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A PERRON-TYPE INTEGRAL OF ORDER 2 FOR RIESZ SPACES

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ABSTRACT. In this paper we introduce a Perron-type integral of order two in Riesz spaces. We prove some fundamental properties and give applications to Fourier series.

1. Introduction

In this paper we introduce a Perron-type integral of order two for functions defined on a subinterval $[a,b] \subset \mathbb{R}$ and taking values in a Dedekind complete Riesz space R, we prove some properties of this integral and compare it with the Perron integral of order one, which was introduced in [3] and which is equivalent to the Henstock-Kurzweil integral introduced by B. Riečan in [10] (see also [11; Chapter 5] and its bibliography). In the case $R = \mathbb{R}$, the Perron integrals of order two and order n were investigated by several authors: among them we recall Bullen [5], James [7] and Sklyarenko [12].

In general, even in the real case, the classical Perron integral of order one is not sufficient in order to integrate all convergent trigonometric series. In this paper we show that, if f is a map defined on $[a,b]$ and with values in $L^0(X,\mathcal{B},\mu)$, where $(X,\mathcal{B},\mu)$ is a measure space with μ positive, σ-additive and σ-finite and X is the "time" space, and f can be represented as a sum of a trigonometric series, convergent pointwise with respect to the "space" variable and almost everywhere with respect to the "time" variable, then f is Perron integrable of order two. We note that the technique here used are slightly different than the classical ones, in the sense that in the classical case it is "natural" to assume

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that the major and minor functions of order 2 satisfy some suitable conditions up to the complement of countable sets or of sets of Lebesgue measure zero. The classical approach is not "natural" in Riesz spaces, substantially because there are some Riesz spaces R and Lipschitz functions $f\colon [a,b] \to R$ that are not differentiable in any point of $]a,b[$ (see [1; pp. 311–312, Remark 4.38]). So we suppose that the involved major and minor functions satisfy weaker conditions: however we have to require some hypothesis on the "nature" of major and minor functions, in order that our Perron integral of order 2 is well-defined (in Riesz spaces it is not trivial to prove this last property). Moreover we want that at least some elementary properties of our Perron 2 integral are satisfied: for instance, the fact that every function Perron 1 integrable is Perron 2 integrable too. To require only uniform continuity of the major and minor functions allows us to give a positive answer to the above problems and questions, by giving a definition of Perron integrability of order 2 that looks quite "simple". Furthermore, we remark that in the literature, in the classical case, to require different properties for the considered major and minor functions of order 2 can "generate" different types of Perron integrals of order 2 (see [5; p. 226]).

2. Preliminaries

Throughout this paper, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+$ is the set of all (strictly) positive real numbers, $\mathbb{R}_0^+$ is the set of all non negative real numbers, and R is a generic Dedekind complete Riesz space. Moreover, we denote by Δ the set $(\mathbb{R}^+)^{[a,b]}$, where $[a,b]$ is a subinterval of $\mathbb{R}$. The set Δ is endowed with the following ordering:

$$(\forall x \in [a,b])\big(\Delta \ni \delta_1 \succeq \delta_2 \iff \delta_1(x) \le \delta_2(x)\big)\,. \tag{1}$$

We observe that $(\Delta, \succeq)$ is a directed set.

DEFINITIONS 2.1. Let $(\Lambda, \succeq)$ be a directed set. A net $(r_\lambda)_{\lambda\in\Lambda}$ of elements of R is said to be *order-convergent* (or *(o)-convergent*) to $r \in R$ if $\sup\limits_{\rho\in\Lambda}\inf\limits_{\lambda\succeq\rho} r_\lambda = \inf\limits_{\rho\in\Lambda}\sup\limits_{\lambda\succeq\rho} r_\lambda = r$. An *$(o)$-net* $(p_\lambda)_{\lambda\in\Lambda}$ is any decreasing net of positive elements of R, order-convergent to zero. A sequence $(q_n)_n$ in R is called *(o)-sequence* if it is decreasing and $\inf\limits_n q_n = 0$.

A Dedekind complete Riesz space is said to be *super Dedekind complete* if every subset $R_1 \subset R$, $R \neq \emptyset$, bounded from above contains a countable subset having the same supremum as R_1. We observe that, if $(X, \mathcal{B}, \mu)$ is a measure space with $\mu\colon \mathcal{B} \to \tilde{\mathbb{R}}_0^+$ σ-additive and σ-finite, then the space $L^0(X, \mathcal{B}, \mu)$ is super Dedekind complete, where $L^0(X,\mathcal{B},\mu)$ is the space of all extended real-valued μ-almost everywhere finite and μ-measurable functions, modulo μ-almost everywhere equal functions (see [8; pp. 126–127, Example 23.3.(iv)]).

A Dedekind complete Riesz space is called *algebra* if there exists a "product" map $\cdot : R \times R \to R$, compatible with respect to sum, order and order limits.

Given a function $f\colon [a,b] \to R$, we say that f is *linear* in $[a,b]$ if there exist two elements of R, α_1 and α_2, such that

$$(\forall x \in [a,b])\big(f(x) = x\alpha_1 + \alpha_2\big). \tag{2}$$

Given a map $f\colon [a,b] \to R$, we say that f is *convex* in $[a,b]$ if for every $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$ and for each $t \in [0,1]$ we have:

$$f\big(tx_1 + (1-t)x_2\big) \le tf(x_1) + (1-t)f(x_2). \tag{3}$$

A map $f\colon [a,b] \to R$ is said to be *uniformly continuous* in $[a,b]$ if there exists an (o)-sequence $(p_n)_n$ such that we have:

$$(\forall n \in \mathbb{N})\,(\forall t,x \in [a,b])\big(|t-x| \le 1/n \implies |f(t)-f(x)| \le p_n\big). \tag{4}$$

A mapping $f\colon [a,b] \to R$ is said to be *uniformly differentiable* in $[a,b]$ if there exists a function $g\colon [a,b] \to R$ such that

$$(\mathrm{o})\text{-}\lim_n \Big(\sup\Big\{ \big|\tfrac{f(v)-f(u)}{v-u} - g(x)\big| : \ u,v,x \in A_n\Big\}\Big) = 0,$$

where $A_n \equiv \{u,v,x \in [a,b] : \ u \ne v \text{ and } x - 1/n \le u \le x \le v \le x + 1/n\}$ for all $n \in \mathbb{N}$. As a consequence of the properties of (o)-convergence, we see that such a function g is unique. In this case, the map g will be called the *uniform derivative* of f, or simply *derivative*, when no confusion can arise.

We now recall the Maeda-Ogasawara-Vulikh representation theorem for Riesz spaces.

THEOREM 2.2. *Given an Archimedean (Dedekind complete) Riesz space R, there exists a compact extremely disconnected topological space Ω, unique up to homeomorphisms, such that R can be embedded order densely as a (solid) subspace of $\mathcal{C}_\infty(\Omega) = \{f \in \tilde{\mathbf{R}}^\Omega : \ f$ is continuous, and $\{\omega : \ |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if $\{a_\lambda\}_{\lambda\in\Lambda}$ is any net such that $a_\lambda \in R$ for all λ, and $a = \sup_\lambda a_\lambda \in R$ (where the supremum is taken with respect to R), then $a = \sup_\lambda a_\lambda$ with respect to $\mathcal{C}_\infty(\Omega)$, and the set $\Big\{\omega \in \Omega : \ \Big(\sup_\lambda a_\lambda\Big)(\omega) \ne \sup_\lambda a_\lambda(\omega)\Big\}$ is meager in Ω.*

The following result holds:

PROPOSITION 2.3. *Let R be a Dedekind complete Riesz space, Ω be as in Theorem 2.2. Assume that $f\colon [a,b] \to R$ is a bounded function. For all $\omega \in \Omega$, define $f_\omega\colon [a,b] \to \tilde{\mathbb{R}}$ by setting*

$$(\forall x \in [a,b])\big(f_\omega(x) = f(x)(\omega)\big). \tag{5}$$

Then f is uniformly continuous if and only if the set

$$\{\omega \in \Omega : \ f_\omega \ \textit{is not continuous in} \ [a,b]\}$$

is meager in Ω.

The proof of Proposition 2.3 is analogous to the one of [1; Proposition 3.8].

3. The Perron integrals of order 1 and 2.

We now recall the Perron integral of order 1 in Dedekind complete Riesz spaces, which was introduced in [3] and the Henstock-Kurzweil integral in Dedekind complete Riesz spaces, and we will introduce the Perron integral of order 2. For a survey of the main properties of the Perron integral of order two in the real case, see [7] and [14; pp. 86–91].

DEFINITION 3.1. Let R be a Dedekind complete Riesz space, and $f\colon [a,b] \to R$. We say that $G\colon [a,b] \to R$ is a *major function of order* 1 for f if

3.1.1) $G(a) = 0$;
3.1.2) there exists $\delta \in \Delta$ such that $f(\xi)(v-u) \le G(v) - G(u)$ for every choice of points $u, v, \xi \in [a,b]$ such that $\xi - \delta(\xi) \le u \le \xi \le v \le \xi + \delta(\xi)$.

A map $K\colon [a,b] \to R$ is said to be a *minor function of order* 1 for f if

3.1.3) $K(a) = 0$;
3.1.4) there exists $\delta \in \Delta$ such that $f(\xi)(v-u) \ge K(v) - K(u)$ for all $u, v, \xi \in [a,b]$ such that $\xi - \delta(\xi) \le u \le \xi \le v \le \xi + \delta(\xi)$.

From now on, if G and K are a major and a minor function of order 1 for f, and δ_1 and δ_2 satisfy 3.1.2) and 3.1.4) respectively, then we say that δ_1 and δ_2 are *compatible* with G and K respectively; moreover we denote by $\mathcal{G}_1$ and $\mathcal{K}_1$ the classes of all major functions and all minor functions of order 1 for f respectively.

DEFINITION 3.2. A function $f\colon [a,b] \to R$ is said to be *Perron integrable* (shortly *$(\mathcal{P})$-integrable*) *of order* 1 in $[a,b]$ if f has both major and minor functions and

$$\inf_{G \in \mathcal{G}_1} [G(b)] = \sup_{K \in \mathcal{K}_1} [K(b)] \in R. \tag{6}$$

The common value I_f in (6) will be called *$(\mathcal{P})$-integral of f*, and we write $(\mathcal{P})\int_a^b f(t)\,\mathrm{d}t = I_f$ or $(\mathcal{P})\int_a^b f = I_f$.

We now recall the Henstock-Kurzweil integral in Riesz spaces (see [10], [1]).

A *decomposition of* $[a,b]$ is a set of the type $\{(A_i,\xi_i):\ i=1,\dots,n\}$, where $\{A_i\}_{i=1}^n$ is a family of pairwise nonoverlapping intervals of $[a,b]$ and $\xi_i \in A_i$ for all $i=1,\dots,n$. If $\bigcup_{i=1}^n A_i = [a,b]$, then the decomposition $\{(A_i,\xi_i):\ i=1,\dots,n\}$ is called *partition.* Given a decomposition $E=\{([x_{i-1},x_i],\xi_i):\ i=1,\dots,n\}$ and a function $\delta\in\Delta$, we say that E is *δ-fine* if $x_i - x_{i-1} \le \delta(\xi_i)$ for all $i=1,\dots,n$.

Definition 3.3. A function $f\colon [a,b]\to R$ is said to be *$(\mathcal{H})$-integrable* (*Henstock-Kurzweil integrable*) if there exists an element $Y\in R$ such that

$$(\mathrm{o})\text{-}\lim_{\delta\in\Delta}\big[\sup\{|S(f,E)-Y|:\ E \text{ is a } \delta\text{-fine partition of } [a,b]\}\big]=0,$$

where $E=\big([x_{i-1},x_i],\xi_i\big)_{i=1}^n$, $S(f,E)\equiv\sum_{i=1}^n f(\xi_i)(x_i-x_{i-1})$. The element Y will be called *$(\mathcal{H})$-integral of f*, and we write $(\mathcal{H})\int_a^b f(t)\,\mathrm{d}t = Y$ or $(\mathcal{H})\int_a^b f = Y$.

Given a $(\mathcal{H})$-integrable functions $f\colon [a,b]\to R$, we observe that f is integrable on every subinterval of $[a,b]$, and set:

$$F^{(1)}(x)=\begin{cases}(\mathcal{H})\int_a^x f & \text{if } a<x\le b,\\ 0 & \text{if } x=a.\end{cases}\tag{7}$$

The map $F^{(1)}$ will be called the *$(\mathcal{H})$-integral function* associated to f.

In [3] we proved that the Henstock-Kurzweil integral and the Perron integral are equivalent in Dedekind complete Riesz spaces.

We now prove the following result, which in [2] was proved for super Dedekind complete Riesz spaces, and which will be useful in the sequel.

Lemma 3.4. *Let R be any Dedekind complete Riesz space, $f\colon [a,b]\to R$ be a Henstock-Kurzweil integrable function and $F^{(1)}$ be as in (7). Then $F^{(1)}$ is uniformly continuous in $[a,b]$.*

Proof. First of all we observe that $F^{(1)}$ is bounded, by virtue of [1; Propositions 3.4, 4.37]. Let Ω be as in 2.2, and let $\mathcal{G}_1$ and $\mathcal{K}_1$ be the classes of all major functions and all minor functions of order 1 for f respectively. For all

$\omega \in \Omega$ and $x \in [a,b]$, let $f_\omega \colon [a,b] \to \tilde{\mathbb{R}}$ be as in (5). Given $G \in \mathcal{G}_1$ and $K \in \mathcal{K}_1$ and $\omega \in \Omega$, let us define $G_\omega, K_\omega \colon [a,b] \to \tilde{\mathbb{R}}$ as follows:

$$G_\omega(x) = G(x)(\omega), \quad K_\omega(x) = K(x)(\omega) \qquad \text{for all} \quad x \in [a,b]. \tag{8}$$

As f is $(\mathcal{H})$-integrable, by Henstock's Lemma (see [1]) there exists an (o)-net $(p_\delta)_{\delta \in \Delta}$ such that for all $\delta \in \Delta$ we have:

$$\sum_{i=1}^{n} \left| f(\xi_i)(x_{i-1} - x_i) - (\mathcal{H}) \int_{x_{i-1}}^{x_i} f \right| \le p_\delta$$

for each δ-fine partition $E = \{([x_{i-1}, x_i], \xi_i) : i = 1, \dots, n\}$.

For every $\delta \in \Delta$, define $\chi_\delta \colon [a,b] \to R$ by setting $\chi_\delta(a) = 0$ and:

$$(\forall x \in]a,b]) \Big(\chi_\delta(x) \equiv \sup \Big\{ \sum_{i=1}^{n} \left| f(\xi_i)(x_i - x_{i-1}) - (\mathcal{H}) \int_{x_{i-1}}^{x_i} f \right| : \\ \{([x_{i-1}, x_i], \xi_i) : i = 1, \dots, n\} \text{ is a } \delta\text{-fine partition of } [a,x] \Big\} \Big). \tag{9}$$

It is easy to check that χ_δ is nondecreasing, $\chi_\delta(b) \le p_\delta$, and that $F^{(1)} + \chi_\delta$ is a major function of order 1 and $F^{(1)} - \chi_\delta$ is a minor function of order 1 for f.

We denote by Δ_1 the following family:

$$\Delta_1 = \Big\{ \delta_1 \colon [a,b] \to \mathbb{R} : \ (\forall x \in [a,b]) (\delta_1(x) = \min\{\delta(x), 1\}) \Big\}_{\delta \in \Delta}, \tag{10}$$

and by $\overline{\mathcal{G}_1}$ and $\overline{\mathcal{K}_1}$ the subclasses of $\mathcal{G}_1$ and $\mathcal{K}_1$ such that for every $G \in \overline{\mathcal{G}_1}$ there exists $\overline{\delta_1} \in \Delta_1$ such that $(\forall x)(G(x) = F^{(1)}(x) + \chi_{\overline{\delta}_1}(x))$ and for each $K \in \overline{\mathcal{K}_1}$ there exists $\overline{\delta_1} \in \Delta_1$ such that $(\forall x)(K(x) = F^{(1)}(x) - \chi_{\overline{\delta}_1}(x))$ respectively. From the fact that the net $(p_\delta)_{\delta \in \Delta}$ is an (o)-net it follows that

$$0 \le p_{\delta_1} \le p_{\underline{1}} \qquad \text{for all} \quad \delta_1 \in \Delta_1, \tag{11}$$

where $\underline{1}$ is the function which associates the real number 1 to every $x \in [a,b]$, and

$$\inf_{\delta_1 \in \Delta_1} p_{\delta_1} = 0. \tag{12}$$

Set now

$$r = \sup_{x \in [a,b]} F^{(1)}(x) + p_{\underline{1}}, \tag{13}$$

$$s = \inf_{x \in [a,b]} F^{(1)}(x) - p_{\underline{1}}. \tag{14}$$

Let Ω be as in Theorem 2.2 and N^* be a nowhere dense subset of Ω such that $r(\omega) \in \mathbb{R}$ and $s(\omega) \in \mathbb{R}$ for all $\omega \in \Omega \setminus N^*$. We have:

$$\inf_{G \in \overline{\mathcal{G}_1}} G(b) - \sup_{K \in \overline{\mathcal{K}_1}} K(b) = 0 \tag{15}$$

and thus there exists a meager set $N_0 \supset N^*$ such that:

$$\inf_{G\in\overline{\mathcal{G}_1}} G(b)(\omega) - \sup_{K\in\overline{\mathcal{K}_1}} K(b)(\omega) = 0 \tag{16}$$

for all $\omega \in \Omega \setminus N_0$. Since the difference between any major function of order 1 and any minor function of order 1 is nondecreasing (see [3; Lemma 2.5]), from (15) and (16) we get:

$$\inf_{G\in\overline{\mathcal{G}_1}} G(x) - \sup_{K\in\overline{\mathcal{K}_1}} K(x) = 0 \tag{17}$$

uniformly with respect to $x \in [a,b]$ and

$$\inf_{G\in\overline{\mathcal{G}}_1} G(x)(\omega) - \sup_{K\in\overline{\mathcal{K}}_1} K(x)(\omega) = 0 \tag{18}$$

for all $\omega \in \Omega \setminus N_0$ and uniformly with respect to $x \in [a,b]$.

Since G and K are major and minor functions of order 1 for f respectively, then for all $\omega \in \Omega \setminus N_0$ the maps G_ω and K_ω are major and minor functions of order 1 for f_ω respectively (where the space R is replaced by $\mathbb{R}$): we denote the class of such major and minor functions by the symbols $\mathcal{G}_1^\omega$ and $\mathcal{K}_1^\omega$ respectively; furthermore, let $\mathcal{G}_*^\omega$ and $\mathcal{K}_*^\omega$ be the classes of those functions $G_*^\omega \in \mathcal{G}_1^\omega$ and $K_*^\omega \in \mathcal{K}_1^\omega$ such that there exists $G \in \overline{\mathcal{G}_1}$ with $G_*^\omega(x) = G(x)(\omega)$ for all $x \in [a,b]$ and such that there exists $K \in \overline{\mathcal{K}_1}$ with $K_*^\omega(x) = K(x)(\omega)$ for all $x \in [a,b]$ respectively. For all $\omega \in \Omega \setminus N_0$ we have:

$$\begin{aligned} 0 &\le \inf_{G_1\in\mathcal{G}_1^\omega} G_1(x) - \sup_{K_1\in\mathcal{K}_1^\omega} K_1(x) \\ &\le \inf_{G_*^\omega\in\mathcal{G}_*^\omega} G_*^\omega(x) - \sup_{K_*^\omega\in\mathcal{K}_*^\omega} K_*^\omega(x) \\ &= \inf_{G\in\overline{\mathcal{G}_1}} [G(x)(\omega)] - \sup_{K\in\overline{\mathcal{K}_1}} [K(x)(\omega)] \\ &= \Big[\inf_{G\in\overline{\mathcal{G}_1}} G(x)\Big](\omega) - \Big[\sup_{K\in\overline{\mathcal{K}_1}} K(x)\Big](\omega) = 0 \end{aligned} \tag{19}$$

uniformly with respect to $x \in [a,b]$. From (19) it follows that for all $\omega \in \Omega \setminus N_0$ we get:

$$\left((\mathcal{H})\int_a^x f(t)\,\mathrm{d}t\right)(\omega) = (\mathcal{H})\int_a^x f_\omega(t)\,\mathrm{d}t \qquad \text{for all} \quad x \in [a,b]. \tag{20}$$

Now for each $\omega \in \Omega \setminus N_0$ and $x \in [a,b]$ let

$$F_\omega(x) = F^{(1)}(x)(\omega)\,, \qquad \tilde{F}_\omega(x) = (\mathcal{H})\int_a^x f_\omega(t)\,\mathrm{d}t\,. \tag{21}$$

Since $\tilde{F}_\omega(x) = F_\omega(x)$ for every $\omega \in \Omega \setminus N_0$ and $x \in [a,b]$ and since $\tilde{F}_\omega$ is continuous for each $\omega \in \Omega \setminus N_0$, then obviously F_ω is continuous for each $\omega \in \Omega \setminus N_0$. From this and Proposition 2.3 it follows that $F^{(1)}$ is uniformly continuous in $[a,b]$. □

We now introduce the major and minor functions of order 2, which will be useful in order to define the Perron integral of order 2 in Riesz spaces.

DEFINITION 3.5. Let R be a Dedekind complete Riesz space, and $f\colon [a,b] \to R$. We say that a uniformly continuous function $\Psi\colon [a,b] \to R$ is a *major function of order* 2 for f if

3.5.1) $\Psi(a) = \Psi(b) = 0$;

3.5.2) there exists $\delta \in \Delta$ such that

$$\frac{\Psi(\xi+h) - 2\Psi(\xi) + \Psi(\xi-h)}{h^2} \geq f(\xi) \tag{22}$$

for every $\xi \in [a,b]$ and for all $h \in \mathbb{R}^+$ such that ξ, $\xi+h$, $\xi-h \in [a,b]$ and $|h| \leq \delta(\xi)$.

From now on, we denote by $\mathcal{G}_2$ the class of all major functions of order 2 for f. A uniformly continuous function Ψ_* satisfying only 3.5.2) is called *pre-major function of order* 2 for f.

We say that a uniformly continuous function $\Phi\colon [a,b] \to R$ is a *minor function of order* 2 for f if

3.5.3) $\Phi(a) = \Phi(b) = 0$;

3.5.4) there exists $\delta \in \Delta$ such that

$$\frac{\Phi(\xi+h) - 2\Phi(\xi) + \Phi(\xi-h)}{h^2} \leq f(\xi) \tag{23}$$

for every $\xi \in [a,b]$ and for all $h \in \mathbb{R}^+$ such that ξ, $\xi+h$, $\xi-h \in [a,b]$ and $|h| \leq \delta(\xi)$.

From now on, we denote by $\mathcal{K}_2$ the set of all minor functions of order 2 for f. A uniformly continuous function Φ_* satisfying only 3.5.4) is called *pre-minor function of order* 2 for f.

Analogously as in the case of major and minor functions of order 1, from now on, if Ψ and Φ are a major and a minor function of order 2 for f, and δ_1 and δ_2 satisfy 3.5.2) and 3.5.4) respectively, then we say that δ_1 and δ_2 are said to be *compatible* with Ψ and Φ respectively.

We now state the following two Lemmas, which are essential in order that our Perron-type integral of order 2 in Riesz spaces is well-defined. For the sake of clearness, we first state these Lemmas, we give the definition of our integral and in the sequel we will prove the Lemmas.

LEMMA 3.6. *Let $f\colon [a,b] \to R$ have both major and minor functions of order 2. If $\Psi \in \mathcal{G}_2$ and $\Phi \in \mathcal{K}_2$, then we have:*

$$\Psi(x) \le \Phi(x) \qquad \text{for all} \quad x \in [a,b]. \tag{24}$$

LEMMA 3.7. *Under the same hypotheses as in Lemma 3.6, if $\emptyset \neq \mathcal{G}_2$, $\emptyset \neq \mathcal{K}_2$ and there exists $c \in]a,b[$ such that $\sup_{\Psi \in \mathcal{G}_2} \Psi(c) = \inf_{\Phi \in \mathcal{K}_2} \Phi(c)$, then we have:*

$$\sup_{\Psi \in \mathcal{G}_2} \Psi(x) = \inf_{\Phi \in \mathcal{K}_2} \Phi(x) \tag{25}$$

uniformly with respect to $x \in [a,b]$.

DEFINITION 3.8. A map $f\colon [a,b] \to R$ is said to be *Perron integrable of order* 2 (or briefly $(\mathcal{P}^2)$*-integrable*) in $[a,b]$ if f has both major and minor functions of order 2, and if the following equality holds:

$$\sup_{\Psi \in \mathcal{G}_2} \Psi(x) = \inf_{\Phi \in \mathcal{K}_2} \Phi(x) \tag{26}$$

uniformly with respect to $x \in [a,b]$. In this case, the map $F^{(2)}\colon [a,b] \to R$ which associates to every $x \in [a,b]$ the common value in (26) is called the $(\mathcal{P}^2)$*-integral function* associated to f, and we will write

$$(\mathcal{P}^2) \int_{a,b,x} f(t)\, \mathrm{d}t = F^{(2)}(x), \qquad x \in [a,b], \tag{27}$$

or

$$(\mathcal{P}^2) \int_{a,b,x} f = F^{(2)}(x), \qquad x \in [a,b]. \tag{28}$$

Proof of Lemma 3.6. Let Ω be as in Theorem 2.2, and let Ψ and Φ be any major function and any minor function of order 2 for f respectively. Let us define $T\colon [a,b] \to R$ as follows:

$$T(x) = \Psi(x) - \Phi(x) \qquad \text{for all} \quad x \in [a,b]. \tag{29}$$

Let $\delta \in \Delta$ be compatible with both Ψ and Φ. For each $\xi \in [a,b]$ and $h \in \mathbb{R}^+$ such that $\xi + h$, $\xi - h \in [a,b]$ and $|h| \le \delta(\xi)$ we have:

$$\Psi(\xi + h) - 2\Psi(\xi) + \Psi(\xi - h) \ge \Phi(\xi + h) - 2\Phi(\xi) + \Phi(\xi - h), \tag{30}$$

that is:

$$T(\xi + h) - 2T(\xi) + T(\xi - h) \ge 0. \tag{31}$$

Since Ψ and Φ are uniformly continuous, then T is uniformly continuous too, and hence bounded. Thus there exist a nowhere dense set $N^{**} \subset \Omega$ and $L^{**} \in R$ such that:

$$(\forall\, x \in [a,b])\,(\forall\, \omega \in \Omega \setminus N^{**})\,\big(|T(x)(\omega)| \le L^{**}(\omega) \in \mathbb{R}\big)\,. \tag{32}$$

For each $\omega \in \Omega \setminus N^{**}$ and $x \in [a,b]$, define $T_\omega \colon [a,b] \to \mathbb{R}$ as follows:

$$T_\omega(x) = T(x)(\omega)\,. \tag{33}$$

We observe that by virtue of Proposition 2.3 there exists a meager set $N \subset \Omega$, $N \supset N^{**}$, such that T_ω is continuous in $[a,b]$ for all $\omega \in \Omega \setminus N$; moreover for every $\omega \in \Omega$ we have:

$$T_\omega(\xi + h) - 2T_\omega(\xi) + T_\omega(\xi - h) \ge 0 \tag{34}$$

for each $\xi \in [a,b]$ and $h \in \mathbb{R}^+$ such that $\xi + h,\ \xi - h \in [a,b]$ and $|h| \le \delta(\xi)$. Thus, by virtue of a well-known theorem on *real-valued* convex function (see [13; p. 23, Theorem 10.7]), we have that for every $\omega \in \Omega \setminus N$ the map T_ω is a convex function such that $T_\omega(a) = T_\omega(b) = 0$. From this it follows that $T_\omega(x) \le 0$ for all $\omega \in \Omega \setminus N$ and $x \in [a,b]$. Since the complement of a meager subset of Ω is a dense subset of Ω, we get that $T_\omega(x) \le 0$ for all $\omega \in \Omega$ and $x \in [a,b]$ and hence $T(x) \le 0$ for all $x \in [a,b]$. This completes the proof. □

P r o o f o f L e m m a 3.7. Let Ω, Ψ, Φ, T, N be as in the proof of Lemma 3.6. Since T_ω is convex for all $\omega \in \Omega \setminus N$ and the complement of a meager subset of Ω is a dense subset of Ω, it follows that T is convex in $[a,b]$.

Fix now arbitrarily $c \in \,]a,b[$, and pick $x \in [a,c[$. From convexity of T in $[a,b]$ it follows that:

$$T(x)\frac{b-c}{b-x} + T(b)\frac{c-x}{b-x} \ge T(c)\,. \tag{35}$$

Since $T(b) = 0$, we get:

$$T(c)\frac{b-x}{b-c} \le T(x) \qquad \text{for all} \quad x \in [a,c[\,. \tag{36}$$

Since $T(c) \le 0$ and $b - x \le b - a$, from (36) we obtain:

$$T(c)\frac{b-a}{b-c} \le T(x) \qquad \text{for all} \quad x \in [a,c[\,. \tag{37}$$

Proceeding analogously as above it is possible to prove that:

$$T(c)\frac{b-a}{c-a} \le T(y) \qquad \text{for all} \quad y \in \,]c,b]\,. \tag{38}$$

From (37) and (38) it follows that for every $c \in \,]a,b[$ there exists a positive real number C^* such that for every $x \in [a,b]$ and for every $\Psi \in \mathcal{G}_2$ and $\Phi \in \mathcal{K}_2$ we have:

$$C^* \cdot \big[\Psi(c) - \Phi(c)\big] \le \Psi(x) - \Phi(x) \le 0\,. \tag{39}$$

From (39) the assertion of Lemma 3.7 follows. □

We now investigate some properties of the $(\mathcal{P}^2)$-integral function.

PROPOSITION 3.9. *Let $f\colon [a,b] \to R$ be $(\mathcal{P}^2)$-integrable, $F^{(2)}$ be as in* (27), *$\Psi \in \mathcal{G}_2$ and $\Phi \in \mathcal{K}_2$. Then $\Psi - F^{(2)}$ and $F^{(2)} - \Phi$ are convex.*

Proof. Proceeding analogously as in the real case, it is easy to check that the pointwise supremum of any arbitrary family of convex functions is a convex function. Now, as $\Psi - \Phi$ is convex for every $\Psi \in \mathcal{G}_2$ and for each $\Phi \in \mathcal{K}_2$, we get:

$$\begin{aligned} \Psi(x) - F^{(2)}(x) &= \sup_{\Phi \in \mathcal{K}_2} \left[\Psi(x) - \Phi(x)\right], \\ F^{(2)}(x) - \Phi(x) &= \sup_{\Psi \in \mathcal{G}_2} \left[\Psi(x) - \Phi(x)\right] \end{aligned} \tag{40}$$

uniformly with respect to $x \in [a,b]$. This concludes the proof of Proposition 3.9. □

PROPOSITION 3.10. *Let $f\colon [a,b] \to R$ be $(\mathcal{P}^2)$-integrable, and $F^{(2)}$ be as in* (27). *Then $F^{(2)}$ is uniformly continuous in $[a,b]$.*

Proof. The assertion follows from the fact that the major and minor functions of order 2 are uniformly continuous by definition and that suprema and infima of uniformly continuous functions are uniformly continuous, as it is readily seen. □

The proof of the following proposition is straightforward.

PROPOSITION 3.11. *If $f_1, f_2\colon [a,b] \to R$ are $(\mathcal{P}^2)$-integrable, then $f_1 + f_2$ is $(\mathcal{P}^2)$-integrable too, and we have:*

$$(\mathcal{P}^2)\int_{a,b,x} (f_1 + f_2) = (\mathcal{P}^2)\int_{a,b,x} f_1 + (\mathcal{P}^2)\int_{a,b,x} f_2 \,. \tag{41}$$

Moreover, if $f\colon [a,b] \to R$ is $(\mathcal{P}^2)$-integrable and $c \in \mathbb{R}$, then cf is $(\mathcal{P}^2)$-integrable too, and we get:

$$(\mathcal{P}^2)\int_{a,b,x} cf = c\,(\mathcal{P}^2)\int_{a,b,x} f \,. \tag{42}$$

We now prove the following:

PROPOSITION 3.12. *If $f\colon [a,b] \to R$ is $(\mathcal{P}^2)$-integrable in $[a,b]$, then f is $(\mathcal{P}^2)$-integrable in every subinterval $[a',b'] \subset [a,b]$.*

Proof. Let us denote by $\mathcal{G}_2^{[a,b]}$, $\mathcal{K}_2^{[a,b]}$, $\mathcal{G}_2^{[a',b']}$, $\mathcal{K}_2^{[a',b']}$ the classes of all major functions of order 2 for f, all minor functions of order 2 for f, all major functions of order 2 for $f\big|_{[a',\,b']}$ and all minor functions of order 2 for

$f\,|_{[a',b']}$, where $f\,|_{[a',b']}$ is the restriction of f to $[a',b']$. For every $\Psi \in \mathcal{G}_2^{[a,b]}$ and $\Phi \in \mathcal{K}_2^{[a,b]}$, let us define $\Psi_0 \colon [a,b] \to R$, $\Phi_0 \colon [a,b] \to R$ as follows:

$$\Psi_0(x) = \Psi(x) - \Psi(a') - \frac{\Psi(b') - \Psi(a')}{b' - a'}(x - a') \qquad \text{for all} \quad x \in [a,b], \tag{43}$$

$$\Phi_0(x) = \Phi(x) - \Phi(a') - \frac{\Phi(b') - \Phi(a')}{b' - a'}(x - a') \qquad \text{for all} \quad x \in [a,b]. \tag{44}$$

Since no confusion can arise, we denote still by Ψ_0 and Φ_0 the restrictions to $[a',b']$ of Ψ_0 and Φ_0 respectively. Let $\delta \in \Delta$ be compatible with Ψ and Φ. It is easy to check that

$$\Psi_0(a') = \Psi_0(b') = \Phi_0(a') = \Phi_0(b') = 0$$

and that for every $\xi \in [a,b]$ and $h \in \mathbb{R}^+$ such that $\xi + h$, $\xi - h \in [a,b]$ and $|h| \le \delta(\xi)$ we have:

$$\begin{aligned} \Psi_0(\xi + h) - 2\Psi_0(\xi) + \Psi_0(\xi - h) &= \Psi(\xi + h) - 2\Psi(\xi) + \Psi(\xi - h), \\ \Phi_0(\xi + h) - 2\Phi_0(\xi) + \Phi_0(\xi - h) &= \Phi(\xi + h) - 2\Phi(\xi) + \Phi(\xi - h). \end{aligned} \tag{45}$$

From (45) it follows that $\Psi_0 \in \mathcal{G}_2^{[a',b']}$ and $\Phi_0 \in \mathcal{K}_2^{[a',b']}$.

Let now $\mathcal{G}_0^{[a',b']}$ and $\mathcal{K}_0^{[a',b']}$ be the subclasses of $\mathcal{G}_2^{[a',b']}$ and $\mathcal{K}_2^{[a',b']}$ whose elements Ψ_0 and Φ_0 are such that there exists $\Psi \in \mathcal{G}_2^{[a,b]}$ such that Ψ_0 can be represented as in (43) and there exists $\Phi \in \mathcal{K}_2^{[a,b]}$ such that Φ_0 can be represented as in (44) respectively. For every $x \in [a,b]$ we have:

$$\begin{aligned} 0 &\le \inf_{\Phi \in \mathcal{K}_2^{[a',b']}} \Phi(x) - \sup_{\Psi \in \mathcal{G}_2^{[a',b']}} \Psi(x) \\ &\le \inf_{\Phi_0 \in \mathcal{K}_0^{[a',b']}} \Phi_0(x) - \sup_{\Psi_0 \in \mathcal{G}_0^{[a',b']}} \Psi_0(x) \\ &\le \inf_{\Phi \in \mathcal{K}_2^{[a,b]}} \left[\Phi(x) - \Phi(a') - \frac{\Phi(b') - \Phi(a')}{b' - a'}(x - a')\right] \\ &\qquad - \sup_{\Psi \in \mathcal{G}_2^{[a,b]}} \left[\Psi(x) - \Psi(a') - \frac{\Psi(b') - \Psi(a')}{b' - a'}(x - a')\right] = 0. \end{aligned} \tag{46}$$

This completes the proof of Proposition 3.12. □

We now compare the Perron integral of order 1 with the Perron integral of order 2 in Riesz spaces. We begin with a proposition, whose proof is straightforward.

PROPOSITION 3.13. *Let $f\colon [a,b] \to R$ be such that Ψ_* is a pre-major function of order 2 for f and Φ_* is a pre-minor function of order 2 for f. Then the maps $\Psi\colon [a,b] \to R$, $\Phi\colon [a,b] \to R$, defined by setting:*

$$\Psi(x) = \Psi_*(x) - \Psi_*(a) - \frac{\Psi_*(b) - \Psi_*(a)}{b-a}(x-a) \qquad \textit{for all} \quad x \in [a,b], \tag{47}$$

$$\Phi(x) = \Phi_*(x) - \Phi_*(a) - \frac{\Phi_*(b) - \Phi_*(a)}{b-a}(x-a) \qquad \textit{for all} \quad x \in [a,b], \tag{48}$$

are a major and a minor function of order 2 for f respectively.

In the sequel we will say that Ψ and Φ are the *naturalized major functions* of Ψ_*, Φ_* respectively.

We now prove the following:

THEOREM 3.14. *Let R be any Dedekind complete Riesz space. If $f\colon [a,b] \to R$ is Perron integrable of order 1, then f is Perron integrable of order 2. Moreover we have:*

$$(\mathcal{P}^2) \int\limits_{a,b,x} f(t)\ \mathrm{d}t = -\frac{x-a}{b-a}\,(\mathcal{H}) \int\limits_a^b F^1(t)\ \mathrm{d}t + (\mathcal{H}) \int\limits_a^x F^1(t)\ \mathrm{d}t\,, \tag{49}$$

where $F^{(1)}$ is as in (7).

P r o o f. First of all we observe that the $(\mathcal{H})$-integral function $F^{(1)}$ is $(\mathcal{H})$-integrable in $[a,b]$ because it is uniformly continuous (see Lemma 3.4) and every uniformly continuous Riesz-space-valued function is $(\mathcal{H})$-integrable (see also [6; Theorem 2]). Let $G \in \mathcal{G}_1$, $K \in \mathcal{K}_1$ and $\delta \in \Delta$ be an element compatible with both G and K. Set

$$\Psi(x) = (\mathcal{H}) \int\limits_a^x G(t)\ \mathrm{d}t\,, \quad \Phi(x) = (\mathcal{H}) \int\limits_a^x K(t)\ \mathrm{d}t \qquad \text{for all} \quad x \in [a,b]\,. \tag{50}$$

We observe that Ψ and Φ are uniformly continuous, by virtue of Lemma 3.4. Fix now arbitrarily $\xi, u, v \in [a,b]$ with $\xi - \delta(\xi) \le u \le \xi \le v \le \xi + \delta(\xi)$, and suppose that $v - \xi = \xi - u$ and denote this common value by h.

Let $y \in [\xi, v]$. Taking the Henstock-Kurzweil integral in 3.1.2) (where u and v are replaced by ξ and y respectively) and integrating with respect to $y \in [\xi, v]$ we get:

$$\Psi(v) - \Psi(\xi) - G(\xi)(v-\xi) \ge \frac{f(\xi)}{2}(\xi - v)^2\,. \tag{51}$$

Proceeding analogously as above it is possible to check that

$$\Psi(u) - \Psi(\xi) + G(\xi)(\xi - u) \ge \frac{f(\xi)}{2}(u - \xi)^2\,. \tag{52}$$

From (51) and (52) we have:

$$\Psi(v) - 2\Psi(\xi) + \Psi(u) \geq h^2 f(\xi) . \tag{53}$$

From (53) it follows that Ψ is a pre-major function of order 2 for f. Proceeding analogously as above, it is possible to check that Φ is a pre-minor function of order 2 for f. Let $\mathcal{G}_\iota$ and $\mathcal{K}_\iota$ be the subclasses of $\mathcal{G}_2$ and $\mathcal{K}_2$ such that every element of $\mathcal{G}_\iota$ is the naturalized major function of the $(\mathcal{H})$-integral function of some maps of $\mathcal{G}_1$ and each element of $\mathcal{K}_\iota$ is the naturalized minor function of the $(\mathcal{H})$-integral function of some mappings of $\mathcal{K}_1$ respectively. For every $x \in [a,b]$, we get:

$$\begin{aligned} 0 &\leq \inf_{\Phi \in \mathcal{K}_2} \Phi(x) - \sup_{\Psi \in \mathcal{G}_2} \Psi(x) \\ &\leq \inf_{\Phi_* \in \mathcal{K}_\iota} \Phi_*(x) - \sup_{\Psi_* \in \mathcal{G}_\iota} \Psi_*(x) \\ &= \inf_{G \in \mathcal{G}_1} \left[\int_a^x G(t)\, dt - \frac{\int_a^b G(t)\, dt}{b-a}(x-a) \right] \\ &\quad - \sup_{K \in \mathcal{K}_1} \left[\int_a^x K(t)\, dt - \frac{\int_a^b K(t)\, dt}{b-a}(x-a) \right] = 0 . \end{aligned} \tag{54}$$

The assertion follows from (54) and from the fact that

$$\inf_{G \in \mathcal{G}_1} G(x) = \sup_{K \in \mathcal{K}_1} K(x) = (\mathcal{H}) \int_a^x f(t)\, dt \tag{55}$$

uniformly with respect to $x \in [a,b]$, which implies:

$$\inf_{G \in \mathcal{G}_1} (\mathcal{H}) \int_a^x f(t)\, dt - \sup_{K \in \mathcal{K}_1} (\mathcal{H}) \int_a^x f(t)\, dt = (\mathcal{H}) \int_a^x F^{(1)}(t)\, dt . \tag{56}$$

□

We note that, even in the real case, there exist some $(\mathcal{P}^2)$-integrable functions, which are not $(\mathcal{P})$-integrable. (see [13; pp. 86–87]).

DEFINITION 3.15. We say that a function $f\colon [a,b] \to R$ is *second uniformly differentiable* if there exists a function $g_2\colon [a,b] \to R$ such that

$$\text{(o)-}\lim_{n \to +\infty} \left(\sup\left\{ \left| \frac{f(\xi+h) - 2f(\xi) + f(\xi-h)}{h^2} - g_2(\xi) \right| : \ \xi, h \in B_n \right\} \right) = 0 ,$$

where $B_n \equiv \{x \in [a,b],\ h \in \mathbb{R}^+ : \xi + h \in [a,b],\ \xi - h \in [a,b] \text{ and } |h| \le 1/n\}$ for all $n \in \mathbb{N}$.

The map g_2 will be called the *second uniform derivative* of f, or simply *second derivative* when no confusion can arise.

Remark 3.16. It is easy to see that, if f is uniform differentiable with derivative f_1, and if f_1 is uniform differentiable with derivative f_2, then f is second uniform differentiable, and its second derivative is f_2. The converse in general is not true, even in the case $R = \mathbb{R}$; indeed, the Heaviside function, defined as follows:

$$H(x) = \begin{cases} 1 & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } -1 \le x < 0 \end{cases} \tag{57}$$

is second uniform differentiable, but not continuous, and hence not uniform differentiable.

THEOREM 3.17. *Let $f\colon [a,b] \to R$ be uniformly continuous and second uniformly differentiable, and let f'' be its second derivative. Then f'' is $(\mathcal{P}^2)$-integrable, and for every $x \in [a,b]$ we have:*

$$(b-a)(\mathcal{P}^2) \int_{a,b,x} f''(t)\ \mathrm{d}t = -(b-x)f(a) + (b-a)f(x) - (x-a)f(b)\,. \tag{58}$$

P r o o f. Let $(p_n)_n$ be an (o)-sequence such that for all $n \in \mathbb{N}$ we have:

$$\sup\left\{ \left|\tfrac{f(\xi+h)-2f(\xi)+f(\xi-h)}{h^2} - f''(\xi)\right| :\ x,h \in B_n \right\} \le p_n\,, \tag{59}$$

where B_n, $n \in \mathbb{N}$, is as above. For each $\delta \in \Delta$ and $x \in [a,b]$, set

$$f_\delta(x) = f(x) + p_\delta \frac{x^2}{2}\,, \tag{60}$$

$$g_\delta(x) = f(x) - p_\delta \frac{x^2}{2}\,, \tag{61}$$

$$f_\delta^*(x) = f_\delta(x) - \frac{(b-x)f_\delta(a) + (x-a)f_\delta(b)}{b-a}\,, \tag{62}$$

$$g_\delta^*(x) = g_\delta(x) - \frac{(b-x)g_\delta(a) + (x-a)g_\delta(b)}{b-a}\,. \tag{63}$$

It is easy to check that f_δ^* and g_δ^* are a major function and a minor function for f'' respectively, and that:

$$\sup_{\delta\in\Delta} f_\delta^*(x) = \inf_{\delta\in\Delta} g_\delta^*(x) = f(x) - \frac{(b-x)f(a) + (x-a)f(b)}{b-a} \qquad \text{for all} \quad x \in [a,b]\,. \tag{64}$$

From (64) the assertion follows. □

4. Applications

Let R be a Dedekind complete Riesz space. Analogously as in the classical case it is possible to give the definition of uniform and total convergence of series and of series of functions. Let us denote by L^0 the space $L^0(X, \mathcal{B}, \mu)$, where $(X, \mathcal{B}, \mu)$ is a measure space with $\mu\colon \mathcal{B} \to \tilde{\mathbb{R}}_0^+$ σ-additive and σ-finite. We recall that in L^0 order-convergence coincides with almost everywhere convergence. A *stochastic process* f is an element of $(L^0)^I$, where I is a proper connected subset of $\mathbb{R}$. Throughout this section, we will deal with periodic stochastic processes of period 2π: so we can assume, without loss of generality, that $I = [-2\pi, 2\pi]$. We will consider the Fourier series associated with a stochastic process f, that is (formally) the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,, \tag{65}$$

where

$$a_n \equiv \frac{1}{\pi}(\mathcal{H}) \int_{-\pi}^{\pi} f(x) \cos nx \; \mathrm{d}x\,, \qquad b_n \equiv \frac{1}{\pi}(\mathcal{H}) \int_{-\pi}^{\pi} f(x) \sin nx \; \mathrm{d}x \tag{66}$$

for all $n \in \mathbb{N}$, provided that the right members of (66) make sense.

We now give an application of Theorem 3.17 to the trigonometric series associated with stochastic processes, and we show that the Perron integral of order two is such that, in a certain sense, every trigonometric series everywhere convergent in $[-2\pi, 2\pi]$ and almost everywhere convergent with respect to $t \in X$ is the Fourier series of its sum. We observe that, even in the real case, this property is not satisfied by the Perron integral of order 1 (see [14; p. 86]).

We begin with recalling the definition of regular matrix (see [13; p. 74]):

DEFINITION 4.1. Given an infinite matrix of real numbers $M = [a_{n,\nu}]_{n,\nu}$ $(n, \nu \in \mathbb{N} \cup \{0\})$ and $n \in \mathbb{N}$, set $A_n = \sum\limits_{\nu=0}^{\infty} |a_{n,\nu}|$, $O_n = \sum\limits_{\nu=0}^{\infty} a_{n,\nu}$, provided that these quantities exist in $\mathbb{R}$. We say that M is *regular* if it satisfies the following properties:

4.1.1) $\lim\limits_{n\to+\infty} a_{n,\nu} = 0$ for all $\nu \in \mathbb{N} \cup \{0\}$;
4.1.2) the sequence $(O_n)_n$ is bounded;
4.1.3) $\lim\limits_{n\to+\infty} A_n = 1$.

If, for all $x \in [-2\pi, 2\pi]$, $\underline{p}(x) = \big(p_\nu(x)\big)_{\nu\in\mathbb{N}\cup\{0\}}$ is a sequence of functions defined in $[-2\pi, 2\pi]$ and with valued in L^0, such that (o)-$\lim\limits_{\nu\to+\infty} p_\nu(x) = r(x) \in L^0$

for all $x \in [-2\pi, 2\pi]$, and $M = [a_{n,\nu}]_{n,\nu}$ $(n, \nu \in \mathbb{N} \cup \{0\})$ is a regular matrix, then for every $x \in [-2\pi, 2\pi]$ let $\underline{\sigma}(x) = (\sigma_n(x))_{n \in \mathbb{N} \cup \{0\}}$ be that sequence of elements of L^0 such that

$$\sigma_n(x) = \sum_{j=0}^{\infty} a_{n,j} p_j(x) \qquad \text{for all} \quad n \in \mathbb{N} \cup \{0\}.$$

We write shortly $\underline{\sigma}(x) = M \cdot \underline{p}(x)$ for all $x \in [-2\pi, 2\pi]$. The sequence $(\sigma_n(x))_{n \in \mathbb{N} \cup \{0\}}$ is well-defined for every $x \in [-2\pi, 2\pi]$: indeed, proceeding analogously as in the real case (see [13; p. 74, Theorem 1.2]), it is possible to prove the following:

PROPOSITION 4.2. *Let M be a regular matrix, and assume that S_ν: $[-2\pi, 2\pi] \to L^0$, $\nu \in \mathbb{N}$, are such that there exist S: $[-2\pi, 2\pi] \to L^0$ and $O^* \subset X$ such that $\mu(O^*) = 0$ and*

$$\begin{aligned}(\forall\, \varepsilon > 0)(\forall\, t \in X \setminus O^*)(\forall\, x \in [-2\pi, 2\pi])\,(\exists\, \overline{\nu} = \overline{\nu}(\varepsilon, t, x))(\forall\, \nu \geq \overline{\nu})& \\ (|S_\nu(x)(t) - S(x)(t)| \leq \varepsilon)\,.&\end{aligned} \tag{67}$$

Let $p_\nu(x) = S_\nu(x)$, $\nu \in \mathbb{N}$, $x \in [-2\pi, 2\pi]$, and under the same hypotheses and notations as above let $\underline{\sigma}(x) = (\sigma_n(x))_{n \in \mathbb{N}} = M \cdot \underline{p}(x)$. Then we have:

$$\begin{aligned}(\forall\, \varepsilon > 0)(\forall\, t \in X \setminus O^*)(\forall\, x \in [-2\pi, 2\pi])\,(\exists\, \overline{n} = \overline{n}(\varepsilon, t, x))(\forall\, n \geq \overline{n})& \\ (|\sigma_n(x)(t) - S(x)(t)| \leq \varepsilon)\,,&\end{aligned} \tag{68}$$

where O^* is as in (67).

The following theorem is an application of Theorem 3.17 and shows that in L^0 the sum of every trigonometric series convergent pointwise in $[-2\pi, 2\pi]$ and almost everywhere with respect to $t \in X$ is a $(\mathcal{P}^2)$-integrable function defined in $[-2\pi, 2\pi]$ and with values in L^0.

THEOREM 4.3. *Let*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,, \qquad x \in [-2\pi, 2\pi]\,, \tag{69}$$

be a series of elements of L^0, whose partial sums $S_n(x)$ satisfy (67) *and $\lim_{n \to +\infty} a_n(t) = \lim_{n \to +\infty} b_n(t) = 0$ for almost all $t \in X$. If S is as in* (67), *then S is the sum of the series* (69) *and is $(\mathcal{P}^2)$-integrable in $[-2\pi, 2\pi]$. Moreover, the functions $x \mapsto S(x) \cos kx$ and $x \mapsto S(x) \sin kx$, $k \in \mathbb{N}$, are $(\mathcal{P}^2)$-integrable in $[-2\pi, 2\pi]$, and we have:*

$$a_k = -\frac{1}{\pi^2} (\mathcal{P}^2) \int\limits_{-2\pi, 2\pi, 0} S(t) \cos kt \,\mathrm{d}t \qquad \textit{for all} \quad k \in \mathbb{N} \cup \{0\}\,, \tag{70}$$

$$b_k = -\frac{1}{\pi^2} (\mathcal{P}^2) \int\limits_{-2\pi, 2\pi, 0} S(t) \sin kt \,\mathrm{d}t \qquad \textit{for all} \quad k \in \mathbb{N}. \tag{71}$$

P r o o f . Let $A_n(x)$, $n \in \mathbb{N} \cup \{0\}$, $x \in [-2\pi, 2\pi]$, be the general term of the series (69). Moreover, let us define the function $F\colon [-2\pi, 2\pi] \to L^0$ as follows:

$$F(x) = \frac{1}{4} a_0 x^2 - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}, \qquad x \in [-2\pi, 2\pi]. \tag{72}$$

It is easy to check that F is uniformly continuous in $[-2\pi, 2\pi]$, since the series in (72) is totally convergent. Proceeding analogously as in [13; p. 319], we get:

$$\frac{F(x+2h) - 2F(x) + F(x-2h)}{4h^2} = A_0 + \sum_{\nu=1}^{\infty} A_\nu(x) \left(\frac{\sin \nu h}{\nu h}\right)^2 \tag{73}$$

for all $x \in [-2\pi, 2\pi]$ and $h \in \mathbb{R}^+$ such that $x + 2h$, $x - 2h \in [-2\pi, 2\pi]$. For all $h \in \mathbb{R} \setminus \{0\}$, let

$$u(h) = \frac{\sin^2 h}{h^2}, \tag{74}$$

and let $u(0) = 1$. Fix arbitrarily a sequence (h_n) of positive real numbers, with $\lim\limits_{n\to+\infty} h_n = 0$. From (73), for $n \in \mathbb{N}$ and $x \in [-2\pi, 2\pi]$ such that $x+2h_n$, $x-2h_n \in [-2\pi, 2\pi]$ we obtain:

$$\frac{F(x+2h_n) - 2F(x) + F(x-2h_n)}{4h_n^2} = \sum_{\nu=0}^{\infty} S_\nu(x) \{u(\nu h_n) - u((\nu+1)h_n)\}. \tag{75}$$

We note that the matrix $[a_{n,\nu}]_{n,\nu} = \big(u(\nu h_n) - u((\nu+1)h_n)\big)_{n,\nu}$ is regular (see also [13; p. 320]). Thus, by applying Proposition 4.2, it follows that for every sequence $(h_n)_{n\in\mathbb{N}}$ of positive real numbers with $\lim\limits_{n\to+\infty} h_n = 0$ we have that there exists a set O^* with $\mu(O^*) = 0$, $O^* = O^*((h_n)_n)$, such that for all $t \in X \setminus O^*$, for each $x \in [-2\pi, 2\pi]$, for every $\varepsilon > 0$ there exists $\overline{n} = \overline{n}(t, x, \varepsilon)$ such that for all $n \geq \overline{n}$ we have:

$$\left| \frac{F(x+2h_n)(t) - 2F(x)(t) + F(x-2h_n)(t)}{4h_n^2} - S(x)(t) \right| \leq \varepsilon, \tag{76}$$

where S is as in (67). This means that for all sequences $(h_n)_n$ of real numbers, with $\lim\limits_{n\to+\infty} h_n = 0$, we have:

$$\begin{aligned}(\mathrm{o})\text{-}\lim_{\delta\in\Delta} \Big[\sup\Big\{ \Big|\tfrac{F(x+2h_n)-2F(x)+F(x-2h_n)}{4h_n{}^2} - S(x)\Big| :& \\ x, x+2h_n, x-2h_n \in [-2\pi, 2\pi],\ 0 < |h_n| < \delta(x),\ n \in \mathbb{N}\Big\}\Big] &= 0.\end{aligned} \tag{77}$$

Consider now the following quantity:

$$\begin{aligned}I \equiv (\mathrm{o})\text{-}\lim_{\delta\in\Delta} \Big[\sup\Big\{ \Big|\tfrac{F(x+2h)-2F(x)+F(x-2h)}{4h^2} - S(x)\Big| :& \\ x, x+2h, x-2h \in [-2\pi, 2\pi],\ 0 < |h| < \delta(x)\Big\}\Big].&\end{aligned} \tag{78}$$

From (78), by super Dedekind completeness of L^0, it follows that there exists a sequence $(h_l)_l$ such that:

$$I = (\mathrm{o})\text{-}\lim_{\delta\in\Delta}\Big[\sup\Big\{\Big|\frac{F(x+2h_l)-2F(x)+F(x-2h_l)}{4h_l^2} - S(x)\Big| : \\ x, x+2h_l, x-2h_l \in [-2\pi, 2\pi],\ 0<|h_l|<\delta(x)\Big\}\Big]. \tag{79}$$

From (76) and (79) it follows that $I = 0$. Since L^0 is super Dedekind complete, from [1; Proposition 2.3, Proposition 3.8] it follows that:

$$I = (\mathrm{o})\text{-}\lim_{k\in\mathbb{N}}\Big[\sup\Big\{\Big|\frac{F(x+2h_l)-2F(x)+F(x-2h_l)}{4h_l^2} - S(x)\Big| : \\ x, x+2h_l, x-2h_l \in [-2\pi, 2\pi],\ 0<|h_l|<1/k\Big\}\Big]. \tag{80}$$

This concludes the proof of the first part of the theorem.

The proof of the second part is analogous to the one of [7; p. 305–306, Theorem 6.2], where the theorem is proved in the case $R = \mathbb{R}$. □

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REFERENCES

[1] BOCCUTO, A.: *Differential and integral calculus in Riesz spaces*, Tatra Mountains Math. Publ. **14** (1998), 293–323.

[2] BOCCUTO, A.: *Integration by parts with respect to the Henstock-Stieltjes integral in Riesz spaces*. Preprint, University of Perugia, 1999.

[3] BOCCUTO, A.: *The Perron-Stieltjes integral in Riesz spaces*. Preprint, University of Perugia, 1999.

[4] BULLEN, P. S.: *A survey of integration by parts for Perron integrals*, J. Austral. Math. Soc. Ser. A **40** (1986), 343–363.

[5] BULLEN, P. S.: *The P^n-integral*, Bull. Austral. Math. Soc. **14** (1972), 219–236.

[6] DUCHOŇ, M.—RIEČAN, B.: *On the Kurzweil-Stieltjes integral in ordered spaces*, Tatra Mountains Math. Publ. **8** (1996), 133–142.

[7] JAMES, R. D.: *A generalized integral II*, Canad. J. Math. **2** (1950), 297–306.

[8] LUXEMBURG, W. A. J.—ZAANEN, A. C.: *Riesz Spaces I*, North-Holland Publishing Co., Amsterdam, 1971.

[9] PFEFFER, W. F.: *The Riemann Approach to Integration*, Cambridge Univ. Press, Cambridge, 1993.

[10] RIEČAN, B.: *On the Kurzweil integral for functions with values in ordered spaces I*, Acta Math. Univ. Comenian. **56-57** (1989), 411–424.

[11] RIEČAN, B.—NEUBRUNN, T.: *Integral, Measure and Ordering*, Kluwer Acad. Publ.; Ister Science, Dordrecht; Bratislava, 1997.

[12] SKLYARENKO, V. A.: *Some properties of the P^2 integral*, Mat. Zametki **12** (1972), 693–700 (Russian) [English transl.: Math. USSR Sbornik **12** (1974), 856–860].

[13] ZYGMUND, A.: *Trigonometric Series, Vol. I*, Cambridge Univ. Press, Cambridge, 1968.

[14] ZYGMUND, A.: *Trigonometric Series, Vol. II*, Cambridge Univ. Press, Cambridge, 1968.

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