\}$ such that $x_n \to x_0$ and $f(x_n) \to 0$. We can assume that $x_n \to x_0$ from the right (for $x_n \to x_0$ from the left the proof is analogous). Then $x' - x_0 + x_n \to x'$ from the right and since f is (s.c.)$^+$ at $x' - x_0$, $f(x' - x_0 + x_n) \to f(x' - x_0)$. Hence $f(x'-x_0) \in C'(f, x')^+$.

Similarly if f is (s.c.)$^-$ at $x' + x_0$, then $f(x'+x_0) \in C'(f, x')^-$.

Our assertion now follows, since $\{f(x')\} = C'(f, x')^- = C'(f, x')^+$ — by continuity of f at x'.

If $f(x'+x_0) \neq f(x'-x_0)$ then at least one of the sets $C'(f, x')^-$ and $C'(f, x')^+$ is different from $\{f(x')\}$, hence x' is a point of discontinuity of f.

There is a connection between the notions of subadditive continuity and almost periodicity or periodicity.

Let us remind that a function $f\colon R \to R$ is called almost periodic if it is continuous and for each $\varepsilon > 0$ there is a number $l(\varepsilon) > 0$ such that each interval $a < x < a + l$ contains at least one number $\tau = \tau(\varepsilon)$ for which $|f(x+\tau) - f(x)| < \varepsilon$ for each $x \in (-\infty, +\infty)$.

Namely from the preceding considerations we have

Proposition 6. *Let f be* (u.s.c.) *and let $0 \in \bigcup_{x \in R - \{0\}} C(f, x)$. Then f is almost-periodic and has the property: For every $\varepsilon > 0$ there exists an almost-period x_ε ($x_\varepsilon \neq 0$) such that $|f(x + x_\varepsilon) - f(x)| < \varepsilon$, for every $x \in R$.*

When f is a periodic function, we denote by A_f the set of all periods of f, i.e. the set of all real numbers x for which: $f(x+y) = f(y)$ for every real y.

Theorem 6. *Let there exist $x \in R$, $x \neq 0$ such that $f(x) = 0$ and let f be* (s.c.)$^+$. *Then f is periodic and $A_f = f^{-1}(0)$.*

Proof. Let $x \in f^{-1}(0)$, $x \neq 0$ and suppose that $x \notin A_f$. Then there exists a y such that $|f(x+y) - f(y)| > 0$. Choose ε so that $0 < \varepsilon < |f(y+x) - f(y)|$. Then evidently there exists no $\delta > 0$ corresponding to y and ε according to (s.c.)$^+$ of f, hence f is not (s.c.)$^+$, a contradiction.

Thus f is periodic and $f^{-1}(0) \subset A_f$ (obviously $f(0) = 0$).

Let now $x \in A_f$. Then $f(0+x) = f(0) = 0$ and so $x \in f^{-1}(0)$.

Corollary 1. *Let $0 \in \bigcup_{x \in R - \{0\}} C'(f, x)$ and let f be continuous and* (s.c.)$^+$. *Then f is periodic and $A_f = f^{-1}(0)$.*

Corollary 2. *Let f be a non-constant* (s.c.)$^+$ *function. Then the set of its null points ($f^{-1}(0)$) contains no interval. If there exists an $x_0 \in R$, $x_0 \neq 0$ such that $f(x_0) = 0$ and if f has at least one point of continuity, then $f^{-1}(0)$ is a set of type $\{kx\colon k = 0, \pm 1, \pm 2, \pm 3, \ldots\}$, where $x = \min\{|y|, y \in f^{-1}(0), y \neq 0\}$.*

It is obvious that in Theorem 6 and in its corollaries the property (s.c.)$^+$ can be replaced by the properties (s.c.)$^-$ or (s.c.).

Now arises the question whether the assumption $A_f = f^{-1}(0)$ of Theorem 6 can be used to characterize this class of (s.c.) functions.

As the next example shows, this is not possible.

Example 1. Let us define the function f first on the interval $\langle 0, 1)$: Put $f(0) = 0$, $f(1/2^n) = 1/2^n$, for $n = 1, 2, 3, \ldots$, and $f(x) = 1$ elsewhere. Let us extend it periodically to the whole R. Thus f is periodic and $A_f = f^{-1}(0)$. But obviously f is neither (s.c.)$^+$ nor (s.c.)$^-$. Let for example $x = 1/2$ and $0 < \varepsilon < 1/2$. It is clear that for arbitrary $\delta > 0$ we can find a number of the form $1/2^k$ such that $f(1/2^k) = 1/2^k < \delta$, while $|f(1/2 \pm 1/2^k) - f(1/2)| = 1/2 > \varepsilon$.

This situation cannot arise for a continuous function:

Theorem 7. *Let f be a continuous periodic function such that $f^{-1}(0) \subset A_f$. Then f is* (u.s.c.).

Proof. In the case $f^{-1}(0) = \emptyset$ the assertion of the theorem follows from Theorem 1.

Otherwise clearly $f^{-1}(0) = A_f$. Take $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta_1 > 0$ such that $|x - x'| < \delta_1$ implies $|f(x) - f(x')| < \varepsilon$. Denote $y_0 = \min\{x: x > 0,\ x \in A_f\}$. Let $0 < \delta_2 < \min(\delta_1, y_0/2)$ and put $\delta = \inf\{|f(x)|: x \in \langle \delta_2, y_0 - \delta_2 \rangle\}$. Evidently $\delta > 0$. Then $|f(y)| < \delta$ implies $y \in \bigcup_{x \in A_f} (x - \delta_2, x + \delta_2)$. Hence if $x_0 \in A_f$ is the element of A_f for which $y \in (x_0 - \delta_2, x_0 + \delta_2)$, then $|f(x + y) - f(x)| \leq |f(x + y) - f(x + x_0)| + |f(x + x_0) - f(x)| = |f(x + y) - f(x + x_0)| < \varepsilon$, for every $x \in R$.

The theorem is proved.

The Dirichlet function is an example of a discontinuous (u.s.c.) and periodic function with $A_f = f^{-1}(0)$.

As the next theorem shows, there is a condiction on f which together with (s.c.)$^+$ or (s.c.)$^-$ implies its periodicity.

We shall say, that a sequence $\{x_n\}$, $x_n \to \infty$ is of type (A), if there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} = c + kd$, $k = 1, 2, 3, \ldots$, where $c \geq 0$, $d > 0$ are some constants.

Theorem 8. *Let f be* (s.c.)$^+$ *or* (s.c.)$^-$ *and let there exist a sequence $\{x_n\}$ of type (A) such that $f(x_n) \to 0$. Then f is periodic.*

Proof. Let f be (s.c.)$^+$ and let $x_0 \in R$. Since $\{x_n\}$ is of type (A), by assumption

$\lim_{k\to\infty} f(c + kd) = \lim_{k\to\infty} f(x_{n_k}) = 0$ for some constants c, d.

Thus $f(x_0) = \lim_{k\to\infty} f(x_0 + c + kd)$ by (s.c.)$^+$ of f at x_0.

For the same reasons

$$f(x_0+d)=\lim_{k\to\infty} f(x_0+d+c+kd)=\lim_{k\to\infty} f(x_0+c+(k+1)d)= \\ +\lim_{k\to\infty} f(x_0+c+kd)=f(x_0),$$

hence f is periodic with period d. (d is not necessarily the smallest positive period.)

The proof for an (s.c.)$^-$ function is analogous.

Remark. If in the preceding proof $c\neq 0$, then c must be also a period of the function f. Namely by assumptions,
$f(k_0d)=\lim_{k\to\infty} f(k_0d+c+kd)=\lim_{k\to\infty} f(c+(k_0+k)d)=0$ for each $k_0=0, 1, 2, 3, \ldots$,
hence $f(c)=\lim_{k\to\infty} f(c+kd)=0$ by (s.c.)$^+$ of f at c. The fact that c is a period of f now follows from Theorem 6.

The next example shows that if $f\colon \langle 0, \infty)\to R$ is (s.c.)$^+$, $0\in C(f, \infty)$ but there exists no sequence $\{x_n\}$ of type (A) such that $f(x_n)\to 0$, then f may be non-periodic.

Example 2. Define the function $f\colon \langle 0, \infty)\to R$ as follows: $f(0)=0$; if $x\in(0, \infty)$ is a number of the form $(+)$ $x=\alpha_k\ 10^k+\alpha_{k-1}\ 10^{k-1}+\ldots+\alpha_1\ 10$, with $\alpha_1, \ldots, \alpha_{k-1}$ equal 0 or 1, $\alpha_k=1$, then put $f(x)=1/10^{\alpha_x}$, where $\alpha_x=\min\{i\colon \alpha_i\neq 0\}$; and $f(x)=1$ elsewhere. Obviously $0\in C(f, \infty)$ and f is not periodic.

Let us prove that f is (s.c.)$^+$, that is $\forall\varepsilon>0$, $\forall x$, $\exists\delta>0$: $|f(y)|<\delta\Rightarrow|f(x+y)-f(x)|<\varepsilon$.

If x is not an integer, then for any $\varepsilon>0$ it is enough to choose $\delta<1$. Namely, if $|f(y)|<\delta<1$, then y is an integer, hence $x+y$ is not an integer and thus $|f(x+y)-f(x)|=|1-1|<\varepsilon$.

For $x=0$ f is clearly (s.c.)$^+$. (Since $f(0)=0$.)

If $x\neq 0$ is an integer number not expressible in the form $(+)$ and if $\varepsilon>0$, then there is a minimal natural number $r=r(x)$ such that $x<10^r$. Take $\delta<1/10^r$. Then $|f(y)|<\delta$ implies again $|f(x+y)-f(x)|=|1-1|<\varepsilon$.

Finally, let x be a number of the form $(+)$ and let $\varepsilon>0$. Then there is a minimal natural number r such that $x<10^r$. Take $\delta<1/10^r$. If y is such that $|f(y)|<\delta$, then $x+y$ is again a number of the form $(+)$ and moreover $f(x+y)=f(x)$. Thus $|f(x+y)-f(x)|=0<\varepsilon$.

Problem: Does there exist a (s.c.)$^+$, continuous and non periodic function f: $\langle 0, \infty)\to R$ ($f\colon R\to R$) such that $0\in C(f, \infty)$?

Let us conclude with some remarks about algebraic, lattice and topological structures of our classes of (s.c.)$^+$, (s.c.)$^-$, (s.c.) and (u.s.c.) functions.

Nonnegative (nonpositive) (s.c.)$^+$, (s.c.)$^-$ and therefore also (s.c.) functions are clearly closed under the addition.

The same is true for (u.s.c.) functions.

In general this is not true. For instance, the function $\sin x$ is neither (s.c.)$^+$ nor (s.c.)$^-$ but it can be written as the sum of two (u.s.c.) functions: $\sin x+2$ and -2.

The multiple of some (s.c.)$^+$, (s.c.)$^-$, (s.c.) or (u.s.c.) function by an arbitrary scalar is also an (s.c.)$^+$, (s.c.)$^-$, (s.c.) or (u.s.c.) function, respectively.

Analogously the maximum and the minimum of (s.c.)$^+$, (s.c.)$^-$, (s.c.) or (u.s.c.) functions is also an (s.c.)$^+$, (s.c.)$^-$, (s.c.) or (u.s.c.) function, respectively. This follows from the inequalities:

$$|\max(a, b) - \max(c, d)| \leq |a - c| + |b - d|,$$

and

$$|\min(a, b) - \min(c, d)| \leq |a - c| + |b - d|, \; a, b, c, d \text{ —}$$

real.

Concerning limits, the sequence of (u.s.c.) functions $\{e^x + 1/n\}$ uniformly converges to the function e^x, which is neither (s.c.)$^+$ nor (s.c.)$^-$ at any point.

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КЛАСС СУБАДДИТИВНО НЕПРЕРЫВНЫХ ВЕЩЕСТВЕННЫХ ФУНКЦИЙ

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Резюме

В этой статье исследуются классы вещественных функций вещественной переменной, которые удовлетворяют условиям типа: для всякого $\varepsilon > 0$ и всякого вещественного x существует $\delta > 0$, такое, что для всякого вещественного y из $|f(y)| < \delta$ вытекает $|f(x \pm y) - f(x)| < \varepsilon$.